On the number of representations in the ternary Goldbach problem with one prime number in a given residue class

Karin Halupczok

Mathematisches Institut der Albert-Ludwigs-Universität Freiburg, Eckerstraße 1, 79104 Freiburg, Germany

Abstract

For

$$J_3(n) = \sum_{\substack{p_1+p_2+p_3=n \\ p_1 \equiv a_1 \pmod{q_1}}} \log p_1 \, \log p_2 \, \log p_3$$

it is shown that

$$\sum_{q_1 \leq n^\theta} \max_{(a_1,q_1)=1} \left| J_3(n) - \frac{n^2}{2\varphi(q_1)} \mathcal{S}_3(n) \right| \ll_A n^2 (\log n)^{-A},$$

for any A and any $\theta < 1/2$, what improves a work of D. I. Tolev; $\mathcal{S}_3(n)$ is the corresponding singular series. A special form of a sieve of Montgomery is used.

Key words: Ternary Goldbach problem, Hardy-Littlewood method, sieve methods
1991 MSC: 11P32, 11N36, 11P55

1 Intoduction

Let $q_1 > 1$ and a_1 with $0 \leq a_1 < q_1$ be pairwise prime integers. We investigate the number of representations of a sufficiently large number $n \in \mathbb{N}$ as a sum of three prime numbers p_1, p_2, p_3, the first one lying in the residue class of a_1 modulo q_1. For the number

$$J_3(n) := \sum_{\substack{p_1+p_2+p_3=n \\ p_1 \equiv a_1 \pmod{q_1}}} \log p_1 \, \log p_2 \, \log p_3,$$

Email address: `karin.halupczok@math.uni-freiburg.de` (Karin Halupczok).

Preprint submitted to Elsevier Science *28 June 2005*

which goes closely with the number we are looking for, weighted just with $\log p_1 \log p_2 \log p_3$, Zulauf shows in [4] and [5] for $q_1 \le \log^D n$ ($D > 0$ fixed) the asymptotical formula

$$J_3(n) = \frac{n^2 \mathcal{S}_3(n)}{2\varphi(q_1)} + O\left(\frac{n^2}{\log^A n}\right), \qquad \text{for all } A > 0.$$

Here the O-constant is depending only on D and A, the singular series $\mathcal{S}_3(n)$ only on a_1 and q_1, in product form it is

$$\mathcal{S}_3(n) := \prod_{p \nmid nq_1} \left(1 + \frac{1}{(p-1)^3}\right) \prod_{\substack{p \nmid q_1 \\ p \mid n}} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p \mid q_1 \\ p \nmid n - a_1}} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p \mid q_1 \\ p \mid n - a_1}} \frac{p}{p-1}.$$

We consider now the question whether one can find larger bounds than $\log^D n$ for the modulus q_1, such that the difference of $J_3(n)$ and the main term including the singular series stays very small in average. We show the following

Theorem 1 *For every positive fixed $\theta_1 < \frac{1}{2}$ and $A > 0$ we have*

$$\mathcal{E} := \sum_{q_1 \le n^{\theta_1}} \max_{(a_1, q_1) = 1} \left| J_3(n) - \frac{n^2 \mathcal{S}_3(n)}{2\varphi(q_1)} \right| \ll \frac{n^2}{\log^A n}$$

Tolev shows this in [2] for $\theta_1 < \frac{1}{3}$.

Notation. We denote by $\varphi(n)$, $\mu(n)$ and $d(n)$ the functions of Euler, Möbius and the divisor function.

