On the Ternary Goldbach Problem with Primes in independent Arithmetic Progressions

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Abstract

We show that for every fixed $A > 0$ and $\theta > 0$ there is a $\vartheta = \vartheta(A, \theta) > 0$ with the following property. Let n be odd and sufficiently large, and let $Q_1 = Q_2 := n^{1/2}(\log n)^{-\vartheta}$ and $Q_3 := (\log n)^{\theta}$. Then for all $q_3 \le Q_3$, all reduced residues $a_3 \bmod q_3$, almost all $q_2 \le Q_2$, all admissible residues $a_2 \bmod q_2$, almost all $q_1 \le Q_1$ and all admissible residues $a_1 \bmod q_1$, there exists a representation $n = p_1 + p_2 + p_3$ with primes $p_i \equiv a_i\ (q_i)$, $i = 1, 2, 3$.

1 Introduction and results

1.1 Preliminaries

Let n be a sufficiently large integer, and for every $i = 1, 2, 3$ let a_i, q_i be relatively prime integers with $q_i \ge 1$ and $0 \le a_i < q_i$.

We consider the ternary Goldbach problem of writing n as

$$n = p_1 + p_2 + p_3$$

with primes p_1, p_2 and p_3 satisfying the three congruences

$$p_i \equiv a_i \bmod q_i,\ i = 1, 2, 3.$$

A necessary condition for solvability is

$$n \equiv a_1 + a_2 + a_3 \bmod (q_1, q_2, q_3),$$

where (q_1, q_2, q_3) denotes the greatest common divisor of the q_i. Otherwise no such representation of n is possible.

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We precise our consideration in the following way. Let

$$J_3(n) := \sum_{\substack{m_1+m_2+m_3=n \\ m_i \equiv a_i\ (q_i), \\ i=1,2,3}} \Lambda(m_1)\,\Lambda(m_2)\,\Lambda(m_3),$$

where Λ is von Mangoldt's function. $J_3(n)$ goes closely with the number of representations of n in the way mentioned.

In this paper we prove that the deviation of $J_3(n)$ from its expected main term is uniformly small for large moduli, namely:

Theorem 1. *For every fixed $A > 0$ and $\theta > 0$ there is a $\vartheta = \vartheta(A, \theta) > 0$ such that for all $q_3 \leq (\log n)^\theta$ and a_3 with $(a_3, q_3) = 1$ we have*

$$\sum_{q_2 \leq \frac{n^{1/2}}{(\log n)^\vartheta}} \max_{\substack{a_2 \\ (a_2,q_2)=1}} \sum_{q_1 \leq \frac{n^{1/2}}{(\log n)^\vartheta}} \max_{\substack{a_1 \\ (a_1,q_1)=1}} \left| J_3(n) - \frac{n^2 \mathcal{S}_3(n)}{2\varphi(q_1)\varphi(q_2)\varphi(q_3)} \right| \ll \frac{n^2}{(\log n)^A}.$$

The O-constant depends on the parameters A and θ.

Here $\mathcal{S}_3(n)$ denotes the singular series for this special Goldbach problem and depends on a_i and q_i likewise $J_3(n)$ does.

We set $S_3(n) = 0$ if $n \not\equiv a_1 + a_2 + a_3 \mod (q_1, q_2, q_3)$, the case where trivially $J_3(n) = 0$ occurs. Then a summand $= 0$ in the formula of Theorem 1 is given, therefore we can assume in the proof without loss of generality that $n \equiv a_1 + a_2 + a_3 \mod (q_1, q_2, q_3)$ holds. We refer to this as "general condition", under this, $\mathcal{S}_3(n)$ is defined and investigated later in paragraphs 2.2 and 2.3.

Definition. For any given q_1, q_2, q_3 we call a triplet a_1, a_2, a_3 of residues mod q_1, q_2, q_3 *admissible* for q_1, q_2, q_3, if $(a_i, q_i) = 1$ for $i = 1, 2, 3$, if $n \equiv a_1 + a_2 + a_3 \mod (q_1, q_2, q_3)$ and if $\mathcal{S}_3(n) > 0$.
For given q_3, a_3, q_2, a_2 and q_1 we call a_1 *admissible*, if a_1, a_2, a_3 is admissible for q_1, q_2, q_3. For given q_3, a_3, q_2 we call a_2 *admissible*, if there exists an admissible a_1 for every positive integer q_1.

We prove in paragraph 2.3

Lemma 1. *If n is odd, then for given q_3, a_3 with $(a_3, q_3) = 1$ and q_2 there exists an admissible a_2 (such that for every q_1 there exists an admissible a_1). For even n and given q_1, q_2, q_3 there exists* no *admissible triplet a_1, a_2, a_3.*

Theorem 1 provides

Theorem 2. *Let $A, \theta, \vartheta > 0$ as above and $n \in \mathbb{N}$ odd and sufficiently large. Let $Q_1, Q_2 := n^{1/2}(\log n)^{-\vartheta}$, $Q_3 := (\log n)^{\theta}$. Then for all $q_3 \leq Q_3$, all a_3, almost all $q_2 \leq Q_2$, all admissible a_2, almost all $q_1 \leq Q_1$ and all admissible a_1 there exists a representation $n = p_1 + p_2 + p_3$ with primes $p_i \equiv a_i \ (q_i)$, $i = 1, 2, 3$. Here the number of exceptions for q_2 is $\ll Q_2(\log n)^{-A}$ resp. for q_1 is $\ll Q_1(\log n)^{-A}$.*

Theorem 2 as corollary of Theorem 1 is proved in section 6.

Theorem 1 is shown by the circle method. It seems that it also should hold with the larger bound $q_3 \leq n^{1/2}(\log n)^{-\vartheta}$, which is the case on the major arcs. It is not possible to achieve this on the minor arcs by the given methods.

Notation. We denote by φ, μ, Λ and τ the functions of Euler, Möbius, von Mangoldt and the divisor function. Other occuring functions are given in their context. By $q_i \sim Q_i$ we abbreviate $Q_i < q_i \leq 2Q_i$. By p and p_i we denote primes. As usual, $e(\alpha) := e^{2\pi i \alpha}$ for $\alpha \in \mathbb{R}$.

1.2 Proceeding by the circle method

Let $A > 0$ and $\theta > 0$. Let $R := (\log n)^B$ with $B = B(A, \theta) := \max\{A + \eta + 3, D(8A + 2\theta + 74)\}$, where $\eta > 0$ is some absolute constant (see end of paragraph 2.2), and $D(8A + 2\theta + 74) > 0$ is some constant depending just on A and θ, its definition is given in the proof of Lemma 5. Further let $\vartheta > \max\{A + 4B + 16, \theta + A + 3\}$, so ϑ depends also on A and θ.

We define major arcs $\mathfrak{M} \subseteq \mathbb{R}$ by

$$\mathfrak{M} := \bigcup_{q \leq R} \bigcup_{\substack{0 < a < q \\ (a,q)=1}} \left] \frac{a}{q} - \frac{R}{qn}, \frac{a}{q} + \frac{R}{qn} \right[$$

and minor arcs by $\mathfrak{m} := \left] -\frac{R}{n}, 1 - \frac{R}{n} \right[\setminus \mathfrak{M}$.

For $\alpha \in \mathbb{R}$ and $j = 1, 2, 3$ let

$$S_j(\alpha) := \sum_{\substack{m \le n \\ m \equiv a_j\ (q_j)}} \Lambda(m)\, e(\alpha m).$$

From the orthogonal relations for $e(\alpha m)$ it follows that

$$J_3(n) = \int_{-\frac{R}{n}}^{1-\frac{R}{n}} S_1(\alpha) S_2(\alpha) S_3(\alpha)\, e(-n\alpha)\, d\alpha.$$

Analogously, denote for $m \le n$

$$J_2(m) := \sum_{\substack{m_2 + m_3 = m \\ m_2 \equiv a_2\ (q_2) \\ m_3 \equiv a_3\ (q_3)}} \Lambda(m_2)\Lambda(m_3) = \int_{-\frac{R}{n}}^{1-\frac{R}{n}} S_2(\alpha) S_3(\alpha)\, e(-m\alpha)\, d\alpha.$$

By

$$J_3^{\mathfrak{M}}(n) := \int_{\mathfrak{M}} S_1(\alpha) S_2(\alpha) S_3(\alpha)\, e(-n\alpha)\, d\alpha$$

and

$$J_2^{\mathfrak{M}}(m) := \int_{\mathfrak{M}} S_2(\alpha) S_3(\alpha)\, e(-m\alpha)\, d\alpha$$

denote the values of $J_3(n)$ and $J_2(m)$ on the major arcs $\mathfrak{M}$ and by

$$J_3^{\mathfrak{m}}(n) := J_3(n) - J_3^{\mathfrak{M}}(n), \quad J_2^{\mathfrak{m}}(m) := J_2(m) - J_2^{\mathfrak{M}}(m)$$

the values on the minor arcs $\mathfrak{m}$.

Concerning the major arcs we get

Theorem 3. *For $Q_1, Q_2, Q_3 \le n^{1/2}/(\log n)^{\vartheta}$ we have*

$$\mathcal{E}_{Q_1,Q_2,Q_3}^{\mathfrak{M}} := \sum_{\substack{q_i \sim Q_i, \\ i=1,2,3}} \max_{\substack{a_i, (a_i,q_i)=1, \\ i=1,2,3}} \left| J_3^{\mathfrak{M}}(n) - \frac{n^2 \mathcal{S}_3(n)}{2\varphi(q_1)\varphi(q_2)\varphi(q_3)} \right| \ll \frac{n^2}{(\log n)^{A+3}}.$$

We prove Theorem 3 in the following section 2.

In section 3 a lemma containing a special form of Montgomery's sieve is proven. Section 4 delivers a proof of Theorem 1 using Theorem 3 and the lemma from section 3. Further used lemmas concerning estimations on the minor arcs are proven afterwards in section 5.

2 Estimations on the major arcs

2.1 Getting the main term and the error term

We have

$$J_3^{\mathfrak{M}}(n) = \sum_{q \le R} \sum_{\substack{0<a<q \\ (a,q)=1}} I(a,q),$$

where

$$I(a,q) := \int_{-\frac{R}{qn}}^{\frac{R}{qn}} S_1\left(\frac{a}{q}+\alpha\right) S_2\left(\frac{a}{q}+\alpha\right) S_3\left(\frac{a}{q}+\alpha\right)\ e\left(-n\left(\frac{a}{q}+\alpha\right)\right) d\alpha.$$

For $j = 1, 2, 3$ we have for $\alpha \in [-\frac{R}{qn}, \frac{R}{qn}]$

$$\begin{aligned}
S_j\left(\frac{a}{q}+\alpha\right) &= \sum_{\substack{m \le n \\ m \equiv a_j\,(q_j)}} \Lambda(m)\ e(\alpha m)\, e\left(\frac{a}{q}m\right) \\
&= \sum_{\substack{m \le n \\ m \equiv a_j\,(q_j) \\ (m,q)=1}} \Lambda(m)\ e(\alpha m)\, e\left(\frac{a}{q}m\right) + \sum_{\substack{m \le n \\ m \equiv a_j\,(q_j) \\ (m,q)>1}} \Lambda(m)\ e(\alpha m)\, e\left(\frac{a}{q}m\right) \\
&= \sum_{\substack{1 \le k \le q \\ (k,q)=1}} \sum_{\substack{m \le n \\ m \equiv a_j\,(q_j) \\ m \equiv k\,(q)}} \Lambda(m)\ e(\alpha m)\, e\left(\frac{a}{q}k\right) + O((\log n)^2)
\end{aligned}$$

since

$$\sum_{\substack{m \le n \\ m \equiv a_j\,(q_j) \\ (m,q)>1}} \Lambda(m) = \sum_{\substack{p^e \le n \\ p^e \equiv a_j\,(q_j) \\ p|q}} \log p \le \sum_{p|q} \log p \cdot \frac{\log n}{\log p} \ll \log n \sum_{p|q} 1 \ll (\log n)^2.$$

So

$$S_j\left(\frac{a}{q}+\alpha\right) = \sum_{\substack{1 \le k \le q \\ (k,q)=1 \\ k \equiv a_j\,((q_j,q))}} e\left(\frac{a}{q}k\right) T_{j,k}(\alpha) + O((\log n)^2)$$

with

$$T_{j,k}(\alpha) := \sum_{\substack{m\le n\\ m\equiv a_j\,(q_j)\\ m\equiv k\,(q)}} \Lambda(m)\, e(\alpha m) = \sum_{\substack{m\le n\\ m\equiv f_{j,k}\,([q_j,q])}} \Lambda(m)\, e(\alpha m).$$

Here $T_{j,k}$ depends on k with $1 \le k \le q$, $(k,q)=1$, $k \equiv a_j\,((q_j,q))$. For such a k there exists an integer $f_{j,k}$ such that the congruence $m \equiv f_{j,k}\,([q_j,q])$ is equivalent to the system $m \equiv a_j\,(q_j)$, $m \equiv k\,(q)$, so the last step follows.

