On the number of representations in the ternary Goldbach problem with one prime number in a given residue class

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Abstract

For

\[ J_3(n) = \sum_{p_1 + p_2 + p_3 = n \atop p_1 \equiv a_1 \pmod{q_1}} \log p_1 \log p_2 \log p_3 \]

it is shown that

\[ \sum_{q_1 \leq n^\theta} \max_{(a_1,q_1)=1} \left| J_3(n) - \frac{n^2}{2\varphi(q_1)} S_3(n) \right| \ll_A n^2 (\log n)^{-A}, \]

for any \( A \) and any \( \theta < 1/2 \), what improves a work of D. I. Tolev; \( S_3(n) \) is the corresponding singular series. A special form of a sieve of Montgomery is used.

Key words: Ternary Goldbach problem, Hardy-Littlewood method, sieve methods


1 Introduction

Let \( q_1 > 1 \) and \( a_1 \) with \( 0 \leq a_1 < q_1 \) be pairwise prime integers. We investigate the number of representations of a sufficiently large number \( n \in \mathbb{N} \) as a sum of three prime numbers \( p_1, p_2, p_3 \), the first one lying in the residue class of \( a_1 \mod q_1 \). For the number

\[ J_3(n) := \sum_{p_1 + p_2 + p_3 = n \atop p_1 \equiv a_1 \pmod{q_1}} \log p_1 \log p_2 \log p_3, \]

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which goes closely with the number we are looking for, weighted just with 
log \( p_1 \) log \( p_2 \) log \( p_3 \), Zulauf shows in [4] and [5] for \( q_1 \leq \log D n \) \((D > 0 \text{ fixed})\) the asymptotical formula

\[
J_3(n) = \frac{n^2 S_3(n)}{2\varphi(q_1)} + O \left( \frac{n^2}{\log^A n} \right), \quad \text{for all } A > 0.
\]

Here the \( O \)-constant is depending only on \( D \) and \( A \), the singular series \( S_3(n) \) only on \( a_1 \) and \( q_1 \), in product form it is

\[
S_3(n) := \prod_{p \nmid q_1} \left( 1 + \frac{1}{(p-1)^3} \right) \prod_{p \mid n} \left( 1 - \frac{1}{(p-1)^3} \right) \prod_{p \nmid q_1} \left( 1 - \frac{1}{(p-1)^2} \right) \prod_{p \mid n-a_1} \frac{p}{p-1}.
\]

We consider now the question whether one can find larger bounds than \( \log D n \) for the modulus \( q_1 \), such that the difference of \( J_3(n) \) and the main term including the singular series stays very small in average. We show the following

**Theorem 1** For every positive fixed \( \theta_1 < \frac{1}{2} \) and \( A > 0 \) we have

\[
\mathcal{E} := \sum_{q_1 \leq n^{\theta_1}} \max_{(a_1,q_1)=1} \left| J_3(n) - \frac{n^2 S_3(n)}{2\varphi(q_1)} \right| \ll \frac{n^2}{\log^A n}
\]

Tolev shows this in [2] for \( \theta_1 < \frac{1}{3} \).

**Notation.** We denote by \( \varphi(n) \), \( \mu(n) \) and \( d(n) \) the functions of Euler, Möbius and the divisor function.

## 2 Approach by the circle method

Let \( S_1(\alpha) := \sum_{\substack{p \leq n \\text{ \( p \equiv a_1 \quad \text{(mod} \quad q_1) \)}}} \log p e(\alpha p) \) and \( S(\alpha) := \sum_{p \leq n} \log p e(\alpha p) \), then

\[
J_3(n) = \int_{-\frac{R}{n}}^{\frac{R}{n}} S_1(\alpha) S^2(\alpha) e(-n\alpha) d\alpha,
\]

where \( R := \log^B n \) with \( B := 6A + 27 > 0 \). Now we decompose the real interval \([-\frac{R}{n}, 1 - \frac{R}{n}]\) into major arcs

\[
\mathfrak{M} := \bigcup_{q \leq \frac{R}{n}} \bigcup_{\substack{a < q \\text{ \( q \nmid a \) = 1}}} \left( \frac{a}{q} - \frac{R}{qn}, \frac{a}{q} + \frac{R}{qn} \right)
\]

