Goldbach's problem with primes in arithmetic progressions and in short intervals

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Abstract: We study the number of solutions in Goldbach's problem, where the primes are taken from arithmetic progressions as well as from short intervals. For this, we show mean value theorems in the style of Bombieri-Vinogradov's theorem and apply them in a sieve argument to get results for Goldbach's problem with primes from short intervals such that $p_i + 2$, $i = 1, 2, 3$, are almost-primes.

Binary mean value theorems and ternary consequences

Corollaries using Sieve Theory

The limits when putting conditions on the primes

A binary theorem of Kawada

As a special case of [Kawada 1993], the following is known:

Theorem: Let $X_1^{2/3} L^C < Y$, $Q \leq Y X_2^{-1/2} L^{-B}$, $X_2 + Y \leq X_1$. Then

$$\sum_{q \leq Q} \max_{a\,(q)}{}^{*} \sum_{k \in [X_1, X_1+Y]} \Bigg| \sum_{\substack{p_2+p_3=2k \\ p_2 \equiv a\,(q) \\ p_2 \in [X_2, X_2+Y]}} \log p_2 \log p_3 - \mathfrak{S}(2k, q, a) Y \Bigg| \ll Y^2 L^{-A}.$$

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We can put easily an additional condition on $2k$, namely such that $2k$ lies in an arithmetic progression (AP):

Corollary with $2k$ in AP

Theorem 1:
Let $X_1^{2/3}L^C < Y$, $Q_2 \leq YX_2^{-1/2}L^{-B}$ and $X_2 + Y \leq X_1$. Then, for fixed $a_1, a_2 \in \mathbb{N}$ and $X_1 > a_1$, we have

$$\sum_{q_1 \leq Q_1} \sum_{q_2 \leq Q_2} \sum_{\substack{k \in [X_1, X_1+Y] \\ 2k \equiv a_1\ (q_1)}} \Bigg| \sum_{\substack{p_2+p_3=2k \\ p_2 \equiv a_2\ (q_2) \\ p_2 \in [X_2, X_2+Y]}} \log p_2 \log p_3 - \mathfrak{S}(2k, q_2, a_2)Y \Bigg| \ll Y^2 L^{-A}.$$

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If in addition $Q_1 \le Y^{1/2}L^D$ holds for some fixed $D > 0$, then the estimate holds true with $\max_{a_1\ (q_1)}$ inserted after $\sum_{q_1 \le Q_1}$. Also $\max_{a_2\ (q_2)}$ can be inserted after $\sum_{q_2 \le Q_2}$.

Simple tools

This can be proven quickly using the following Lemmas:

Lemma 1:

Let $a \in \mathbb{N}$ be fixed, $Q \geq 1$, $N \in \mathbb{N}$ with $N > a$ and $v_N, \ldots, v_{2N-1} \in \mathbb{C}$. Then

$$\sum_{Q \leq q < 2Q} \Big| \sum_{\substack{N \leq n < 2N \\ n \equiv a\ (q)}} v_n \Big| \ll N^{1/2} (\log N)^{3/2} \Big(\sum_{N \leq n < 2N} |v_n|^2 \Big)^{1/2}.$$

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Lemma 2:

Let $Q \geq 1$, $N \in \mathbb{N}$ with $N \geq 1$ and $v_N, \ldots, v_{2N-1} \in \mathbb{C}$. Then

$$\sum_{Q \leq q < 2Q} \max_{a\ (q)} \Big| \sum_{\substack{N \leq n < 2N \\ n \equiv a\ (q)}} v_n \Big| \ll (N \log^2 Q + Q^2)^{1/2} \Big(\sum_{N \leq n < 2N} |v_n|^2 \Big)^{1/2}.$$

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Both Lemmas are simple consequences of Halasz-Montgomery's inequality.