Now for positive integers x and $h \le x$ let

$$\Delta(x,h) := \max_{y\le x}\max_{(l,h)=1}\left| \sum_{\substack{m\le y\\ m\equiv l\,(h)}} \Lambda(m) - \frac{y}{\varphi(h)}\right|.$$

This expression is ≥ 1 for $h \le x$. (Take $y = \varphi(h)$ and $l = 1$).

Note that by the Theorem of Bombieri and Vinogradov (see, for example, Brüdern [2]) we have

$$\sum_{h\le U} \Delta(x,h) \ll \frac{x}{(\log x)^D} + U\sqrt{x}(\log(Ux))^6$$

for any fixed $D \ge 1$. This yields that if $U \le x^{1/2}/(\log x)^{D+6}$, then

$$\sum_{h\le U} \Delta(x,h) \ll \frac{x}{(\log x)^D}.$$

Now we compute $T_{j,k}(\alpha)$ by partial summation and by introducing Δ. We get

$$\begin{aligned}
T_{j,k}(\alpha) &= \sum_{\substack{m\le n\\ m\equiv f_{j,k}([q_j,q])}} \Lambda(m)\, e(\alpha m)\\
&= -\int_0^n \Bigg(\sum_{\substack{m\le y\\ m\equiv f_{j,k}([q_j,q])}} \Lambda(m)\Bigg) \frac{d}{dy}(e(\alpha y))dy + \Bigg(\sum_{\substack{m\le n\\ m\equiv f_{j,k}([q_j,q])}} \Lambda(m)\Bigg) e(\alpha n)\\
&= -\int_0^n \left(\frac{y}{\varphi([q_j,q])} + O(\Delta(n,[q_j,q]))\right) \frac{d}{dy} e(\alpha y)dy\\
&\quad + \left(\frac{n}{\varphi([q_j,q])} + O(\Delta(n,[q_j,q]))\right) e(\alpha n)
\end{aligned}$$

$$= \frac{1}{\varphi([q_j,q])}\left(-\int_0^n y\left(\frac{d}{dy}e(\alpha y)\right)dy + ne(\alpha n)\right) + O\Big((1+|\alpha|n)\Delta(n,[q_j,q])\Big)$$
$$= \frac{1}{\varphi([q_j,q])}\int_0^n e(\alpha y)dy + O\left(\frac{R}{q}\Delta(n,[q_j,q])\right),$$

since $|\alpha| \leq \frac{R}{qn}$ and $1 \leq \frac{R}{q}$.

This yields, using

$$\int_0^n e(\alpha y)dy = M(\alpha) + O(1), \quad M(\alpha) := \sum_{m=1}^{n} e(\alpha m),$$

the expression

$$T_{j,k}(\alpha) = \frac{M(\alpha)}{\varphi([q_j,q])} + O\left(\frac{R}{q}\Delta(n,[q_j,q])\right).$$

We use this term for $T_{j,k}(\alpha)$ to compute $S_j(\frac{a}{q}+\alpha)$ as

$$S_j\left(\frac{a}{q}+\alpha\right) = \sum_{\substack{1\leq k\leq q \\ (k,q)=1 \\ k\equiv a_j \ ((q_j,q))}} e\left(\frac{a}{q}k\right)\left(\frac{M(\alpha)}{\varphi([q_j,q])} + O\left(\frac{R}{q}\Delta(n,[q_j,q])\right)\right) + O((\log n)^2)$$
$$= \frac{c_j(a,q)}{\varphi([q_j,q])}M(\alpha) + O\left(\frac{R}{q}\Delta(n,[q_j,q])\right) + O((\log n)^2)$$
$$= \frac{c_j(a,q)}{\varphi([q_j,q])}M(\alpha) + O\left(\frac{R}{q}(\log n)^2\Delta(n,[q_j,q])\right)$$

since $(\log n)^2 \geq 1$ and $\frac{R}{q}\Delta(n,[q_j,q]) \geq 1$, with Ramanujan sums

$$c_j(a,q) := \sum_{\substack{1\leq k\leq q \\ (k,q)=1 \\ k\equiv a_j \ ((q,q_j))}} e\left(\frac{a}{q}k\right) \text{ for } j = 1,2,3.$$

We used here that $|c_j(a,q)| = 1$ or $c_j(a,q) = 0$, see paragraph 2.2.

This provides

$$I(a,q) = \int_{-\frac{R}{qn}}^{\frac{R}{qn}} S_1\left(\frac{a}{q}+\alpha\right) S_2\left(\frac{a}{q}+\alpha\right) S_3\left(\frac{a}{q}+\alpha\right) e\left(-n\left(\frac{a}{q}+\alpha\right)\right) d\alpha$$

$$= H_{a,q}(n) + \mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3$$

with

$$H_{a,q}(n) := \frac{(c_1c_2c_3)(a,q)}{\varphi([q_1,q])\varphi([q_2,q])\varphi([q_3,q])} e\left(-n\frac{a}{q}\right) \int_{-\frac{R}{qn}}^{\frac{R}{qn}} M^3(\alpha)e(-n\alpha)d\alpha,$$

$$\mathcal{O}_1 := \sum_{j,k,l} \frac{1}{\varphi([q_j,q])\varphi([q_k,q])} \int_{-\frac{R}{qn}}^{\frac{R}{qn}} |M^2(\alpha)|d\alpha \cdot O\left(\frac{R}{q}(\log n)^2 \Delta(n,[q_l,q])\right),$$

$$\mathcal{O}_2 := \sum_{j,k,l} \frac{1}{\varphi([q_j,q])} \int_{-\frac{R}{qn}}^{\frac{R}{qn}} |M(\alpha)|d\alpha \cdot O\left(\frac{R^2}{q^2}(\log n)^4 \Delta(n,[q_k,q])\Delta(n,[q_l,q])\right),$$

$$\mathcal{O}_3 := O\left(\frac{R^3}{q^3}(\log n)^6 \Delta(n,[q_1,q])\Delta(n,[q_2,q])\Delta(n,[q_3,q])\frac{R}{qn}\right).$$

Note that we abbreviated $(c_1c_2c_3)(a,q) := c_1(a,q)c_2(a,q)c_3(a,q)$. The sum $\sum_{j,k,l}$ is over all triplets (j,k,l) of pairwise different $j,k,l \in \{1,2,3\}$.

So we managed to show

$$J_3^{\mathfrak{M}}(n) = \sum_{q \le R} \sum_{\substack{a<q \\ (a,q)=1}} I(a,q) = \sum_{q \le R} \sum_{\substack{a<q \\ (a,q)=1}} (H_{a,q}(n) + \mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3).$$

The main term of $J_3^{\mathfrak{M}}(n)$ is contained in

$$H(n) := \sum_{q \le R} \sum_{\substack{a<q \\ (a,q)=1}} H_{a,q}(n).$$

We have to show now that for each $i = 1,2,3$ the error term $\mathcal{O}_i$ fulfills

$$\sum_{q_1,q_2,q_3} \sum_{q \le R} \sum_{\substack{a<q \\ (a,q)=1}} \mathcal{O}_i \ll \frac{n^2}{(\log n)^{A+3}},$$

then it will follow that

$$\sum_{q_1,q_2,q_3} \max_{a_1,a_2,a_3} \left| J_3^{\mathfrak{M}}(n) - H(n) \right| \ll \frac{n^2}{(\log n)^{A+3}}.$$

The main term $H(n)$ will be considered later.

So we first consider the error term with $\mathcal{O}_1$. It is (since $\varphi(q) \gg q/(\log\log q)$)

$$\ll \sum_{j,k,l} \sum_{q \le R} \sum_{q_j, q_k} \frac{1}{\varphi([q_j, q])\varphi([q_k, q])} \sum_{\substack{a<q \\ (a,q)=1}} \frac{R^2}{q^2} n(\log n)^2 \sum_{q_l} \Delta(n, [q_l, q])$$

$$\ll \sum_{j,k,l} \sum_{q_j} \frac{\log\log n}{q_j} \sum_{q_k} \frac{\log\log n}{q_k} R^2 n (\log n)^2 \sum_{q_l} \sum_{q \le R} \frac{1}{q} \Delta(n, [q_l, q])$$

$$\ll R^2 n (\log n)^5 \sum_{j,k,l} \sum_{h_l \le RQ_l} \omega(h_l) \Delta(n, h_l)$$

with

$$\omega(h_l) := \sum_{q_l} \sum_{\substack{q \le R \\ [q_l, q] = h_l}} \frac{1}{q} = \sum_{d_l \le R} \sum_{q_l} \sum_{\substack{q \le R \\ (q_l, q) = d_l \\ q_l q = h_l d_l}} \frac{1}{q}$$

$$\ll \sum_{d_l \le R} \sum_{\substack{q \le R \\ d_l | q}} \frac{1}{q} \ll \sum_{d_l \le R} \sum_{q \le R} \frac{1}{q d_l} \ll (\log n)^2,$$

so the $\mathcal{O}_1$-error term is

$$\ll R^2 n (\log n)^7 \sum_{j,k,l} \sum_{h \le RQ_l} \Delta(n, h) \ll R^2 n (\log n)^7 \frac{n}{(\log n)^D}$$

$$\ll \frac{n^2}{(\log n)^{D-2B-7}} \ll \frac{n^2}{(\log n)^{A+3}},$$

for some $D \ge A+2B+10$ and $D+6 \le \vartheta - B$, so this holds if $\vartheta \ge A+3B+16$, which is the case. We used the Theorem of Bombieri and Vinogradov with $Q_l \ll n^{1/2}(\log n)^{-\vartheta}$ for $\vartheta > 0$. So we are done for $\mathcal{O}_1$.

We consider now the error term with $\mathcal{O}_2$. It is

$$\ll \sum_{j,k,l} \sum_{q \le R} \sum_{q_j} \frac{1}{\varphi([q_j, q])} \sum_{\substack{a<q \\ (a,q)=1}} \frac{R^3}{q^3} (\log n)^4 \sum_{q_k, q_l} \Delta(n, [q_k, q]) \Delta(n, [q_l, q])$$

$$\ll \sum_{j,k,l} \sum_{q_j} \frac{\log\log n}{q_j} R^3 (\log n)^4 \sum_{q_k, q_l} \sum_{q \le R} \frac{1}{q^2} \Delta(n, [q_k, q]) \Delta(n, [q_l, q])$$

$$\ll R^3 (\log n)^6 \sum_{j,k,l} \sum_{h_k \le RQ_k} \sum_{h_l \le RQ_l} \omega(h_k, h_l) \Delta(n, h_k) \Delta(n, h_l)$$

with

$$\omega(h_k,h_l) := \sum_{q_k,q_l} \sum_{\substack{q\le R\\ [q_k,q]=h_k\\ [q_l,q]=h_l}} \frac{1}{q^2} = \sum_{d_k,d_l\le R} \sum_{q_k,q_l} \sum_{\substack{q\le R\\ (q_k,q)=d_k,(q_l,q)=d_l\\ q_kq=h_kd_k,q_lq=h_ld_l}} \frac{1}{q^2}$$

$$\ll \sum_{d_k,d_l\le R} \sum_{\substack{q\le R\\ [d_k,d_l]\mid q}} \frac{1}{q^2} \le \sum_{d_k,d_l\le R} \sum_{q\le R} \frac{1}{q^2[d_k,d_l]^2}$$

$$= \sum_{d_k,d_l\le R} \sum_{q\le R} \frac{(d_k,d_l)^2}{q^2d_k^2d_l^2} \le R^2 \sum_q \frac{1}{q^2} \sum_{d_k} \frac{1}{d_k^2} \sum_{d_l} \frac{1}{d_l^2} \ll R^2,$$

so the $\mathcal{O}_2$-error term is

$$\ll R^5(\log n)^6 \sum_{h_k\le RQ_k} \Delta(n,h_k) \sum_{h_l\le RQ_l} \Delta(n,h_l)$$

$$\ll R^5(\log n)^6 \cdot \left(\frac{n}{(\log n)^D}\right)^2 = \frac{n^2}{(\log n)^{2D-5B-6}} \ll \frac{n^2}{(\log n)^{A+3}},$$

for some $2D \ge A+5B+9$ and $D+6 \le \vartheta - B$, so this holds if $\vartheta \ge \frac{1}{2}A + \frac{7}{2}B + 11$, which is the case. We used the Theorem of Bombieri and Vinogradov with $Q_k, Q_l \ll n^{1/2}(\log n)^{-\vartheta}$ for $\vartheta > 0$. So we are done for $\mathcal{O}_2$.