2 Approach by the circle method

Let $S_1(\alpha) := \sum_{\substack{p \le n \\ p \equiv a_1 \pmod{q_1}}} \log p \, e(\alpha p)$ and $S(\alpha) := \sum_{p \le n} \log p \, e(\alpha p)$, then

$$J_3(n) = \int_{-\frac{R}{n}}^{1 - \frac{R}{n}} S_1(\alpha) S^2(\alpha) e(-n\alpha) \, d\alpha,$$

where $R := \log^B n$ with $B := 6A + 27 > 0$. Now we decompose the real interval $[-\frac{R}{n}, 1 - \frac{R}{n}]$ into major arcs

$$\mathfrak{M} := \bigcup_{q \le \frac{R}{2}} \bigcup_{\substack{a < q \\ (a,q) = 1}} \left(\frac{a}{q} - \frac{R}{qn}, \frac{a}{q} + \frac{R}{qn}\right)$$

and minor arcs $\mathfrak{m} := \left[-\frac{R}{n}, 1-\frac{R}{n}\right] \setminus \mathfrak{M}$. Let

$$J_3^{\mathfrak{M}}(n) := \int_{\mathfrak{M}} S_1(\alpha) S^2(\alpha) e(-n\alpha)\, d\alpha$$

be the contribution of $J_3(n)$ to the major arcs, and $J_3^{\mathfrak{m}}(n)$ the contribution to the minor arcs. We estimate the mean error $\mathcal{E}$ by the sum of

$$\mathcal{E}^{\mathfrak{M}} := \sum_{q_1 \le n^{\theta_1}} \max_{(a_1,q_1)=1} \left| J_3^{\mathfrak{M}}(n) - \frac{n^2 \mathcal{S}_3(n)}{2\varphi(q_1)} \right|$$

and

$$\mathcal{E}^{\mathfrak{m}} := \sum_{q_1 \le n^{\theta_1}} \max_{(a_1,q_1)=1} \left| J_3^{\mathfrak{m}}(n) \right| .$$

Tolev shows in [2] that $\mathcal{E}^{\mathfrak{M}} \ll \frac{n^2}{\log^A n}$ for $\theta_1 < \frac{1}{2}$ using the theorem of Bombieri and Vinogradov where $R = \log^B n$ with $B \ge A+5$. The problematic estimation $\mathcal{E}^{\mathfrak{m}} \ll \frac{n^2}{\log^A n}$ on the minor arcs is shown for $\theta_1 < \frac{1}{3}$.

3 Proof of the Theorem

So for proving Theorem 1 we show now the estimation $\mathcal{E}^{\mathfrak{m}} \ll \frac{n^2}{\log^A n}$ for $\theta_1 \le \frac{1}{2}$.

Besides $J_3(n)$ we also consider for $m \le n$

$$J_2(m) := \sum_{p_2+p_3=m} \log p_2 \, \log p_3$$

and decompose $J_2(m)$ into the parts on $\mathfrak{M}$ and $\mathfrak{m}$ according to

$$\begin{aligned} J_2(m) &= \int_{-\frac{R}{n}}^{1-\frac{R}{n}} S^2(\alpha)\, e(-m\alpha)\, d\alpha \\ &= \left(\int_{\mathfrak{M}} + \int_{\mathfrak{m}} \right) S^2(\alpha)\, e(-m\alpha)\, d\alpha =: J_2^{\mathfrak{M}}(m) + J_2^{\mathfrak{m}}(m). \end{aligned}$$

For $J_2^{\mathfrak{m}}(m)$ we derive two bounds, first the trivial one

$$J_2^{\mathfrak{m}}(m) \ll \int_0^1 |S(\alpha)|^2 d\alpha = \sum_{p \le n} \log^2 p \ll n \log n, \tag{1}$$

and secondly, using Bessels inequality,

$$\begin{aligned} \sum_{m \le n} |J_2^{\mathfrak{m}}(m)|^2 &\le \int_{\mathfrak{m}} |S^2(\alpha)|^2 d\alpha \le \int_0^1 |S(\alpha)|^2 d\alpha \cdot \max_{\alpha \in \mathfrak{m}} |S(\alpha)|^2 \\ &\ll n \log n \cdot \frac{n^2}{\log^{B-8} n} = \frac{n^3}{\log^{B-9} n} . \end{aligned} \tag{2}$$

For the last estimation of $|S(\alpha)|$ on $\mathfrak{m}$ see Vaughan [3], Theorem 3.1. There, $R = \log^B n$ is used.