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and minor arcs \( m := \left[ -\frac{R}{n}, 1 - \frac{R}{n} \right] \setminus \mathcal{M} \). Let

\[
J_3^m(n) := \int_{\mathcal{M}} S_1(\alpha) S_2(\alpha) e(-n\alpha) \, d\alpha
\]

be the contribution of \( J_3(n) \) to the major arcs, and \( J_3^m(n) \) the contribution to the minor arcs. We estimate the mean error \( E \) by the sum of

\[
E^m := \sum_{q_1 \leq n^{\theta_1}} \max_{(a_1, q_1) = 1} \left| J_3^m(n) - \frac{n^2 S_3(n)}{2\varphi(q_1)} \right|
\]

and

\[
E^m := \sum_{q_1 \leq n^{\theta_1}} \max_{(a_1, q_1) = 1} |J_3^m(n)|.
\]

Tolev shows in [2] that \( E^m \ll \frac{n^2}{\log^4 n} \) for \( \theta_1 < \frac{1}{2} \) using the theorem of Bombieri and Vinogradov where \( R = \log \frac{B}{n} n \) with \( B \geq A+5 \). The problematic estimation \( E^m \ll \frac{n^2}{\log^4 n} \) on the minor arcs is shown for \( \theta_1 < \frac{1}{3} \).

3 Proof of the Theorem

So for proving Theorem 1 we show now the estimation \( E^m \ll \frac{n^2}{\log^4 n} \) for \( \theta_1 \leq \frac{1}{2} \).

Besides \( J_3(n) \) we also consider for \( m \leq n \)

\[
J_2(m) := \sum_{p_2 + p_3 = m} \log p_2 \log p_3
\]

and decompose \( J_2(m) \) into the parts on \( \mathcal{M} \) and \( m \) according to

\[
J_2(m) = \int_{\mathcal{M}} S_2(\alpha) e(-m\alpha) \, d\alpha
= \left( \int_{\mathcal{M}} + \int_{m} \right) S_2(\alpha) e(-m\alpha) \, d\alpha =: J_2^m(m) + J_2^m(m).
\]

For \( J_2^m(m) \) we derive two bounds, first the trivial one

\[
J_2^m(m) \ll \int_0^1 |S(\alpha)|^2 \, d\alpha = \sum_{p \leq n} \log^2 p \ll n \log n,
\]

and secondly, using Bessels inequality,

\[
\sum_{m \leq n} |J_2^m(m)|^2 \leq \int_{m} |S_2(\alpha)|^2 \, d\alpha \leq \int_0^1 |S(\alpha)|^2 \, d\alpha \cdot \max_{\alpha \in m} |S(\alpha)|^2
\ll n \log n \cdot \frac{n^2}{\log \frac{B-8}{n}} = \frac{n^3}{\log \frac{B-9}{n}}.
\]
For the last estimation of $|S(\alpha)|$ on $m$ see Vaughan [3], Theorem 3.1. There, $R = \log^B n$ is used.

With this we estimate a part of the sum in $E_m$, for fixed $Q_1 \leq n^{\alpha_1}$, namely

$$E_{Q_1} := \sum_{Q_1 < q_1 \leq 2Q_1} \max_{(a_1, q_1) = 1} |J_3^m(n)| \leq \sum_{q_1} \max_{a_1} \sum_{p_1 \leq n} \log p_1 |J_2^m(n - p_1)|$$

$$\leq \sum_{q_1} \max_{a_1} \sum_{m \leq n} \log n |J_2^m(m)|.$$

For $0 \leq a_1 < q_1$ we consider the number

$$N(a_1, q_1) := \# \left\{ m \leq n; \ m \equiv n - a_1 \ (\text{mod} \ q_1), \ |J_2^m(m)| > \frac{n}{\log^{A+2} n} \right\}$$

of natural $m \leq n$ with $m \equiv n - a_1 \ (\text{mod} \ q_1)$ for which $|J_2^m(m)|$ is big. Then