Now to the error term with $\mathcal{O}_3$, it is

$$\ll \sum_{q\le R} \sum_{\substack{a<q\\ (a,q)=1}} \frac{R^4}{q^4 n}(\log n)^6 \sum_{q_1,q_2,q_3} \Delta(n,[q_1,q])\Delta(n,[q_2,q])\Delta(n,[q_3,q])$$

$$\ll \frac{R^4}{n}(\log n)^6 \sum_{\substack{h_1\le RQ_1\\ h_2\le RQ_2\\ h_3\le RQ_3}} \omega(h_1,h_2,h_3)\Delta(n,h_1)\Delta(n,h_2)\Delta(n,h_3)$$

with

$$\omega(h_1,h_2,h_3) := \sum_{q_1,q_2,q_3} \sum_{\substack{q\le R\\ [q_i,q]=h_i}} \frac{1}{q^3} = \sum_{d_1,d_2,d_3\le R} \sum_{q_1,q_2,q_3} \sum_{\substack{q\le R\\ (q_i,q)=d_i\\ q_iq=h_id_i}} \frac{1}{q^3} \ll \sum_{d_1,d_2,d_3\le R} \sum_{\substack{q\le R\\ [d_1,d_2,d_3]\mid q}} \frac{1}{q^3}$$

$$\ll \sum_{d_1,d_2,d_3\le R} \sum_{q\le R} \frac{1}{q^3[d_1,d_2,d_3]^3} = \sum_{d_1,d_2,d_3\le R} \sum_{q\le R} \frac{(d_1,[d_2,d_3])^3(d_2,d_3)^3}{q^3d_1^3d_2^3d_3^3}$$

$$\ll \sum_{d_1,d_2\le R}\sum_{d_3\le R}\frac{1}{d_3^3}\sum_{q\le R}\frac{1}{q^3}\ll R^2,$$

so the $\mathcal{O}_3$-error term is

$$\ll \frac{R^6}{n}(\log n)^6\sum_{h_1\le RQ_1}\Delta(n,h_1)\sum_{h_2\le RQ_2}\Delta(n,h_2)\sum_{h_3\le RQ_3}\Delta(n,h_3)$$
$$\ll \frac{R^6}{n}(\log n)^6\frac{n^3}{(\log n)^{3D}}=\frac{n^2}{(\log n)^{3D-6B-6}}\ll\frac{n^2}{(\log n)^A},$$

for some $3D\ge A+6B+9$ and $D+6\le\vartheta-B$, so this holds if $\vartheta\ge\frac{1}{3}A+3B+9$, which is the case. We used the Theorem of Bombieri and Vinogradov with $Q_1,Q_2,Q_3\ll n^{1/2}(\log n)^{-\vartheta}$ for $\vartheta>0$. So we are done for $\mathcal{O}_3$.

What is now left is the consideration of the main term $H(n)$. Since

$$\int_{-\frac{R}{qn}}^{\frac{R}{qn}} M^3(\alpha)e(-n\alpha)d\alpha=\frac{n^2}{2}+O\left(\frac{q^2n^2}{R^2}\right)$$

(see for example Vaughan [5]) we have

$$H(n)=\sum_{q\le R}\sum_{\substack{a<q\\(a,q)=1}}\frac{(c_1c_2c_3)(a,q)e\big(-n\frac{a}{q}\big)}{\varphi([q_1,q])\varphi([q_2,q])\varphi([q_3,q])}\left(\frac{n^2}{2}+O\bigg(\frac{q^2n^2}{R^2}\bigg)\right).$$

Now let

$$\lambda(q):=\frac{\varphi(q_1)\varphi(q_2)\varphi(q_3)}{\varphi([q_1,q])\varphi([q_2,q])\varphi([q_3,q])}b(q)$$

with

$$b(q):=\sum_{\substack{a<q\\(a,q)=1}}(c_1c_2c_3)(a,q)\;e\bigg(-n\frac{a}{q}\bigg)$$

and let

$$\mathcal{S}_3(n):=\sum_{q=1}^{\infty}\lambda(q)$$

be the singular series. In the next paragraph we show that it is absolutely convergent.

Therefore we have

$$H(n) = \sum_{q\le R} \frac{\lambda(q)n^2}{2\varphi(q_1)\varphi(q_2)\varphi(q_3)} + O\left(\frac{n^2}{R^2}\sum_{q\le R}\frac{q^2|\lambda(q)|}{\varphi(q_1)\varphi(q_2)\varphi(q_3)}\right)$$
$$= \frac{n^2}{2\varphi(q_1)\varphi(q_2)\varphi(q_3)}\mathcal{S}_3(n) + O(e_1) + O(e_2)$$

with

$$e_1 := \frac{n^2}{\varphi(q_1)\varphi(q_2)\varphi(q_3)}\sum_{q>R}|\lambda(q)|,$$
$$e_2 := \frac{n^2}{R^2\varphi(q_1)\varphi(q_2)\varphi(q_3)}\sum_{q\le R}q^2|\lambda(q)|.$$

For the two occuring error terms e_1 and e_2 we have to show that

$$\sum_{q_1,q_2,q_3}\max_{a_1,a_2,a_3} e_j \ll \frac{n^2}{(\log n)^{A+3}},$$

then Theorem 3 follows. This is done in the next paragraph.

2.2 Estimations with the singular series

Now we need estimations for the λ-series. These show the absolute convergence of $\mathcal{S}_3(n)$ and can also be used to deal with e_1 and e_2.

First we state that the Ramanujan sums $c_j(a,q)$ for fixed $a_j, q_j,\ j=1,2,3$, can be computed by

$$c_j(a,q) = c_{a_j,q_j}(a,q) = \begin{cases} \mu\left(\frac{q}{d_j}\right)e\left(\frac{au_ja_j}{d_j}\right), & \text{if } \left(d_j, \frac{q}{d_j}\right)=1,\\ 0, & \text{else,}\end{cases}$$

where $d_j := (q_j, q)$ and u_j is the solution of the congruence $\frac{q}{d_j}u_j \equiv 1\ (d_j)$, with $0 \le u_j < d_j$. (For a proof see [6]). From this result we already used that $|c_j(a,q)| = 1$ or $c_j(a,q) = 0$ in the paragraph before.

We are now going to show that b is multiplicative in q. We prove a proposition about the c_j first.

Proposition 1. *Let* $q = \bar{q}\tilde{q}$*,* $(\bar{q}, \tilde{q}) = 1$*,* $(a, q) = 1$*, and let* $a = \tilde{a}\bar{q} + \bar{a}\tilde{q}$ *with* $(\tilde{a}, \tilde{q}) = 1$*,* $(\bar{a}, \bar{q}) = 1$*. Then* $c_j(a, q) = c_j(\tilde{a}, \tilde{q}) \cdot c_j(\bar{a}, \bar{q})$ *for* $j = 1, 2, 3$*.*

Proof. Let $\tilde{a}_j\bar{q} + \bar{a}_j\tilde{q} \equiv a_j \ ((q_j, q))$ with $\tilde{a}_j$ a residue mod $(q_j, \tilde{q})$ and $\bar{a}_j$ a residue mod $(q_j, \bar{q})$. Then we have for $j = 1, 2, 3$

$$c_j(a,q) = \sum_{\substack{m<q\\(m,q)=1\\m\equiv a_j((q_j,q))}} e\Big(m\frac{a}{q}\Big) = \sum_{\substack{\tilde{m}<\tilde{q}\\(\tilde{m},\tilde{q})=1\\\tilde{m}\equiv\tilde{a}_j\ ((q_j,\tilde{q}))}} \sum_{\substack{\bar{m}<\bar{q}\\(\bar{m},\bar{q})=1\\\bar{m}\equiv\bar{a}_j\ ((q_j,\bar{q}))}} e\Big(\frac{(\tilde{m}\bar{q}+\bar{m}\tilde{q})(\tilde{a}\bar{q}+\bar{a}\tilde{q})}{\tilde{q}\bar{q}}\Big)$$

by substituting $m = \tilde{m}\bar{q} + \bar{m}\tilde{q}$ with $\tilde{m} \equiv \tilde{a}_j \ ((q_j, \tilde{q}))$ and $\bar{m} \equiv \bar{a}_j \ ((q_j, \bar{q}))$, and we have $\tilde{a}_j \equiv a_j\bar{q}^{-1} \ ((q_j, \tilde{q}))$ and $\bar{a}_j \equiv a_j\tilde{q}^{-1} \ ((q_j, \bar{q}))$. Therefore we get

$$\begin{aligned}
c_j(a,q) &= \sum_{\substack{\tilde{m}<\tilde{q}\\(\tilde{m},\tilde{q})=1\\\tilde{m}\equiv a_j\bar{q}^{-1}\ ((q_j,\tilde{q}))}} e\Big(\frac{\tilde{m}\tilde{a}\bar{q}}{\tilde{q}}\Big) \sum_{\substack{\bar{m}<\bar{q}\\(\bar{m},\bar{q})=1\\\bar{m}\equiv a_j\tilde{q}^{-1}\ ((q_j,\bar{q}))}} e\Big(\frac{\bar{m}\bar{a}\tilde{q}}{\bar{q}}\Big)\\
&= \sum_{\substack{\tilde{m}<\tilde{q}\\(\tilde{m},\tilde{q})=1\\\tilde{m}\equiv a_j\ ((q_j,\tilde{q}))}} e\Big(\frac{\tilde{m}\tilde{a}}{\tilde{q}}\Big) \sum_{\substack{\bar{m}<\bar{q}\\(\bar{m},\bar{q})=1\\\bar{m}\equiv a_j\ ((q_j,\bar{q}))}} e\Big(\frac{\bar{m}\bar{a}}{\bar{q}}\Big)\\
&= c_j(\tilde{a},\tilde{q}) \cdot c_j(\bar{a},\bar{q}).
\end{aligned}$$

□

Proposition 1 provides the multiplicativity of b:

Proposition 2. *Let* $(\bar{q}, \tilde{q}) = 1$*. Then* $b(\bar{q}\tilde{q}) = b(\bar{q})b(\tilde{q})$*.*

Proof. We have

$$\begin{aligned}
b(\bar{q}\tilde{q}) &= \sum_{\substack{a<\bar{q}\tilde{q}\\(a,\bar{q}\tilde{q})=1}} (c_1c_2c_3)(a,\bar{q}\tilde{q})e\Big(-n\frac{a}{\bar{q}\tilde{q}}\Big)\\
&= \sum_{\substack{\tilde{a}<\tilde{q}\\(\tilde{a},\tilde{q})=1}} \sum_{\substack{\bar{a}<\bar{q}\\(\bar{a},\bar{q})=1}} (c_1c_2c_3)(\tilde{a},\tilde{q}) \cdot (c_1c_2c_3)(\bar{a},\bar{q}) \cdot e\Big(-n\frac{\tilde{a}\bar{q}+\bar{a}\tilde{q}}{\bar{q}\tilde{q}}\Big)
\end{aligned}$$

by substituting $a = \tilde{a}\bar{q} + \bar{a}\tilde{q}$ in the last step. We further get

$$b(\bar{q}\tilde{q}) = \sum_{\substack{\bar{a}<\bar{q}\\(\bar{a},\bar{q})=1}} (c_1c_2c_3)(\bar{a},\bar{q})e\Big(-n\frac{\bar{a}}{\bar{q}}\Big) \sum_{\substack{\tilde{a}<\tilde{q}\\(\tilde{a},\tilde{q})=1}} (c_1c_2c_3)(\tilde{a},\tilde{q})e\Big(-n\frac{\tilde{a}}{\tilde{q}}\Big)$$

$$= b(\bar{q}) \cdot b(\tilde{q}). \qquad \square$$

Proposition 2 shows that it suffices to evaluate b at prime powers p^k, p prime and $k \geq 1$, to obtain formulas for b and λ. It may happen that $b(p^k) = 0$, what we study now.