With this we estimate a part of the sum in $\mathcal{E}^{\mathfrak{m}}$, for fixed $Q_1 \le n^{\theta_1}$, namely

$$\begin{aligned}\mathcal{E}^{\mathfrak{m}}_{Q_1} &:= \sum_{Q_1<q_1\le 2Q_1} \max_{(a_1,q_1)=1} |J_3^{\mathfrak{m}}(n)| \le \sum_{q_1} \max_{a_1} \sum_{\substack{p_1\le n\\ p_1\equiv a_1 \,(\mathrm{mod}\, q_1)}} \log p_1 \, |J_2^{\mathfrak{m}}(n-p_1)| \\ &\le \sum_{q_1} \max_{a_1} \sum_{\substack{m\le n\\ m\equiv n-a_1 \,(\mathrm{mod}\, q_1)}} \log n \, |J_2^{\mathfrak{m}}(m)|.\end{aligned}$$

For $0 \le a_1 < q_1$ we consider the number

$$N(a_1,q_1) := \# \left\{ m \le n\,; \quad m \equiv n - a_1 \,(\mathrm{mod}\, q_1), \quad |J_2^{\mathfrak{m}}(m)| > \frac{n}{\log^{A+2} n} \right\}$$

of natural $m \le n$ with $m \equiv n - a_1 \,(\mathrm{mod}\, q_1)$ for which $|J_2^{\mathfrak{m}}(m)|$ is big. Then

$$\begin{aligned}\mathcal{E}^{\mathfrak{m}}_{Q_1} &\le \sum_{q_1} \max_{a_1} \Bigg(\sum_{\substack{m\le n\\ m\equiv n-a_1 \,(\mathrm{mod}\, q_1)\\ |J_2^{\mathfrak{m}}(m)|>\frac{n}{\log^{A+2} n}}} \log n \, |J_2^{\mathfrak{m}}(m)| + \sum_{\substack{m\le n\\ m\equiv n-a_1 \,(\mathrm{mod}\, q_1)\\ |J_2^{\mathfrak{m}}(m)|\le\frac{n}{\log^{A+2} n}}} \log n \, \frac{n}{\log^{A+2} n} \Bigg) \\ &\overset{(1)}{\ll} \sum_{q_1} \max_{a_1} \; n \log^2 n \; N(a_1,q_1) + \sum_{q_1} \frac{n}{q_1} \log n \, \frac{n}{\log^{A+2} n} \\ &\ll n \log^2 n \; \sum_{q_1} \max_{a_1} N(a_1,q_1) + \frac{n^2}{\log^{A+1} n} \\ &\ll n \log^2 n \; Q_1^{\frac{1}{2}} \left(\sum_{q_1} \max_{a_1} N(a_1,q_1)^2 \right)^{\frac{1}{2}} + \frac{n^2}{\log^{A+1} n},\end{aligned}$$

the last step follows by application of the Cauchy-Schwarz-inequality and $Q_1 < q_1 \le 2Q_1$. Now we split the sum over the modules q_1 according to their number $d(q_1)$ of divisors, namely

$$\mathcal{E}^{\mathfrak{m}}_{Q_1} \ll n \log^2 n \; E_1^{\frac{1}{2}} + n \log^2 n \; E_2^{\frac{1}{2}} + \frac{n^2}{\log^{A+1} n}$$

with

$$E_1 := \sum_{\substack{q_1\\ d(q_1)\le \log^{2A+7} n}} q_1 \; \max_{a_1} N(a_1,q_1)^2$$

and

$$E_2 := \sum_{\substack{q_1\\ d(q_1)> \log^{2A+7} n}} Q_1 \; \max_{a_1} N(a_1,q_1)^2.$$

If it is now possible to show for $Q_1 \le n^{\frac{1}{2}}$ the estimations

$$E_1, E_2 \ll \frac{n^2}{\log^{2A+6} n}, \tag{3}$$

then

$$\mathcal{E}^{\mathfrak{m}}_{Q_1} \ll \frac{n^2}{\log^{A+1} n}$$

and finally

$$\mathcal{E}^{\mathfrak{m}} \le \sum_{i, Q_1 = 2^i \le n^{\theta_1}} \mathcal{E}^{\mathfrak{m}}_{Q_1} \ll \log n \cdot \frac{n^2}{\log^{A+1} n} \ll \frac{n^2}{\log^{A} n} \; ,$$

as was stated.