$$E_{Q_1} \leq \sum_{q_1} \max_{a_1} \left( \sum_{m \leq n, m \equiv n - a_1 \ (\text{mod} \ q_1), |J_2^m(m)| > \frac{n}{\log^{A+2} n}} \log n |J_2^m(m)| + \sum_{m \leq n, m \equiv n - a_1 \ (\text{mod} \ q_1), |J_2^m(m)| \leq \frac{n}{\log^{A+2} n}} \log n \frac{n}{\log^{A+2} n} \right)$$

$$\leq \sum_{q_1} \max_{a_1} n \log^2 n N(a_1, q_1) + \sum_{q_1} \frac{n}{q_1} \log n \frac{n}{\log^{A+2} n}$$

$$\leq n \log^2 n \sum_{q_1} \max_{a_1} N(a_1, q_1) + \frac{n^2}{\log^{A+1} n}$$

$$\leq n \log^2 n \ \frac{1}{Q_1^\frac{1}{2}} \left( \sum_{q_1} \max_{a_1} N(a_1, q_1)^2 \right)^{\frac{1}{2}} + \frac{n^2}{\log^{A+1} n},$$

the last step follows by application of the Cauchy-Schwarz-inequality and $Q_1 < q_1 \leq 2Q_1$. Now we split the sum over the modules $q_1$ according to their number $d(q_1)$ of divisors, namely

$$E_{Q_1} \ll n \log^2 n \ E_1^\frac{1}{2} + n \log^2 n \ E_2^\frac{1}{2} + \frac{n^2}{\log^{A+1} n}$$

with

$$E_1 := \sum_{d(q_1) \leq \log^{2A+7} n} q_1 \max_{a_1} N(a_1, q_1)^2$$

and

$$E_2 := \sum_{d(q_1) > \log^{2A+7} n} Q_1 \max_{a_1} N(a_1, q_1)^2.$$
If it is now possible to show for $Q_1 \leq n^{\frac{1}{2}}$ the estimations

$$E_1, E_2 \ll \frac{n^2}{\log^{2A+6} n},$$  \hspace{1cm} (3)$$

then

$$E_m^{Q_1} \ll \frac{n^2}{\log^{A+1} n}$$

and finally

$$E_m \leq \sum_{i,Q_1=2^i \leq n^{1/2}} E_m^{Q_1} \ll \log n \cdot \frac{n^2}{\log^{A+1} n} \ll \frac{n^2}{\log^A n},$$

as was stated.

4 Estimations for $E_1$ and $E_2$

Now we look at the estimations in (3). First consider the easier part, namely $E_2$. Let

$$A_{Q_1} := \#\{q_1 ; Q_1 < q_1 \leq 2Q_1, d(q_1) > \log^{2A+7} n\},$$

then

$$A_{Q_1} \log^{2A+7} n \ll \sum_{q_1 \leq 2Q_1 \atop d(q_1) > \log^{2A+7} n} d(q_1) \ll \sum_{q_1 \leq 2Q_1} d(q_1) \ll Q_1 \log n,$$

so

$$A_{Q_1} \ll \frac{Q_1}{\log^{2A+6} n}.$$  

Since $N(a_1,q_1) \ll \frac{n}{q_1}$ we get

$$E_2 \ll \sum_{Q_1 < q_1 \leq 2Q_1 \atop d(q_1) > \log^{2A+7} n} Q_1 \frac{n^2}{q_1^2} \ll A_{Q_1} \frac{n^2}{Q_1} \ll \frac{n^2}{\log^{2A+6} n}.$$  

Now to $E_1$. We set for $m \leq n$

$$b_m := \begin{cases} 1, & \text{if } |J_m^m(m)| > \frac{n}{\log^{A+1} n} \\ 0, & \text{otherwise} \end{cases},$$

thus

$$N(a_1, q_1) = \sum_{m \leq n \atop m \equiv n-a_1(q_1)} b_m \quad \text{and let } N := \sum_{0 \leq a_1 < q_1} N(a_1, q_1),$$
for $\alpha \in \mathbb{R}$ let

$$T(\alpha) := \sum_{m \leq n} b_m e(\alpha m).$$

For $0 \leq h < q_1$ let

$$f_h(q_1) := \sum_{d|q_1} \mu(d) \frac{q_1}{d} N \left( h, \frac{q_1}{d} \right),$$