We first show:

Proposition 3. *Let $j \in \{1, 2, 3\}$. If $p^k \nmid q_j$ and ($p \mid q_j$ or $k \neq 1$), then $c_j(a, p^k) = 0$.*

Proof.

Firstly, if $p^k \nmid q_j$ and $p \mid q_j$, we have $d_j = (q_j, p^k) = p^r$ with $1 \leq r < k$ and $(d_j, \frac{p^k}{d_j}) = (p^r, p^{k-r}) \geq p$, so $c_j(a, p^k) = 0$.

Secondly, if $p^k \nmid q_j$ and $k \neq 1$, then $d_j = (q_j, p^k) = p^r$ with $0 \leq r < k$, and

$$\left(d_j, \frac{p^k}{d_j}\right) = (p^r, p^{k-r}) = p^{\min(r,k-r)}.$$

For $r > 0$ this is $\geq p$, and so $c_j(a, p^k) = 0$. For $r = 0$ we have $d_j = 1$ and $\mu(\frac{p^k}{d_j}) = \mu(p^k) = 0$ since $k \neq 1$, so $c_j(a, p^k) = 0$, too. $\square$

Therefore $c_j(a, p^k) = 0$ holds unless $p^k \mid q_j$ or ($p \nmid q_j$ and $k = 1$). This shows that

$$b(p^k) = \sum_{\substack{a < p^k \\ (a,p)=1}} c_1(a, p^k) c_2(a, p^k) c_3(a, p^k)\, e\left(-n\frac{a}{p^k}\right) = 0,$$

unless $p^k \mid q_j$ or ($p \nmid q_j$ and $k = 1$) for every $j = 1, 2, 3$. We now have to consider only these cases.

Case 1. If $k \geq 1$, $p^k \mid (q_1, q_2, q_3)$, then

$$\begin{aligned} b(p^k) &= \sum_{\substack{a < p^k \\ (a,p)=1}} c_1(a, p^k) c_2(a, p^k) c_3(a, p^k)\, e\left(-n\frac{a}{p^k}\right) \\ &= \sum_{\substack{a < p^k \\ (a,p)=1}} e\left(-n\frac{a}{p^k}\right) \prod_{i=1,2,3} e\left(\frac{a a_i}{p^k}\right) \quad \text{(since } d_i = (q_i, p^k) = p^k \text{ so } u_i = 1) \end{aligned}$$

$$= \sum_{\substack{a<p^k \\ (a,p)=1}} e\left(\frac{a_1+a_2+a_3-n}{p^k}a\right),$$

so $b(p^k)=\varphi(p^k)$ since $p^k \mid a_1+a_2+a_3-n$ by the general condition.

Case 2. If $k=1$ and $(p,q_1)=(p,q_2)=(p,q_3)=1$ then

$$b(p) = \sum_{a=1}^{p-1} c_1(a,p)c_2(a,p)c_3(a,p)\; e\left(-n\frac{a}{p}\right)$$
$$= \sum_{a=1}^{p-1} e\left(-n\frac{a}{p}\right) \prod_{i=1,2,3} \underbrace{\sum_{m=1}^{p-1} e\left(m\frac{a}{p}\right)}_{=-1} = \begin{cases} 1-p, & \text{if } p \mid n, \qquad (A) \\ 1, & \text{if } p \nmid n. \qquad (B) \end{cases}$$

Case 3. If $k=1$, $p \mid q_1$ (so $d_1=p$) and $(p,q_2)=(p,q_3)=1$ (analogously the cases with permuted indices), then

$$b(p) = \sum_{a=1}^{p-1} c_1(a,p)c_2(a,p)c_3(a,p)\; e\left(-n\frac{a}{p}\right)$$
$$= \sum_{a=1}^{p-1} e\left(-n\frac{a}{p}\right) e\left(\frac{aa_1}{p}\right) \Bigg(\underbrace{\sum_{m=1}^{p-1} e\left(m\frac{a}{p}\right)}_{=-1}\Bigg)^2$$
$$= \sum_{a=1}^{p-1} e\left(\frac{a_1-n}{p}a\right) = \begin{cases} p-1, & \text{if } p \mid a_1-n, \qquad (C) \\ -1, & \text{if } p \nmid a_1-n. \qquad (D) \end{cases}$$

Case 4. If $k=1$, $p \mid q_1$, $p \mid q_2$ and $(p,q_3)=1$ (analogously the cases with permuted indices), then

$$b(p) = \sum_{a=1}^{p-1} c_1(a,p)c_2(a,p)c_3(a,p)\; e\left(-n\frac{a}{p}\right)$$
$$= \sum_{a=1}^{p-1} e\left(-n\frac{a}{p}\right) e\left(\frac{aa_1}{p}\right) e\left(\frac{aa_2}{p}\right) \underbrace{\sum_{m=1}^{p-1} e\left(m\frac{a}{p}\right)}_{=-1}$$

$$= -\sum_{a=1}^{p-1} e\left(\frac{a_1 + a_2 - n}{p} a\right) = \begin{cases} 1 - p, & \text{if } p \mid a_1 + a_2 - n, \qquad (E) \\ 1, & \text{if } p \nmid a_1 + a_2 - n. \qquad (F) \end{cases}$$

If we combine all these cases, we have shown

1. If $k \geq 1$ and $p^k \mid (q_1, q_2, q_3)$: $b(p^k) = \varphi(p^k)$, furthermore $\lambda(p^k) = b(p^k)$.
2. If $p \nmid (q_1, q_2, q_3)$:

$$b(p) = \begin{cases} 1 - p, & \text{if } (p, q_1) = (p, q_2) = (p, q_3) = 1,\ p \mid n, \qquad (A) \\ 1, & \text{if } (p, q_1) = (p, q_2) = (p, q_3) = 1,\ p \nmid n, \qquad (B) \\ p - 1, & \text{if } p \mid q_1,\ (p, q_2) = (p, q_3) = 1,\ p \mid a_1 - n, \qquad (C) \\ & \quad \text{also with permuted indices,} \\ -1, & \text{if } p \mid q_1,\ (p, q_2) = (p, q_3) = 1,\ p \nmid a_1 - n, \qquad (D) \\ & \quad \text{also with permuted indices,} \\ 1 - p, & \text{if } p \mid q_1,\ p \mid q_2,\ (p, q_3) = 1,\ p \mid a_1 + a_2 - n, \ (E) \\ & \quad \text{also with permuted indices,} \\ 1, & \text{if } p \mid q_1,\ p \mid q_2,\ (p, q_3) = 1,\ p \nmid a_1 + a_2 - n, \ (F) \\ & \quad \text{also with permuted indices,} \end{cases}$$

so $b(p) \in \{\pm 1, \pm(p-1)\}$. Expressed in λ we have

$$\lambda(p) = \begin{cases} \frac{1}{(p-1)^3}, & (B) \\ -\frac{1}{(p-1)^2}, & (A), (D) \\ \frac{1}{p-1}, & (C), (F) \\ -1. & (E) \end{cases}$$

3. In any other case: $b(p^k) = \lambda(p^k) = 0$.

In the following let $d := (q_1, q_2, q_3)$ where the q_j are fixed. For a prime p let γ_p such that $p^{\gamma_p} \| d$, that is $p^{\gamma_p} \mid d$ but $p^{\gamma_p + 1} \nmid d$.

Now with b, λ is multiplicative too, since $\varphi(q_i)\varphi([q_i, \bar{q}\tilde{q}]) = \varphi([q_i, \bar{q}])\varphi([q_i, \tilde{q}])$ for $(\bar{q}, \tilde{q}) = 1$, $i = 1, 2, 3$. This multiplicativity for λ shows for $Q \geq 1$:

$$\sum_{q=1}^{Q} q|\lambda(q)| \leq \prod_{\substack{p \leq Q \\ \text{prime}}} \left(1 + \sum_{k=1}^{2\log Q} p^k |\lambda(p^k)|\right)$$

$$
\begin{aligned}
&= \Big(\prod_{p\le Q, p|d} (1+p|\lambda(p)|+p^2|\lambda(p^2)|+\cdots+p^{\gamma_p}|\lambda(p^{\gamma_p})|) \Big) \cdot \Big(\prod_{p\le Q,(p,d)=1} (1+p|\lambda(p)|) \Big) \\
&\le \Big(\prod_{p|d} (1+p(p-1)+p^2(p^2-p)+\cdots+p^{\gamma_p}(p^{\gamma_p}-p^{\gamma_p-1})) \Big) \cdot \Big(\prod_{p\le Q,(p,d)=1} (1+p|\lambda(p)|) \Big) \\
&\le \Big(\prod_{p|d} p^{2\gamma_p} \Big) \cdot \Big(\prod_{p\le Q,(p,d)=1} (1+p|\lambda(p)|) \Big) = d^2 \cdot \mathcal{A} \cdot \mathcal{B} \cdot \mathcal{C} \cdot \mathcal{D},
\end{aligned}
$$

where

$$
\mathcal{A} := \prod_{p\le Q,(B)} \left(1+\frac{p}{\varphi(p)^3}\right) \le \sum_{q, p|q \Rightarrow p\le Q} \frac{q\mu^2(q)}{\varphi(q)^3} \ll \sum_q \frac{\mu^2(q)}{q^2}(\log\log q)^3 \ll 1,
$$

$$
\begin{aligned}
\mathcal{B} &:= \prod_{p\le Q,(A)} \left(1+\frac{p}{\varphi(p)^2}\right) \cdot \prod_{i=1,2,3} \prod_{\substack{p\le Q, \\ (D) \text{ for } q_i}} \left(1+\frac{p}{\varphi(p)^2}\right) \\
&\le \left(\sum_{\substack{q\le n \\ p|q\Rightarrow p|n}} \frac{q\mu^2(q)}{\varphi(q)^2} \right) \prod_{i=1,2,3} \left(\sum_{\substack{q\le q_i \\ p|q\Rightarrow p|q_i}} \frac{q\mu^2(q)}{\varphi(q)^2} \right) \ll \left(\sum_{q\le n} \frac{\mu^2(q)}{q} (\log\log q)^2 \right)^4 \\
&\ll (\log n)^8,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{C} &:= \prod_{i=1,2,3} \prod_{\substack{p\le Q \\ (C) \text{ for } q_i}} \left(1+\frac{p}{p-1}\right) \cdot \prod_{\substack{i,j\in\{1,2,3\} \\ i\ne j}} \prod_{\substack{p\le Q \\ (F) \text{ for } q_i,q_j}} \left(1+\frac{p}{p-1}\right) \\
&\le \prod_{i=1,2,3} \prod_{p|q_i} (1+2) \cdot \prod_{\substack{i,j\in\{1,2,3\} \\ i\ne j}} \prod_{p|(q_i,q_j)} (1+2) \le \prod_{i=1,2,3} 2^{2\omega(q_i)} \cdot \prod_{\substack{i,j\in\{1,2,3\} \\ i\ne j}} 2^{2\omega((q_i,q_j))} \\
&\le \tau^2(q_1)\tau^2(q_2)\tau^2(q_3)\tau^2((q_1,q_2))\tau^2((q_1,q_3))\tau^2((q_2,q_3)) \le \tau^4(q_1)\tau^4(q_2)\tau^4(q_3),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D} &:= \prod_{\substack{i,j,k\in\{1,2,3\} \\ i,j,k \text{ p.w.d.}}} \prod_{\substack{p\le Q \\ (E) \text{ for } q_i,q_j,q_k}} (1+p) \le \prod_{\substack{i,j,k\in\{1,2,3\} \\ i,j,k \text{ p.w.d.}}} \prod_{\substack{p|(q_i,q_j) \\ p\nmid q_k}} (1+p) \\
&= \prod_{\substack{i,j,k\in\{1,2,3\} \\ i,j,k \text{ p.w.d.}}} \sigma\left(\prod_{\substack{p|(q_i,q_j) \\ p\nmid q_k}} p \right) \le \prod_{\substack{i,j,k\in\{1,2,3\} \\ i,j,k \text{ p.w.d.}}} \sigma\left(\frac{(q_i,q_j)}{(q_1,q_2,q_3)} \right) \ll \prod_{\substack{i,j,k\in\{1,2,3\} \\ i,j,k \text{ p.w.d.}}} \frac{(q_i,q_j)}{d} \log n \\
&= \frac{1}{d^3}(q_1,q_2)(q_1,q_3)(q_2,q_3)(\log n)^3,
\end{aligned}
$$

where $\sigma(t) := \sum_{t|q} t$ is the divisor sum function, for which $\sigma(t) \ll t\log t$ holds, and $\omega(t)$ is the number of distinct prime factors of t.