4 Estimations for E_1 and E_2

Now we look at the estimations in (3). First consider the easier part, namely E_2. Let

$$A_{Q_1} := \#\{q_1 \; ; \; Q_1 < q_1 \le 2Q_1, d(q_1) > \log^{2A+7} n\},$$

then

$$A_{Q_1} \log^{2A+7} n < \sum_{\substack{q_1 \le 2Q_1 \\ d(q_1) > \log^{2A+7} n}} d(q_1) \le \sum_{q_1 \le 2Q_1} d(q_1) \ll Q_1 \log n,$$

so

$$A_{Q_1} \ll \frac{Q_1}{\log^{2A+6} n}.$$

Since $N(a_1, q_1) \ll \frac{n}{q_1}$ we get

$$E_2 \ll \sum_{\substack{Q_1 < q_1 \le 2Q_1 \\ d(q_1) > \log^{2A+7} n}} Q_1 \frac{n^2}{q_1^2} \ll A_{Q_1} \frac{n^2}{Q_1} \ll \frac{n^2}{\log^{2A+6} n}.$$

Now to E_1. We set for $m \le n$

$$b_m := \begin{cases} 1, & \text{if } |J_2^{\mathfrak{m}}(m)| > \frac{n}{\log^{A+2} n}, \\ 0, & \text{otherwise,} \end{cases}$$

thus

$$N(a_1, q_1) = \sum_{\substack{m \le n \\ m \equiv n - a_1 \, (q_1)}} b_m \qquad \text{and let} \qquad N := \sum_{0 \le a_1 < q_1} N(a_1, q_1),$$

for $\alpha \in \mathbb{R}$ let

$$T(\alpha) := \sum_{m \le n} b_m e(\alpha m).$$

For $0 \le h < q_1$ let

$$f_h(q_1) := \sum_{d | q_1} \mu(d) \, \frac{q_1}{d} \, N\left(h, \frac{q_1}{d}\right),$$

by Möbius' inversion formula we get

$$q_1 N(h, q_1) = \sum_{d | q_1} f_h(d)$$

for all h. So

$$\begin{aligned} E_1 &= \sum_{\substack{q_1 \\ d(q_1) \le \log^{2A+7} n}} \frac{1}{q_1} \max_{a_1} \; q_1^2 \, N(a_1, q_1)^2 \\ &= \sum_{\substack{q_1 \\ d(q_1) \le \log^{2A+7} n}} \frac{1}{q_1} \max_{a_1} \left(\sum_{d | q_1} f_{a_1}(d) \right)^2 \\ &\le \sum_{\substack{q_1 \\ d(q_1) \le \log^{2A+7} n}} \frac{1}{q_1} \max_{a_1} \left(\sum_{d | q_1} |f_{a_1}(d)| \right)^2 \\ &\le \sum_{\substack{q_1 \\ d(q_1) \le \log^{2A+7} n}} \frac{1}{q_1} \left(\sum_{d | q_1} 1 \right) \left(\sum_{d | q_1} \max_{a_1} |f_{a_1}(d)|^2 \right) \end{aligned}$$

using the Cauchy-Schwarz-inequality.