A ternary mean value theorem as corollary

The following theorem can be deduced for the ternary Goldbach problem with extra conditions on the primes:

Corollary 1:
Let $X_1^{2/3}L^C \leq Y$, $X_2^{3/5+\varepsilon} \leq Y$, and $Q_i \leq YX_i^{-1/2}L^{-B}$ for $i = 1, 2$.
Let $n \geq X_1 + X_2 + 2Y$ be odd. Then

$$\sum_{q_1 \leq Q_1} \max_{a_1\,(q_1)}{}^{*} \sum_{q_2 \leq Q_2} \max_{a_2\,(q_2)}{}^{*} \left| \sum_{\substack{p_1+p_2+p_3=n \\ p_i \in [X_i, X_i+Y],\ i=1,2 \\ p_i \equiv a_i\ (q_i),\ i=1,2}} \log p_1 \log p_2 \log p_3 - \mathfrak{T}(n, q_1, a_1, q_2, a_2) Y^2 \right| \ll Y^2 L^{-A}.$$

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Proof: Sum up over $2k = n - p_1$ instead of p_1 and insert the estimate of the previous theorem. The correct singular series is obtained.

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The well known Theorem of [Perelli/Pintz 1993] states:
If $X_1^{1/3+\varepsilon} \ll R \ll X_1$, then

$$\sum_{2k \in [X_1, X_1+R]} \left| \sum_{p_2+p_3=2k} \log p_2 \log p_3 - \mathfrak{S}(2k)2k \right|^2 \ll R X_1^2 L^{-A}.$$

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Using minor adaptions, we can deduce:
Theorem 2:
Let $Y^{1/3+\varepsilon} \ll R \ll Y \asymp X_1 - X_2 - Y \geq 0$ and $X_2^{3/5+\varepsilon} \ll Y \ll X_2$, then

$$\sum_{2k\in[X_1,X_1+R]} \left| \sum_{\substack{p_2+p_3=2k \\ p_2\in[X_2,X_2+Y]}} \log p_2 \log p_3 - \mathfrak{S}(2k)Y \right| \ll RYL^{-A}.$$

Putting an AP-condition on $2k$

Using both lemmas, we deduce as above:

Theorem 3:
Let $Y^{1/3+\varepsilon} \ll R \ll Y \asymp X_1 - X_2 - Y \geq 0$ and $X_2^{3/5+\varepsilon} \ll Y \ll X_2$.
Fix $a \in \mathbb{N}$. Then, for $X_1 > a$,

$$\sum_{q \leq Q} \sum_{\substack{2k \in [X_1, X_1+R] \\ 2k \equiv a\ (q)}} \Bigg| \sum_{\substack{p_2+p_3=2k \\ p_2 \in [X_2, X_2+Y]}} \log p_2 \log p_3 - \mathfrak{S}(2k) Y \Bigg| \ll RYL^{-A}.$$

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If in addition $Q \leq R^{1/2} L^D$, the same estimate holds with $\max_{a\ (q)}$ inserted after $\sum_{q \leq Q}$.

Another ternary corollary

Again, we deduce for the ternary Goldbach problem in the same way as above (with $Y_1 = R$ and $Y_2 = Y$):

Corollary 2:
Let n be odd, $Y_2^{1/3+\varepsilon} \ll Y_1 \ll Y_2 \asymp n - X_1 - Y_1 - X_2 - Y_2 \geq 0$ and $X_i^{3/5+\varepsilon} \ll Y_i$ for $i = 1, 2$. Let $n \geq X_1 + Y_1 + X_2 + Y_2$ be odd. Then, for $Q \leq Y_1 X_1^{-1/2} L^{-B}$,

$$\sum_{q \leq Q} \max_{a\,(q)}{}^{*} \Bigg| \sum_{\substack{p_1+p_2+p_3=n \\ p_i \in [X_i, X_i+Y_i],\ i=1,2 \\ p_1 \equiv a\ (q)}} \log p_1 \log p_2 \log p_3 - \mathfrak{T}(n, q, a) Y_1 Y_2 \Bigg|$$

$$\ll Y_1 Y_2 L^{-A}.$$

Corollaries

Corollary 3:

For all but $\ll YL^{-A}$ even integers $2k \not\equiv 2\,(6)$ with $k \in [X_1, X_1 + Y]$, the equation $2k = p_2 + p_3$ is solvable in primes p_2, p_3 such that $p_2 + 2 = P_3$, $p_2 \in [X_2, X_2 + Y]$, where $X_2^{\theta} = Y$ with $\theta \geq 0.861$, and $X_2 + Y \leq X_1$, $X_1^{2/3+\varepsilon} \ll Y$.