Therefore

$$\sum_{q=1}^{Q} q|\lambda(q)| \ll \frac{(q_1,q_2)(q_1,q_3)(q_2,q_3)}{(q_1,q_2,q_3)}\tau^4(q_1)\tau^4(q_2)\tau^4(q_3)(\log n)^{11},$$

also true for $Q \to \infty$. So for any $Q \geq 1$ we have

$$\sum_{q\geq Q} |\lambda(q)| \leq \frac{1}{Q}\sum_{q=1}^{\infty} q|\lambda(q)| \leq \frac{1}{Q}\frac{(q_1,q_2)(q_1,q_3)(q_2,q_3)}{(q_1,q_2,q_3)}\tau^4(q_1)\tau^4(q_2)\tau^4(q_3)(\log n)^{11}.$$

We see that the singular series $\mathcal{S}_3(n) = \sum_{q=1}^{\infty} \lambda(q)$ converges absolutely, and we have

$$\mathcal{S}_3(n) \ll \frac{(q_1,q_2)(q_1,q_3)(q_2,q_3)}{(q_1,q_2,q_3)}\tau^4(q_1)\tau^4(q_2)\tau^4(q_3)(\log n)^{11}.$$

It follows further that

$$\begin{aligned}
\sum_{q_1,q_2,q_3} \max_{a_1,a_2,a_3} e_1 &\ll \sum_{q_1,q_2,q_3} \frac{(q_1,q_2)(q_1,q_3)(q_2,q_3)}{d\varphi(q_1)\varphi(q_2)\varphi(q_3)}\frac{n^2}{R}\tau^4(q_1)\tau^4(q_2)\tau^4(q_3)(\log n)^{11} \\
&\ll \frac{n^2}{R}(\log n)^{12} \sum_{q_1,q_2,q_3} \frac{\tau^4(q_1)\tau^4(q_2)\tau^4(q_3)(q_1,q_2)(q_1,q_3)(q_2,q_3)}{q_1q_2q_3d} \\
&\ll \frac{n^2}{R}(\log n)^{\eta} \ll \frac{n^2}{(\log n)^{A+3}},
\end{aligned}$$

since $B \geq A+\eta+3$ in $R = (\log n)^B$ for some absolute constant $\eta > 0$. This can be proven as follows. By using

$$\sum_{t\leq n} \frac{\tau^m(t)}{t} \leq (\log n)^{2^m},$$

we see that

$$\begin{aligned}
&\sum_{q_1,q_2,q_3} \frac{(q_1,q_2)(q_1,q_3)(q_2,q_3)}{q_1q_2q_3(q_1,q_2,q_3)}\tau^4(q_1)\tau^4(q_2)\tau^4(q_3) \\
&\leq \sum_{d\leq n}\sum_{a,b,c\leq n}\sum_{e,f,g\leq n} \frac{dadbdc}{d^3a^2b^2c^2efgd}\tau^{12}(d)\tau^8(a)\tau^8(b)\tau^8(c)\tau^4(e)\tau^4(f)\tau^4(g)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{d} \sum_{a,b,c} \sum_{e,f,g} \frac{\tau^{12}(d)\tau^8(a)\tau^8(b)\tau^8(c)\tau^4(e)\tau^4(f)\tau^4(g)}{abcdefg} \\
&\ll (\log n)^{\eta}
\end{aligned}
$$

for some absolute constant $\eta > 0$, where we substituted $q_1 = dabe$, $q_2 = dacf$, $q_3 = dcbg$ with pairwise coprime a, b, c and e, f, g.

Further we have

$$
\sum_{q \le R} q^2 |\lambda(q)| \le R \sum_{q=1}^{\infty} q |\lambda(q)| \ll R\tau^4(q_1)\tau^4(q_2)\tau^4(q_3) \frac{(q_1, q_2)(q_1, q_3)(q_2, q_3)}{d} (\log n)^{11},
$$

so also

$$
\begin{aligned}
\sum_{q_1,q_2,q_3} \max_{a_1,a_2,a_3} e_2 &\ll \sum_{q_1,q_2,q_3} \frac{n^2 (q_1, q_2)(q_1, q_3)(q_2, q_3)}{R\varphi(q_1)\varphi(q_2)\varphi(q_3) d} \tau^4(q_1)\tau^4(q_2)\tau^4(q_3)(\log n)^{11} \\
&\ll \frac{n^2}{(\log n)^{A+3}}
\end{aligned}
$$

as above.

So everything concerning Theorem 3 is shown. □

2.3 Discussion of the singular series

Now we consider $\mathcal{S}_3(n)$ under the general condition.

Since $\mathcal{S}_3(n)$ is absolutely convergent and since λ is multiplicative, we see that it has an Eulerproduct, namely

$$
\mathcal{S}_3(n) = \prod_p \Big(1 + \sum_{k=1}^{\infty} \lambda(p^k) \Big).
$$

For $p^{\alpha} \| (q_1, q_2, q_3)$ we have $1 + \lambda(p) + \cdots + \lambda(p^{\alpha}) = p^{\alpha}$ and for other primes p we get factors according to the cases $(A), \ldots, (F)$. Moreover we see that $\mathcal{S}_3(n)$ vanishes if case (E) for a prime p occurs, that is if

(E) : $\exists\, j, k, l \in \{1, 2, 3\}$ pairwise different with

$$p \mid (q_j, q_k),\ p \nmid q_l,\ p \mid n - (a_j + a_k).$$

In all other cases we have

$$\mathcal{S}_3(n) = (q_1, q_2, q_3) \prod_{\substack{p,(A)\\ \text{or } (D)}} \Big(1 - \frac{1}{(p-1)^2}\Big) \prod_{p,(B)} \Big(1 + \frac{1}{(p-1)^3}\Big) \prod_{\substack{p,(C)\\ \text{or } (F)}} \Big(1 + \frac{1}{p-1}\Big)$$

with properties

$(A) : (p, q_1) = (p, q_2) = (p, q_3) = 1, p \mid n,$
$(B) : (p, q_1) = (p, q_2) = (p, q_3) = 1, p \nmid n,$
$(C) : \exists\ j, k, l \in \{1, 2, 3\}$ pwd: $p \mid q_j,\ (p, q_k) = (p, q_l) = 1,\ p \mid n - a_j,$
$(D) : \exists\ j, k, l \in \{1, 2, 3\}$ pwd: $p \mid q_j,\ (p, q_k) = (p, q_l) = 1,\ p \nmid n - a_j,$
$(F) : \exists\ j, k, l \in \{1, 2, 3\}$ pwd: $p \mid q_j,\ p \mid q_k,\ (p, q_l) = 1,\ p \nmid n - (a_j + a_k).$

So we see that $\mathcal{S}_3(n) = 0$ if and only if case (E) occurs or the general condition is not fulfilled. Further if $\mathcal{S}_3(n) > 0$ we see from the Eulerproduct that it is at least some absolute positive constant times (q_1, q_2, q_3), since $\prod_{p>2}(1 - (p-1)^{-2})$ converges and the other products are > 1.

Now we prove

Lemma 1. *If n is odd, then for given q_3, a_3 with $(a_3, q_3) = 1$ and q_2 there exists an admissible a_2 (such that for every q_1 there exists an admissible a_1). For even n and given q_1, q_2, q_3 there exists* no *admissible triplet a_1, a_2, a_3.*

Recall that a_1, a_2, a_3 is admissible for q_1, q_2, q_3, if $(a_i, q_i) = 1$ for $i = 1, 2, 3$, $n \equiv a_1 + a_2 + a_3 \bmod (q_1, q_2, q_3)$ and $\mathcal{S}_3(n) > 0$.

Proof. For the proof, let $q := (q_1, q_2, q_3)$ and denote by $\nu_p(m)$ the exponent of a prime p in m, that is $p^{\nu_p(m)} \mid m$ but $p^{\nu_p(m)+1} \nmid m$.

First let n be even, and consider q_1, q_2, q_3 with

(a) $2 \mid q_j, 2 \nmid q_k, q_l$. Then $(A), (B), (E), (F)$ are not possible, and the condition $2 \mid n - a_j$ is wrong since a_j must be odd. Therefore (D) holds with $p = 2$, and so $\mathcal{S}_3(n) = 0$.

(b) $2 \mid q_j, q_k$ and $2 \nmid q_l$. Then $(A), \ldots, (D)$ are not possible, and condition $2 \mid n - (a_j + a_k)$ in (E) holds since a_j, a_k are odd, so $\mathcal{S}_3(n) = 0$.

(c) Further $2 \mid q_1, q_2, q_3$ is not possible since then a_1, a_2, a_3 are odd and so $n \not\equiv a_1 + a_2 + a_3\ (q)$, so $\mathcal{S}_3(n) = 0$.

(d) Also $2 \nmid q_1, q_2, q_3$ is not possible since then (A) holds for $p = 2$, so $\mathcal{S}_3(n) = 0$ holds.

Now let n be odd and let q_3, a_3 with $(a_3, q_3) = 1$ and q_2 be given. We construct a_2 and q_2 with $(a_2, q_2) = 1$ such that

$$\forall\, p \mid (q_3, q_2) : n \not\equiv a_3 + a_2\ (p).$$

For any $p \mid (q_3, q_2)$ take h_p such that $1 \le h_p \le p-1$ with $n - a_3 + h_p \not\equiv 0\ (p)$. Such a number h_p exists for $p > 2$ since then $p - 1 > 1$, and if $p = 2$ take $h_2 = 1$ since $n - a_3 + 1 \not\equiv 0\,(2)$ holds for $p = 2 \mid (q_3, q_2)$, where q_3 is even and therefore a_3 is odd.

Then take a_2 with $(a_2, q_2) = 1$ and $a_2 \equiv n - a_3 + h_p\ (p)$ for every $p \mid (q_3, q_2)$ via the Chinese Remainder Theorem. Now we prove that this a_2 is admissible. For this, consider now any q_1, and we have to find now an admissible a_1, that means such that

(1) $n \equiv a_1 + a_2 + a_3\ ((q_1, q_2, q_3))$,

(2) $\forall\, p \mid (q_1, q_2),\ p \nmid q_3 : n \not\equiv a_1 + a_2\ (p)$,

(3) $\forall\, p \mid (q_1, q_3),\ p \nmid q_2 : n \not\equiv a_1 + a_3\ (p)$,

(4) $\forall\, p \mid (q_2, q_3),\ p \nmid q_1 : n \not\equiv a_2 + a_3\ (p)$.

Now condition (4) is fulfilled by the choice of a_2. We have to construct now an admissible a_1 mod q_1, $(a_1, q_1) = 1$, namely such that conditions (1) – (3) are fulfilled.

Firstly, a_1 has to be such that $a_1 \equiv n - a_2 - a_3\ ((q_1, q_2, q_3))$. Since $n - a_2 - a_3 \equiv -h_p \not\equiv 0\ (p)$ for any $p \mid (q_2, q_3)$ we see that a_1 mod (q_1, q_2, q_3) may be chosen like that, and it will not contradict to $(a_1, q_1) = 1$, and also condition (1) is fulfilled.

Further a_1 must be $a_1 \equiv n - a_3 + k_p \not\equiv 0\ (p)$ for every $p \mid (q_1, q_3)$, $p \nmid q_2$, where $1 \le k_p \le p-1$ (condition (3)), and also with $a_1 \equiv n - a_2 + l_p \not\equiv 0\ (p)$ for every $p \mid (q_2, q_1)$, $p \nmid q_3$, where $1 \le l_p \le p - 1$ (condition (2)). Here the existence of l_p and k_p can be explained as above for h_p. Then take a_1 with $(a_1, q_1) = 1$ to hold these congruences, again via the Chinese Remainder Theorem. It is admissible by construction. $\square$

By studying property (E), we encounter the following connection with the binary Goldbach problem.