by Möbius’ inversion formula we get

$$q_1 N(h, q_1) = \sum_{d|q_1} f_h(d)$$

for all $h$. So

$$E_1 = \sum_{q_1, d(q_1) \leq \log^{24+7} n} \frac{1}{q_1} \max_{a_1} q_1^2 N(a_1, q_1)^2$$

$$= \sum_{q_1, d(q_1) \leq \log^{24+7} n} \frac{1}{q_1} \max_{a_1} \left( \sum_{d|q_1} f_{a_1}(d) \right)^2$$

$$\leq \sum_{q_1, d(q_1) \leq \log^{24+7} n} \frac{1}{q_1} \max_{a_1} \left( \sum_{d|q_1} |f_{a_1}(d)| \right)^2$$

$$\leq \sum_{q_1, d(q_1) \leq \log^{24+7} n} \frac{1}{q_1} \left( \sum_{d|q_1} 1 \right) \left( \sum_{d|q_1} \max_{a_1} |f_{a_1}(d)|^2 \right)$$

using the Cauchy-Schwarz-inequality.

The maximum is taken over $a_1$ with $0 \leq a_1 < q_1$ and $(a_1, q_1) = 1$. Since $|f_{a_1}(d)|^2$ is $d$-periodic in $a_1$ for $d|q_1$ (as $N(a_1 + d, t) = N(a_1, t)$ for $t|d$), its maximum stays equal if taken only over $a_1$ with $0 \leq a_1 < d$ and $(a_1, d) = 1$. We further get

$$E_1 \leq \sum_{q_1, d(q_1) \leq \log^{24+7} n} \frac{1}{q_1} d(q_1) \left( \sum_{d|q_1} \sum_{0 \leq a_1 < d} |f_{a_1}(d)|^2 \right).$$

By Montgomery in [1], equation (10), we have the formula

$$\frac{1}{d} \sum_{h=0}^{d-1} |f_h(d)|^2 = \sum_{a<d \atop (a,d)=1} \left| T \left( \frac{a}{d} \right) \right|^2$$
that we can apply here. We get

\[ E_1 \leq \sum_{d(q_1) \leq \log^{2A+7} n} \frac{1}{q_1} d(q_1) \sum_d d \sum_{a<d \atop (a,d)=1} \left| T \left( \frac{a}{d} \right) \right|^2 \]

\[ \leq \sum_{d \leq 2Q_1} \log^{2A+7} n \sum_{q_1 \leq 2Q_1} \sum_{d \leq q_1} d \sum_{a<d \atop (a,d)=1} \left| T \left( \frac{a}{d} \right) \right|^2. \]

We have

\[ \sum_{q_1 \leq 2Q_1} \sum_{d \leq q_1} d = \sum_{q_1 \leq 2Q_1} \frac{1}{q_1} \ll \log n \]

so

\[ E_1 \ll \log^{2A+8} n \sum_{d \leq 2Q_1} \sum_{a<d \atop (a,d)=1} \left| T \left( \frac{a}{d} \right) \right|^2 \ll \log^{2A+8} n (n + Q_1^2) N \]

by application of the large sieve. Let

\[ \mathcal{N} := \left\{ m \leq n \mid |J_2^m(m)| > \frac{n}{\log A+2} \right\}, \text{ i.e. } \#\mathcal{N} = N, \]

then

\[ \left( N \frac{n}{\log A+2} \right)^2 \leq \left( \sum_{m \in \mathcal{N}} |J_2^m(m)| \right)^2 \leq N \sum_{m \leq n} |J_2^m(m)|^2 \ll N \frac{n^3}{\log B - 9} \frac{1}{n} \]

by (2), so we have

\[ N \ll \frac{n}{\log B - 2A - 13} n. \]

By this and applying \( Q_1 \leq n^{\frac{1}{2}} \) (since \( \theta_1 \leq \frac{1}{2} \)) it follows that

\[ E_1 \ll \frac{n^2}{\log B - 2A - 13 - 2A - 8} \frac{1}{n} = \frac{n^2}{\log 2A+6} \frac{1}{n} \]

as \( B = 6A + 27 \). So everything is shown.

**Acknowledgements.** The author thanks Jan-Christoph Schlage-Puchta for several helpful ideas.

**References**