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Proof: Theorem 9.3 in [Halberstam/Richert], using Theorem 1 above.

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Corollary 4:

Let Y be large and consider an odd integer $n \not\equiv 1\,(6)$ with $n \geq X_1 + X_2 + 2Y$. Then the equation $n = p_1 + p_2 + p_3$ is solvable in primes p_1, p_2, p_3 such that $(p_1 + 2)(p_2 + 2) = P_9$, where $p_i \in [X_i, X_i + Y]$, $X_i^{\theta_i} = Y$, with $\theta_i \geq 0.933$, $i = 1, 2$.

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Proof: Theorem 10.3 in [Halberstam/Richert], using Corollary 1 (beginning with the result of [Kawada]) above.

A similar result using a counting argument

Corollary 5:
Let $X_1^{\theta_1} = Y = X_2^{\theta_2}$ be large, where $\theta_1 \geq 0.971$ and $\theta_2 \geq 0.861$.
Let n be an odd integer $n \not\equiv 1$ (6) with
$X_1 + X_2 + 2Y \leq n < Y^{3/2-\varepsilon}$. Then the equation $n = p_1 + p_2 + p_3$ is solvable in primes p_1, p_2, p_3 such that $p_1 + 2 = P_2$, $p_2 + 2 = P_3$ and $p_i \in [X_i, X_i + Y]$, $i = 1, 2$.

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Proof: Counting argument using Corollary 3 and the result of [Wu 2004] on the number of Chen primes (= primes p with $p + 2 = P_2$) in short intervals $[X, X + X^\theta]$ with $\theta \geq 0.971$.

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Corollary 5 cannot be deduced from Corollary 4 since $0.971 > 0.933$.

Another similar result with different interval lengths

Corollary 6:
Let $X_1^{\theta_1} = Y_1$ be large with $\theta_1 \geq 0.861$, let $X_2^{\theta_2} = Y_2$ with $\theta_2 > 3/5$, and let $Y_2^{1/3+\varepsilon} \ll Y_1 \ll Y_2$. Let n be an odd integer $n \not\equiv 1\,(6)$ with $Y_2 \asymp n - X_1 - Y_1 - X_2 - Y_2 > 0$. Then the equation $n = p_1 + p_2 + p_3$ is solvable in primes p_1, p_2, p_3 such that $p_1 + 2 = P_3$ and $p_i \in [X_i, X_i + Y_i]$, $i = 1, 2$.

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All theorems so far can be deduced by the following single estimate, what we state as

Conjecture:

$$\sum_{2r\in[X_1,X_1+R]} \sum_{q\le Q} \max_{a(q)}{}^{*} \left| \sum_{\substack{p_2\in[X_2,X_2+Y] \\ p_2\equiv a\ (q) \\ p_2+p_3=2r}} \log p_2 \log p_3 - \mathfrak{S}(2r,q,a)Y \right| \ll \frac{RY}{L^A}$$

for $Q \ll Y^{1/2}L^{-B}$, $Y^{\theta} \ll R \ll Y$ and some $0 < \theta < 1$.
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(Also for a fixed $a \in \mathbb{N}$ with weaker restriction on Q.)
The same should be true in the variant $p_2 - p_3 = 2r$.

This is an estimate in the style of Bombieri-Vinogradov's theorem.

A BDH-variant of the conjecture is true

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Further open: What about extra conditions on the second prime in binary theorems? Or the third prime in ternary theorems?

Thank you!