Let p be any prime > 2 and let n be sufficiently large. We can construct a_i, q_i, with $(a_i, q_i) = 1$ for $i = 1, 2, 3$, and with

$$p \mid (q_1, q_2), \;\; p \nmid q_3, \;\; n \equiv a_1 + a_2 \ (p), \;\; a_1 + a_2 + a_3 \equiv n \ ((q_1, q_2, q_3)),$$

namely take any odd q_1, q_2, q_3 such that $p \mid (q_1, q_2)$, $p \nmid q_3$, and take a_1 with $n - a_1 \not\equiv 0 \ (p)$ relatively prime to q_1, take a_2 relatively prime to q_2 with $a_2 \equiv n - a_1 \ (p)$ and $(n - a_1 - a_2, (q_1, q_2, q_3)) = 1$, and a_3 with $a_3 \equiv n - a_1 - a_2 \ ((q_1, q_2, q_3))$ relatively prime to q_3. If we could show that there exist primes $p_i \equiv a_i \ (q_i)$, $i = 1, 2, 3$, with $n = p_1 + p_2 + p_3$, and so $n \equiv a_1 + a_2 + p_3 \ ((q_1, q_2))$, then since $n \equiv a_1 + a_2 \ (p)$ it follows that $0 \equiv p_3 \ (p)$, so $p_3 = p$ and $n - p = p_1 + p_2$. Then the number $n - p$ would be the sum of two primes.

So if the considered ternary Goldbach problem with primes in independent arithmetic progressions touches the binary Goldbach problem, the circle method fails.

3 A Lemma involving sieve methods

Before considering the minor arcs we show the following Lemma by using the large sieve inequality and a formula of Montgomery in [4]. The method was already presented in [3].

Lemma 2. *For $Q \geq 1$, $H > 0$ and $b_1, \ldots, b_n \in \mathbb{C}$ we have*

$$\sum_{q \sim Q} q \max_{0 \leq a < q} \Bigg| \sum_{\substack{m \leq n \\ m \equiv a(q)}} b_m \Bigg|^2$$
$$\ll (n^2 + Q^2) H^{-1} (\log Q) \max_{m \leq n} |b_m|^2 + (n + Q^2)\, H (\log Q) \sum_{m \leq n} |b_m|^2$$

with an absolute O-constant.

Remark. If Q may be some small power of n the Cauchy-Schwarz-estimate

$$\sum_{q \sim Q} q \max_{0 \leq a < q} \Bigg| \sum_{\substack{m \leq n \\ m \equiv a(q)}} b_m \Bigg|^2 \ll \sum_{q \leq 2Q} q \sum_{m \leq n} |b_m|^2 \, \frac{n}{q} \ll n\, Q \sum_{m \leq n} |b_m|^2$$

is weaker. An approach with the large sieve inequality involving characters does not work either.

Proof of Lemma 2.

For a residue class a mod q we set

$$N(a,q) := \sum_{\substack{m\le n\\ m\equiv a(q)}} b_m.$$

Now the expression on the left hand side in Lemma 2 is E_1+E_2 with

$$E_1 := \sum_{\substack{q\sim Q\\ d(q)>H}} q \max_{0\le a<q} |N(a,q)|^2$$

and

$$E_2 := \sum_{\substack{q\sim Q\\ d(q)\le H}} q \max_{0\le a<q} |N(a,q)|^2.$$

Consider first E_1. Let

$$A_Q := \#\{q\ ;\ q\sim Q,\ d(q)>H\},$$

then

$$A_Q H < \sum_{\substack{q\sim Q\\ d(q)>H}} d(q) \le \sum_{q\le 2Q} d(q) \ll Q\log Q,$$

so

$$A_Q \ll \frac{Q\log Q}{H}.$$

Since $N(a,q)\ll (\frac{n}{q}+1)\max_{m\le n}|b_m|$ we get

$$\begin{aligned} E_1 &\ll \sum_{\substack{q\sim Q\\ d(q)>H}} q\max_a |N(a,q)|^2 \ll \sum_{\substack{q\sim Q\\ d(q)>H}} q\left(\frac{n^2}{q^2}+1\right)\max_{m\le n}|b_m|^2 \\ &\ll A_Q\left(\frac{n^2}{Q}+Q\right)\max_{m\le n}|b_m|^2 \ll \left(\frac{n^2}{H}+\frac{Q^2}{H}\right)(\log Q)\max_{m\le n}|b_m|^2. \end{aligned}$$

This is the first summand on the right hand side of Lemma 2.

Now to E_2.

For any integer $0 \le h < q$ let

$$f_h(q) := \sum_{d|q} \mu(d) \frac{q}{d} N\Big(h, \frac{q}{d}\Big),$$

so Möbius' inversion formula gives

$$qN(h,q) = \sum_{d|q} f_h(d)$$

for all $0 \le h < q$. With this we have

$$E_2 = \sum_{\substack{q \sim Q \\ d(q) \le H}} \frac{1}{q} \max_{0 \le a < q} q^2 |N(a,q)|^2 = \sum_{\substack{q \sim Q \\ d(q) \le H}} \frac{1}{q} \max_{0 \le a < q} \left| \sum_{d|q} f_a(d) \right|^2$$

$$\le \sum_{\substack{q \sim Q \\ d(q) \le H}} \frac{d(q)}{q} \sum_{d|q} \max_{0 \le a < q} |f_a(d)|^2.$$

The maximum is taken over a with $0 \le a < q$. We see that $|f_a(d)|^2$ is d-periodic in a for $d|q$, since $N(a+d,t) = N(a,t)$ for $t|d$, so

$$f_{a+dl}(d) = \sum_{t|d} \mu(t) \frac{d}{t} N\Big(a+dl, \frac{d}{t}\Big) = \sum_{t|d} \mu(t) \frac{d}{t} N\Big(a, \frac{d}{t}\Big) = f_a(d) \text{ for all } l \in \mathbb{Z},$$

therefore the maximum stays equal if taken only over a with $0 \le a < d$. We estimate this maximum by $\sum_{0 \le a < d}$ and get

$$E_2 \le \sum_{\substack{q \sim Q \\ d(q) \le H}} \frac{d(q)}{q} \sum_{d|q} \sum_{0 \le a < d} |f_a(d)|^2.$$

By Montgomery in [4], equation (10), we have for $T(\alpha) := \sum_{m \le n} b_m e(\alpha m)$, $\alpha \in \mathbb{R}$, the formula

$$\frac{1}{d} \sum_{h=0}^{d-1} |f_h(d)|^2 = \sum_{\substack{a < d \\ (a,d)=1}} \left| T\Big(\frac{a}{d}\Big) \right|^2,$$

that we can apply here. We get

$$
\begin{aligned}
E_2 &\leq \sum_{\substack{q\sim Q\\ d(q)\leq H}} d(q) \sum_{d|q} \frac{d}{q} \sum_{\substack{a<d\\ (a,d)=1}} \left| T\left(\frac{a}{d}\right)\right|^2 \\
&\leq H \sum_{d\leq 2Q} \left(\sum_{\substack{q\sim Q\\ d|q}} \frac{d}{q}\right) \sum_{\substack{a<d\\ (a,d)=1}} \left| T\left(\frac{a}{d}\right)\right|^2 \\
&\ll H(\log Q) \sum_{d\leq 2Q} \sum_{\substack{a<d\\ (a,d)=1}} \left| T\left(\frac{a}{d}\right)\right|^2 \\
&\ll H(\log Q)\ (n+Q^2) \sum_{m\leq n} |b_m|^2
\end{aligned}
$$

by the inequality of the large sieve. This is the second term on the right hand side of Lemma 2. □

4 The conclusion with Lemma 2

Now let $A, \theta > 0$ and $\vartheta > 0$ as above. Let $Q_1, Q_2, Q_3 \leq n^{1/2}/(\log n)^{\vartheta}$.

We consider first

$$
\mathcal{E}^{\mathfrak{m}}_{Q_1,Q_2,Q_3} := \sum_{q_3\sim Q_3} \max_{a_3} \sum_{q_2\sim Q_2} \max_{a_2} \sum_{q_1\sim Q_1} \max_{a_1} |J_3^{\mathfrak{m}}(n)| \, .
$$

From the definition of J_3 and J_2 we have

$$
\begin{aligned}
\mathcal{E}^{\mathfrak{m}}_{Q_1,Q_2,Q_3} &\leq \sum_{q_3} \max_{a_3} \sum_{q_2} \max_{a_2} \sum_{q_1} \max_{a_1} \sum_{\substack{m_1\leq n\\ m_1\equiv a_1 (q_1)}} \Lambda(m_1)\, |J_2^{\mathfrak{m}}(n-m_1)| \\
&\leq \sum_{q_3} \max_{a_3} \sum_{q_2} \max_{a_2} \sum_{q_1} \max_{a_1} \sum_{\substack{m\leq n\\ m\equiv n-a_1 (q_1)}} (\log n)\ |J_2^{\mathfrak{m}}(m)|.
\end{aligned}
$$

By Cauchy-Schwarz' inequality we now get

$$
\mathcal{E}^{\mathfrak{m}}_{Q_1,Q_2,Q_3} \leq (\log n) \sum_{q_3} \max_{a_3} \sum_{q_2} \max_{a_2} \left(\sum_{q_1\sim Q_1} q_1 \max_{a_1} \left| \sum_{\substack{m\leq n\\ m\equiv a_1\ (q_1)}} |J_2^{\mathfrak{m}}(m)| \right|^2 \right)^{1/2}
$$

and we apply Lemma 2 to the expression in large brackets.

Since $Q_1 \le n^{1/2}$ we see that

$$\mathcal{E}^{\mathfrak{m}}_{Q_1,Q_2,Q_3} \ll (\log n) \sum_{q_3} \max_{a_3} \sum_{q_2} \max_{a_2} \bigg(\frac{n^2}{H} (\log n) \max_{m \le n} |J_2^{\mathfrak{m}}(m)|^2 + nH (\log n) \sum_{m \le n} |J_2^{\mathfrak{m}}(m)|^2 \bigg)^{1/2}.$$

Now we apply the following two lemmas, which will be proven in the last paragraphs.

Lemma 3. *For $Q_2, Q_3 \le n^{1/2}/(\log n)^{\vartheta}$ we have*

$$\sum_{q_2,q_3} \max_{\substack{m \le n \\ a_2,a_3}} |J_2^{\mathfrak{m}}(m)| \ll n(\log n)^7.$$

Lemma 4. *For $Q_2 \le n^{1/2}/(\log n)^{\vartheta}$ and $Q_3 \le (\log n)^{\theta}$ we have*

$$\sum_{q_3} \max_{a_3} \sum_{q_2} \max_{a_2} \bigg(\sum_{m \le n} |J_2^{\mathfrak{m}}(m)|^2 \bigg)^{\frac{1}{2}} \ll \frac{n^{3/2}}{(\log n)^{2A+16}}.$$

Here the sum over such a small Q_3-range is of course pointless; but we state it here to see why no larger bound for Q_3 is possible to get with the given method in the proof of Lemma 4.

With $H := (\log n)^{2A+23}$ it follows from Lemma 3 and 4 that

$$\mathcal{E}^{\mathfrak{m}}_{Q_1,Q_2,Q_3} \ll \frac{n^2}{(\log n)^{A+3}}.$$

Finally, together with Theorem 3, we get for $Q_2 \le n^{1/2}/(\log n)^{\vartheta}$ and $Q_3 \le (\log n)^{\theta}$ the estimate

$$\sum_{q_3 \sim Q_3} \max_{\substack{a_3 \\ (a_3,q_3)=1}} \sum_{q_2 \sim Q_2} \max_{\substack{a_2 \\ (a_2,q_2)=1}} \sum_{q_1 \sim Q_1} \max_{\substack{a_1 \\ (a_1,q_1)=1}} \left| J_3(n) - \frac{n^2 \mathcal{S}_3(n)}{2\varphi(q_1)\varphi(q_2)\varphi(q_3)} \right|$$
$$\le \sum_{\substack{k, Q_3 = 2^k \\ \le (\log n)^{\theta}}} \sum_{\substack{j, Q_2 = 2^j \\ \le n^{1/2}/(\log n)^{\vartheta}}} \sum_{\substack{i, Q_1 = 2^i \\ \le n^{1/2}/(\log n)^{\vartheta}}} \left(\mathcal{E}^{\mathfrak{m}}_{Q_1,Q_2,Q_3} + \mathcal{E}^{\mathfrak{M}}_{Q_1,Q_2,Q_3} \right)$$

$$\ll (\log n)^3 \cdot \frac{n^2}{(\log n)^{A+3}} = \frac{n^2}{(\log n)^A},$$

and from that follows Theorem 1.

So it remains to show Lemma 3 and Lemma 4.