The maximum is taken over a_1 with $0 \le a_1 < q_1$ and $(a_1, q_1) = 1$. Since $|f_{a_1}(d)|^2$ is d-periodic in a_1 for $d | q_1$ (as $N(a_1 + d, t) = N(a_1, t)$ for $t | d$), its maximum stays equal if taken only over a_1 with $0 \le a_1 < d$ and $(a_1, d) = 1$. We further get

$$E_1 \le \sum_{\substack{q_1 \\ d(q_1) \le \log^{2A+7} n}} \frac{1}{q_1} \, d(q_1) \left(\sum_{d | q_1} \sum_{0 \le a_1 < d} |f_{a_1}(d)|^2 \right).$$

By Montgomery in [1], equation (10), we have the formula

$$\frac{1}{d} \sum_{h=0}^{d-1} |f_h(d)|^2 = \sum_{\substack{a < d \\ (a,d)=1}} \left| T\left(\frac{a}{d}\right) \right|^2$$

that we can apply here. We get

$$E_1 \leq \sum_{\substack{q_1 \\ d(q_1)\leq \log^{2A+7} n}} \frac{1}{q_1}\, d(q_1) \sum_{d|q_1} d \sum_{\substack{a<d \\ (a,d)=1}} \left| T\left(\frac{a}{d}\right)\right|^2$$
$$\leq \sum_{d\leq 2Q_1} \log^{2A+7} n \sum_{\substack{Q_1<q_1\leq 2Q_1 \\ d|q_1}} \frac{d}{q_1} \sum_{\substack{a<d \\ (a,d)=1}} \left| T\left(\frac{a}{d}\right)\right|^2 .$$

We have

$$\sum_{\substack{Q_1<q_1\leq 2Q_1 \\ d|q_1}} \frac{d}{q_1} = \sum_{\frac{Q_1}{d}<q_1'\leq \frac{2Q_1}{d}} \frac{1}{q_1'} \ll \log n$$

so

$$E_1 \ll \log^{2A+8} n \sum_{d\leq 2Q_1} \sum_{\substack{a<d \\ (a,d)=1}} \left| T\left(\frac{a}{d}\right)\right|^2 \ll \log^{2A+8} n\; (n+Q_1^2)\; N$$

by application of the large sieve. Let

$$\mathcal{N} := \left\{ m \leq n\;;\; |J_2^{\mathfrak{m}}(m)| > \frac{n}{\log^{A+2} n} \right\},\ \text{i. e. } \#\mathcal{N} = N,$$

then

$$\left(N\frac{n}{\log^{A+2} n}\right)^2 < \left(\sum_{m\in\mathcal{N}} |J_2^{\mathfrak{m}}(m)|\right)^2 \leq N \sum_{m\leq n} |J_2^{\mathfrak{m}}(m)|^2 \ll N\frac{n^3}{\log^{B-9} n}$$

by (2), so we have

$$N \ll \frac{n}{\log^{B-2A-13} n}\ .$$

By this and applying $Q_1 \leq n^{\frac{1}{2}}$ (since $\theta_1 \leq \frac{1}{2}$) it follows that

$$E_1 \ll \frac{n^2}{\log^{B-2A-13-2A-8} n} = \frac{n^2}{\log^{2A+6} n}$$

as $B = 6A+27$. So everything is shown.

Acknowledgements. The author thanks Jan-Christoph Schlage-Puchta for several helpful ideas.

References

[1] H. L. Montgomery, A note on the large sieve. *J. London Math. Soc.*, 1968, **vol. 43**, pp. 93-98.

[2] D. I. Tolev, On the Number of Representations of an Odd Integer as a Sum of Three Primes, one of which belongs to an Arithmetic Progression. *Proceedings of the Steklov Institute of Mathematics*, 1997, **vol. 218**, pp. 414-432.

[3] R. C. Vaughan, The Hardy-Littlewood Method. *Cambridge: Cambridge Univ. Press*, 1981.

[4] A. Zulauf, Über die Darstellung natürlicher Zahlen als Summen von Primzahlen aus gegebenen Restklassen und Quadraten mit gegebenen Koeffizienten. I: Resultate für genügend große Zahlen. *J. Reine Angew. Math.*, 1954, **vol. 192**, pp. 210-290. II: Die Singuläre Reihe. *J. Reine Angew. Math.*, 1954, **vol. 193**, pp. 39-53.

[5] A. Zulauf, On the Number of Representation of an Integer as a Sum of Primes Belonging to Given Arithmetical Progressions. *Compositio Math.*, 1961, **vol. 15**, pp. 64-90.