5 Two Lemmas on the minor arcs

5.1 Proof of Lemma 3

We have

$$\sum_{q_2,q_3} \max_{\substack{m\le n\\ a_2,a_3}} |J_2^{\mathfrak{m}}(m)| = \sum_{q_2,q_3} \max_{\substack{m\le n\\ a_2,a_3}} |J_2(m) - J_2^{\mathfrak{M}}(m)|.$$

Now we estimate $J_2(m)$ and $J_2^{\mathfrak{M}}(m)$. The reason why we split $J_2^{\mathfrak{m}}(m)$ is that the trivial upper estimate for $J_2^{\mathfrak{m}}(m)$, namely

$$J_2^{\mathfrak{m}}(m) \ll \int_0^1 |S_2(\alpha)S_3(\alpha)| d\alpha,$$

does not suffice.

We have

$$\begin{aligned} J_2(m) &= \int_0^1 S_2(\alpha)S_3(\alpha)\, e(-m\alpha)\, d\alpha \\ &= \sum_{\substack{m_2\le n\\ m_2\equiv a_2\ (q_2)}} \Lambda(m_2) \sum_{\substack{m_3\le n\\ m_3\equiv a_3\ (q_3)}} \Lambda(m_3) \int_0^1 e(\alpha(m_2+m_3-m))\, d\alpha \end{aligned}$$

and by the orthogonal relations for $e(\alpha m)$ we have that the last integral is 1, if $m_2 + m_3 = m$, and 0 otherwise. Therefore we get

$$J_2(m) \ll \sum_{\substack{m_2\le n\\ m_2\equiv a_2\ (q_2)\\ m_2\equiv m-a_3\ (q_3)}} (\log n)^2 \ll \frac{n}{[q_2,q_3]}(\log n)^2 \ll \frac{n}{q_2q_3}(\log n)^2(q_2,q_3),$$

so

$$\sum_{q_2,q_3} \max_{\substack{m\le n\\ a_2,a_3}} |J_2(m)| \ll n(\log n)^2 \sum_{q_2,q_3} \frac{(q_2,q_3)}{q_2q_3}$$

$$\ll n(\log n)^2 \sum_{m,q_2',q_3'} \frac{m}{mq_2'mq_3'} \ll n(\log n)^5.$$

Now we consider the following

Proposition 4. *We have*

$$\sum_{q_2,q_3} \max_{\substack{a_2,a_3 \\ m\le n}} |J_2^{\mathfrak{M}}(m)| \ll n(\log n)^7.$$

By this and together with above estimation we get therefore Lemma 3. $\square$

Proof of Proposition 4.

We have to consider the analogous estimation for $J_2^{\mathfrak{M}}(m)$ as was done in paragraph 2.1 in order to estimate $J_3^{\mathfrak{M}}(m)$.

We get

$$J_2^{\mathfrak{M}}(m) = \sum_{q\le R} \sum_{\substack{a<q \\ (a,q)=1}} I(a,q)$$

with

$$\begin{aligned} I(a,q) &= \int_{-R/qn}^{R/qn} S_2\left(\frac{a}{q}+\alpha\right) S_3\left(\frac{a}{q}+\alpha\right) e\left(-m\left(\frac{a}{q}+\alpha\right)\right) d\alpha \\ &= \frac{(c_2c_3)(a,q)}{\varphi([q_2,q])\varphi([q_3,q])} e\left(-m\frac{a}{q}\right) \int_{-R/qn}^{R/qn} M^2(\alpha) e(-m\alpha) d\alpha \\ &+ \sum_{i,j} \frac{1}{\varphi([q_i,q])} \int_{-R/qn}^{R/qn} |M(\alpha)| d\alpha \cdot O\left(\frac{R}{q}(\log n)^2 \Delta(n,[q_j,q])\right) \\ &+ O\left(\frac{R^3}{nq^3}(\log n)^4 \Delta(n,[q_2,q])\Delta(n,[q_3,q])\right) \\ &=: H_{a,q}(m) + \mathcal{O}_1 + \mathcal{O}_2, \end{aligned}$$

say. Now

$$\begin{aligned} \sum_{q_2,q_3} \sum_{q\le R} \sum_{\substack{a<q \\ (a,q)=1}} \mathcal{O}_1 &\ll \sum_{i,j} \sum_{q\le R} \sum_{q_i} \frac{1}{\varphi([q_i,q])} \sum_{\substack{a<q \\ (a,q)=1}} \frac{R}{q} (\log n)^2 \sum_{q_j} \Delta(n,[q_j,q]) \\ &\ll \sum_{i,j} \sum_{q_i} \frac{\log\log n}{q_i} R(\log n)^2 \sum_{q_j} \sum_{q\le R} \Delta(n,[q_j,q]) \end{aligned}$$

$$\ll (\log n)^4 R \sum_j \sum_{h_j \le RQ_j} \omega(h_j)\Delta(n,h_j)$$

with

$$\omega(h_j) := \sum_{q_j} \sum_{\substack{q\le R\\ [q_j,q]=h_j}} 1 = \sum_{d_j\le R} \sum_{q_j} \sum_{\substack{q\le R\\ (q,q_j)=d_j\\ qq_j=h_jd_j}} 1$$

$$\ll \sum_{d_j\le R} \sum_{\substack{q\le R\\ d_j|q}} 1 \ll R\log R \ll R\log n.$$

So the $\mathcal{O}_1$-error term is

$$\ll R^2(\log n)^5 \sum_j \sum_{h_j\le RQ_j} \Delta(n,h_j) \ll R^2(\log n)^5 \cdot \frac{n}{(\log n)^{\vartheta-B-6}}$$
$$\ll n(\log n)^{3B-\vartheta+11} \ll n(\log n)^{-A-B-2} \ll n,$$

again by using Bombieri-Vinogradov's Theorem and $\vartheta \ge A+4B+13$.

Now to $\mathcal{O}_2$. We have

$$\sum_{q_2,q_3} \sum_{q\le R} \sum_{\substack{a<q\\ (a,q)=1}} \mathcal{O}_2 \ll \sum_{q\le R} \sum_{\substack{a<q\\ (a,q)=1}} \frac{R^3}{nq^3}(\log n)^4 \sum_{q_2,q_3} \Delta(n,[q_2,q])\Delta(n,[q_3,q])$$
$$\ll \frac{R^3}{n}(\log n)^4 \sum_{\substack{h_2\le RQ_2\\ h_3\le RQ_3}} \omega(h_2,h_3)\Delta(n,h_2)\Delta(n,h_3)$$

with

$$\omega(h_2,h_3) := \sum_{q_2,q_3} \sum_{\substack{q\le R\\ [q_i,q]=h_i\\ i=2,3}} \frac{1}{q^2} = \sum_{d_2,d_3\le R} \sum_{q_2,q_3} \sum_{\substack{q\le R\\ (q_i,q)=d_i\\ q_iq=h_id_i\\ i=2,3}} \frac{1}{q^2}$$
$$\ll \sum_{d_2,d_3\le R} \sum_{\substack{q\le R\\ [d_2,d_3]|q}} \frac{1}{q^2} \ll \sum_{d_2,d_3} \sum_{q\le R} \frac{1}{q^2[d_2,d_3]^2}$$
$$= \sum_{d_2,d_3} \sum_{q\le R} \frac{1}{q^2 d_2^2 d_3^2}(d_2,d_3)^2 \ll \sum_{d_3\le R} 1 \ll R,$$

so the $\mathcal{O}_2$-error term is

$$\ll \frac{R^4}{n}(\log n)^4\Big(\sum_{h_2\le RQ_2}\Delta(n,h_2)\Big)\Big(\sum_{h_3\le RQ_3}\Delta(n,h_3)\Big)\ll n(\log n)^{6B-2\vartheta+16}\ll n,$$

again by using Bombieri-Vinogradov's Theorem and $\vartheta \ge A+4B+13$.
Now there remains the main term. Since

$$\int_{-R/qn}^{R/qn} M^2(\alpha)e(-m\alpha)d\alpha = m-1+O\bigg(\frac{qn}{R}\bigg)\ll n$$

for $q\le R$ we can estimate it in the following way. It is

$$\begin{aligned} H &:= \sum_{q_2,q_3}\max_{\substack{m\le n\\ a_2,a_3}}\sum_{q\le R}\sum_{\substack{a<q\\(a,q)=1}}\frac{(c_2c_3)(a,q)}{\varphi([q_2,q])\varphi([q_3,q])}e\bigg(-m\frac{a}{q}\bigg)\int_{-R/qn}^{R/qn}M^2(\alpha)e(-m\alpha)d\alpha\\ &\ll n\sum_{q_2,q_3}\sum_{q\le R}\frac{q(\log n)}{[q_2,q][q_3,q]} = n(\log n)\sum_{q_2,q_3}\sum_{q\le R}\frac{(q_2,q)(q_3,q)}{q_2qq_3}\\ &\ll n(\log n)\sum_{a,b,c,d,e,f,g}\frac{dc\cdot db}{dace\cdot dabf\cdot dbcg}\ll n(\log n)^7, \end{aligned}$$

where we substituted $q_2 = dace$, $q_3 = dabf$, $q = dbcg$ with $a,b,c,d,e,f,g\le n$, $d := (q,q_1,q_3)$, and pairwise relatively prime a,b,c and e,f,g.
This shows the Proposition. □

5.2 Proof of Lemma 4

Since the left hand side of Lemma 4 is

$$\ll\bigg(\sum_{q_3}q_3\max_{a_3}\sum_{q_2}q_2\max_{a_2}\sum_{m\le n}|J_2^{\mathfrak{m}}(m)|^2\bigg)^{\frac{1}{2}},$$

it suffices to show that

$$\sum_{q_3}q_3\max_{a_3}\sum_{q_2}q_2\max_{a_2}\sum_{m\le n}|J_2^{\mathfrak{m}}(m)|^2\ll\frac{n^3}{(\log n)^{4A+32}}$$

for any $A > 0$ in the required regions for Q_2 and Q_3. The left hand side is

$$\sum_{q_3} q_3 \max_{a_3} \sum_{q_2} q_2 \max_{a_2} \sum_{m \le n} \left| \int_{\mathfrak{m}} S_2(\alpha) S_3(\alpha)\, e(-m\alpha) d\alpha \right|^2$$
$$\le \sum_{q_3} q_3 \max_{a_3} \sum_{q_2} q_2 \max_{a_2} \int_{\mathfrak{m}} |S_2(\alpha) S_3(\alpha)|^2 d\alpha$$

by Bessel's inequality. Now

$$\begin{aligned} |S_2(\alpha)|^2 &= \sum_{\substack{m,m' \le n \\ m \equiv m' \equiv a_2\ (q_2)}} \Lambda(m)\Lambda(m')\, e(\alpha(m-m')) \\ &= \sum_{\substack{|r| \le n \\ r \equiv 0\ (q_2)}} e(\alpha r) \sum_{\substack{m \le n \\ m \equiv a_2\ (q_2) \\ m-r \le n}} \Lambda(m)\Lambda(m-r) \\ &=: \sum_{\substack{|r| \le n \\ r \equiv 0\ (q_2)}} e(\alpha r) R(r; a_2, q_2), \end{aligned}$$

say, with $R(r; a_2, q_2) \ll \frac{n}{q_2}(\log n)^2$.
So the left hand side is

$$\ll n(\log n)^2 \sum_{q_3 \sim Q_3} q_3 \max_{a_3} \sum_{q_2 \sim Q_2} \sum_{\substack{|r| \le n \\ r \equiv 0\ (q_2)}} \left| \int_{\mathfrak{m}} |S_3(\alpha)|^2\, e(\alpha r) d\alpha \right|$$
$$\ll n(\log n)^2 \sum_{q_3 \sim Q_3} q_3 \max_{a_3} \sum_{0 < |r| \le n} \tau(|r|) \left| \int_{\mathfrak{m}} |S_3(\alpha)|^2 e(\alpha r) d\alpha \right|$$
$$+ n(\log n)^2\ \ Q_2 \sum_{q_3 \sim Q_3} q_3 \max_{a_3} \int_0^1 |S_3(\alpha)|^2 d\alpha.$$

Now

$$\int_0^1 |S_3(\alpha)|^2 d\alpha \ll \frac{n}{q_3}(\log n)^2,$$

so the second term is $\ll n^2(\log n)^2 Q_2 Q_3 (\log n)^2 \ll n^{5/2}(\log n)^{4+\theta} \ll n^3(\log n)^{-A}$ and therefore in the required bound.

The first term is

$$\ll n(\log n)^2 \sum_{q_3} q_3 \max_{a_3} \Big(\sum_{0<|r|\le n} \tau(|r|)^2 \Big)^{1/2} \Big(\sum_{0<|r|\le n} \Big| \int_{\mathfrak{m}} |S_3(\alpha)|^2 e(\alpha r) d\alpha \Big|^2 \Big)^{1/2}$$
$$\ll n^{3/2}(\log n)^4 \sum_{q_3\sim Q_3} q_3 \max_{a_3} \Big(\int_{\mathfrak{m}} |S_3(\alpha)|^4 d\alpha \Big)^{1/2}$$
$$\ll n^{3/2}(\log n)^4 \Big(\sum_{q_3\sim Q_3} q_3^2 \Big)^{1/2} \Big(\sum_{q_3\sim Q_3} \max_{a_3} \int_{\mathfrak{m}} |S_3(\alpha)|^4 d\alpha \Big)^{1/2}$$
$$\ll n^{3/2}(\log n)^4 \Big(\sum_{q_3\sim Q_3} q_3^3 \max_{a_3} \int_{\mathfrak{m}} |S_3(\alpha)|^4 d\alpha \Big)^{1/2}.$$

Now here is the difficulty to show a nontrivial bound for the expression in large brackets. It should be $\ll n^3/(\log n)^C$ for any large constant $C > 0$ and large Q_3, but however one tries to manage it, there is still some power of Q_3 left. We best can give the bound

$$\ll n^{3/2}(\log n)^4 \Big(\sum_{q_3\sim Q_3} q_3^3 \max_{a_3} \max_{\alpha\in\mathfrak{m}} |S_3(\alpha)|^2 \int_0^1 |S_3(\alpha)|^2 d\alpha \Big)^{1/2}.$$

Now we need another Lemma to estimate $|S_3(\alpha)|^2$ for $\alpha \in \mathfrak{m}$, it is the following.

Lemma 5. *For all $q_3 \sim Q_3$, $(a_3, q_3) = 1$ and $\alpha \in \mathfrak{m}$ we have $|S_3(\alpha)|^2 \ll \frac{n^2}{q_3(\log n)^C}$ for $C = 8A + 2\theta + 74$.*

By using this we get for the above expression

$$\ll n^{3/2}(\log n)^5 \Big(\sum_{q_3\sim Q_3} Q_3 \frac{n^3}{(\log n)^C} \Big)^{1/2}$$
$$\ll \frac{n^3}{(\log n)^{C/2-5}} Q_3 \ll \frac{n^3}{(\log n)^{4A+32}}$$

for $C = 8A + 2\theta + 74$ since $Q_3 \le (\log n)^\theta$.

So we see that Q_3 cannot be chosen as a power of n using the given method.

But this estimation shows Lemma 4 for $Q_2 \le n^{1/2}/(\log n)^{\vartheta}$ and $Q_3 \le (\log n)^{\theta}$ as required. $\square$

Proof of Lemma 5.

By Lemma 2 of A. Balog in [1] we have the validity of the following assertion. For $C > 0$ there exists a $D = D(C) > 0$ such that for any $\alpha \in \mathbb{R}$ with $\|\alpha - \frac{u}{v}\| < \frac{1}{v^2}$ with integers $(u, v) = 1$ and $(\log n)^D \le v \le \frac{n}{(\log n)^D}$ we have

$$\sum_{q_3 \le n^{1/3}/(\log n)^D} q_3 \max_{(a_3,q_3)=1} |S_3(\alpha)|^2 \ll \frac{n^2}{(\log n)^C},$$

and since $Q_3 \le (\log n)^{\theta} \ll \frac{n^{1/3}}{(\log n)^D}$ also

$$\sum_{q_3 \sim Q_3} q_3 \max_{(a_3,q_3)=1} |S_3(\alpha)|^2 \ll \frac{n^2}{(\log n)^C}.$$

By Dirichlet's Approximation Theorem, for $\alpha \in \mathbb{R}$ and $B > 0$ there exist integers u, v, $1 \le v \le n/(\log n)^B$, with $(u, v) = 1$ and $\|\alpha - \frac{u}{v}\| < \frac{(\log n)^B}{vn}$, and for $\alpha \in \mathfrak{m}$ it follows that $v \ge (\log n)^B$.

Therefore the conditions of Balog's Lemma are fulfilled if we take $B \ge D(8A + 2\theta + 74)$, and it can be applied then. It follows that for all $\alpha \in \mathfrak{m}$ we have

$$\sum_{q_3 \sim Q_3} q_3 \max_{a_3} |S_3(\alpha)|^2 \ll \frac{n^2}{(\log n)^{8A+2\theta+74}},$$

and so we have for all $q_3 \sim Q_3$ and $(a_3, q_3) = 1$ the inequality

$$|S_3(\alpha)|^2 \ll \frac{n^2}{q_3(\log n)^{8A+2\theta+74}},$$

since

$$|S_3(\alpha)|^2 \ll \frac{1}{Q_3} \sum_{q_3 \sim Q_3} q_3 \max_{a_3} |S_3(\alpha)|^2 \ll \frac{1}{Q_3} \cdot \frac{n^2}{(\log n)^{8A+2\theta+74}}.$$

That shows Lemma 5. $\square$

6 Proof of Theorem 2

Now we prove Theorem 2 in this last section. Let $A, \theta, \vartheta > 0$ be as in Theorem 2 and let n be odd and sufficiently large.

Besides $J_3(n)$ consider also

$$R_3(n) = \sum_{\substack{p_1,p_2,p_3 \\ p_1+p_2+p_3=n \\ p_i \equiv a_i\ (q_i), \\ i=1,2,3}} \log p_1 \log p_2 \log p_3 \quad \text{and} \quad r_3(n) = \sum_{\substack{p_1,p_2,p_3 \\ p_1+p_2+p_3=n \\ p_i \equiv a_i\ (q_i), \\ i=1,2,3}} 1.$$

Then we have

$$|R_3(n) - J_3(n)| \le (\log n)^3 W,$$

where W denotes the number of solutions of $p^l + q^j + r^k = n$, with p, q, r prime and where l, j or k are at least 2, and $p^l \equiv a_1\ (q_1)$, $q^j \equiv a_2\ (q_2)$, $r^k \equiv a_3\ (q_3)$. Now four cases occur: For $i = 1, 2, 3, 4$ let $W_{(i)}$ be the number of solutions in case (i), namely (1) $l, j \ge 2$, (2) $l = 1, j \ge 2$, (3) $l \ge 2, j = 1$, (4) $l = j = 1, k \ge 2$.

In case (1) there are at most $O(\sqrt{n})$ many possibilities for $p^l, q^j \le n$, so $W_{(1)} \ll n$ and we have $\sum_{q_1,q_2,q_3} \max_{a_1,a_2,a_3} W_{(1)} \ll \frac{n^2}{(\log n)^{2\vartheta-\theta}} \ll \frac{n^2}{(\log n)^{A+3}}$ since $\vartheta > \theta + A + 3$.

In case (4) we have at most $O(\sqrt{n})$ many possibilities for $r^k \le n$ and $\ll \frac{n}{q_2}$ many for q, so $W_{(4)} \ll \frac{n^{3/2}}{q_2}$ and we get $\sum_{q_1,q_2,q_3} \max_{a_1,a_2,a_3} W_{(4)} \ll Q_1 Q_3 n^{3/2} \ll \frac{n^2}{(\log n)^{\vartheta-\theta}} \ll \frac{n^2}{(\log n)^{A+3}}$ since $\vartheta > \theta + A + 3$.

The same estimation comes of course analogously with $W_{(2)}$ in case (2).

In case (3) we consider the number

$$\#\{p^l \le n;\ l \ge 2, p^l \equiv a_1\ (q_1)\} \le \sum_{\substack{m \le n \\ m \equiv a_1 (q_1)}} \Lambda(m)(1 - \mu^2(m)) =: N(a_1, q_1)$$

in the context of section 3, with $b_m := \Lambda(m)(1 - \mu^2(m))$. Then $W_{(3)} \ll N(a_1, q_1) \cdot \frac{n}{q_2}$, and by application of Lemma 2 we get

$$\begin{aligned} \sum_{q_1,q_2,q_3} \max_{a_1,a_2,a_3} W_{(3)} \ll nQ_3 \sum_{q_1} \max_{a_1} N(a_1, q_1) &\ll nQ_3 \Big(\sum_{q_1} q_1 \max_{a_1} N(a_1, q_1)^2 \Big)^{1/2} \\ &\ll nQ_3 \Big(\frac{n^2}{H} + n^{3/2} H \Big)^{1/2} \log n \end{aligned}$$

since

$$\sum_{m\le n}|b_m|^2=\sum_{\substack{m\le n\\ m\equiv a_1\,(q_1)}}\Lambda^2(m)(1-\mu^2(m))^2\ll\sum_{\substack{p^k\le n\\ k\ge 2}}(\log p)^2\ll\sqrt{n}\log n$$

and $Q_1\le\sqrt{n}$.

If we choose the parameter H as $H:=\frac{n^{1/2}}{(\log n)^{2A+6}Q_3^2}$ we get further

$$\sum_{q_1,q_2,q_3}\max_{a_1,a_2,a_3}W_{(3)}\ll nQ_3\Big(n^{3/2}(\log n)^{2A+6}Q_3^2+\frac{n^2}{Q_3^2(\log n)^{2A+6}}\Big)^{1/2}$$
$$\ll n\cdot n^{3/4}Q_3^2(\log n)^{A+3}+\frac{n^2}{(\log n)^{A+3}}\ll\frac{n^2}{(\log n)^{A+3}}.$$

So we get

$$\sum_{q_1,q_2,q_3}\max_{a_1,a_2,a_3}W\ll\frac{n^2}{(\log n)^{A+3}}.$$

Therefore it follows from Theorem 1:

$$\sum_{q_3}\max_{a_3}\sum_{q_2}\max_{a_2}\sum_{q_1}\max_{a_1}\Big|R_3(n)-\frac{n^2\mathcal{S}_3(n)}{2\varphi(q_1)\varphi(q_2)\varphi(q_3)}\Big|$$
$$\le\sum_{q_3}\max_{a_3}\sum_{q_2}\max_{a_2}\sum_{q_1}\max_{a_1}\Big|R_3(n)-J_3(n)\Big|$$
$$+\sum_{q_3}\max_{a_3}\sum_{q_2}\max_{a_2}\sum_{q_1}\max_{a_1}\Big|J_3(n)-\frac{n^2\mathcal{S}_3(n)}{2\varphi(q_1)\varphi(q_2)\varphi(q_3)}\Big|$$
$$\ll\sum_{q_1,q_2,q_3}\max_{a_1,a_2,a_3}W(\log n)^3+\frac{n^2}{(\log n)^A}\ll\frac{n^2}{(\log n)^A}.$$

So the formula of Theorem 1 holds also for $R_3(n)$ instead of $J_3(n)$.

Now let $q_3\le Q_3=(\log n)^\theta$ and $(a_3,q_3)=1$ be fixed.

For given q_2 and admissible a_2 consider

$$\mathcal{Q}_1:=\{q_1\le Q_1;\ \exists\, a_1\text{ adm.}: R_3(n)=0\},\quad E_1:=\#\mathcal{Q}_1,$$

and

$$\mathcal{Q}_2:=\{q_2\le Q_2;\ \exists\, a_2\text{ adm.}: E_1\ge Q_1(\log n)^{-A}\},\quad E_2:=\#\mathcal{Q}_2.$$

We have $\mathcal{S}_3(n) \gg 1$ if it is positive (see the formula for it as Euler product), so we have

$$\begin{aligned} E_2 \cdot \frac{Q_1}{(\log n)^A} \cdot \frac{n^2}{Q_1 Q_2 Q_3} & \\ &\leq \sum_{q_2 \in \mathcal{Q}_2} \max_{\substack{a_2 \text{ adm.} \\ E_1 \geq \frac{Q_1}{(\log n)^A}}} \sum_{q_1 \in \mathcal{Q}_1} \max_{\substack{a_1 \text{ adm.} \\ R_3(n)=0}} \left| \frac{n^2 \mathcal{S}_3(n)}{2\varphi(q_1)\varphi(q_2)\varphi(q_3)} \right| \\ &\ll \frac{n^2}{(\log n)^{2A+\theta}} \end{aligned}$$

by Theorem 1, and it follows that $E_2 \ll Q_2 (\log n)^{-A}$.

So for almost all q_2 and all admissible a_2 we have that $E_1 < Q_1 (\log n)^{-A}$, that means that for almost all q_1 and all admissible a_1 it holds that $R_3(n) > 0$. Since $r_3(n) \geq \frac{R_3(n)}{(\log n)^3}$, it follows that $r_3(n)$ is positive, too, so Theorem 2 follows. $\square$

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