Generating Lamplighter-like Groups with Bireversible Automata

Rachel Skipper
joint with Ben Steinberg

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Regular rooted trees

Let $X$ be a finite set called an *alphabet*, and let $X^*$ be the set of finite words over the alphabet $X$ including the empty word, $\emptyset$. 

Definition

An *automorphism* of $T_X$ is a bijection from $X^*$ to $X^*$ which preserves edge incidences.

Any such function on the finite words uniquely determines a function on $X^\infty$, the set of infinite words. Likewise, any "prefix relation preserving" function on $X^\infty$ defines an automorphism of the tree.
Let $X$ be a finite set called an *alphabet*, and let $X^*$ be the set of finite words over the alphabet $X$ including the empty word, $\emptyset$. $X^*$ has the structure of the vertex set of a rooted tree, $T_X$. 

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![Diagram of a tree with vertices labeled by strings from $X^*$ including the empty string $\emptyset$.]

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\]
\[
00 \quad 01 \quad 10 \quad 11
\]

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Finite State (Mealy) Automata

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A finite state (Mealy) automaton is $A$ is a 4-tuple $A = (Q, X, \delta, \lambda)$ where $Q$ is finite a set of states, $X$ is a finite alphabet, $\delta : Q \times X \rightarrow Q$ is the transition function, and $\lambda : Q \times X \rightarrow X$ is the output function. For each $q \in Q$ and $x \in X$, we will use the notation $\lambda_q(x)$ to mean $\lambda(q, x)$. 
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\]

\( \lambda_q \) extends to a function \( X^* \) and \( X^\omega \):

\[
\lambda_q(x_0, x_1, \ldots, x_n) = \lambda_q(x_0)\lambda_{\delta(q, x_0)}(x_1, \ldots, x_n)
\]

and

\[
\lambda_q(x_0, x_1, \ldots) = \lim_{n \to \infty} \lambda_q(x_0, x_1, \ldots, x_n).
\]
Definition by example: The Alëshin Automaton

Each state gives a function from $X^\omega \to X^\omega$. $A$ is invertible if, for each $q$, $\lambda q$ is a permutation of the alphabet. In this case, states define automorphisms of $T X$.

The group generated by the states of $A$ is called the automaton group for $A$ and denoted $G(A)$. 

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For an $\mathcal{A} = (Q, X, \delta, \lambda)$, its *dual automaton* $\partial\mathcal{A}$ is given by $(X, Q, \lambda, \delta)$, i.e., the alphabet and states are interchanged and the output and transition functions are interchanged.
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If $A$, $\partial A$, and $\partial A^{-1}$ are invertible, $A$ is called bireversible. In this case, all 8 possible automata are invertible.
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Generating Lamplighter-like Groups with Bireversible Automata
Why bireversible automaton groups?

First examples of free groups and virtually free groups as automaton groups were bireversible.
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Bireversible automaton groups act essentially freely on the boundary of the rooted tree hence, one can potentially compute spectral measures for their random walks via the action on the tree.
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Various automaton representations of lamplighter like groups have been found (Grigorchuk, Žuk, Nekrashevych, Sushanksii, Sidki, Savchuk, Steinberg, Silva, Sunik, Juschenko, Wesolek, Bartholdi.....)
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Bondarenko, D’Angeli, and Rodaro constructed $(\mathbb{Z}/3\mathbb{Z}) \wr \mathbb{Z}$ as a bireversible automaton group (2016).

Likewise, Ahmed and Savchuk realized $(\mathbb{Z}/2\mathbb{Z})^2 \wr \mathbb{Z}$ as a bireversible automaton group (2018).
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Identify $R^\omega$ with $R[[t]]$ via

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Then for any $f \in R[[t]]$ define two functions $\mu_f$ and $\alpha_f$ on $R^\omega$ given by

$$\mu_f : g(t) \mapsto f(t)g(t)$$
$$\alpha_f : g(t) \mapsto f(t) + g(t)$$

Exercise: For $f$ invertible, these define automorphisms of $T_R$ and for $\mu_f \alpha_h \mu_f^{-1} = \alpha_{fh}$. 

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Generating Lamplighter-like Groups with Bireversible Automata
Rational power series approach

Fix now \( f = r \left( \frac{1-at}{1-bt} \right) \) with \( r \in R^\times \) and \( a, b \in R \).
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Proposition (S, Steinberg)

Let $f(t) = r \left( \frac{1-at}{1-bt} \right)$ where $r \in R^\times$ and $a, b \in R$. Then $\mu_f$ is finite state with set of states $\{ \alpha_{sra_f} \alpha_{sb} : s \in R \}$. Moreover, for any $s \in R$, the state $\alpha_{sra_f} \alpha_{sb}$ permutes the degree zero terms via:

$$\tilde{s} \longmapsto r(\tilde{s} + (b - a)s).$$
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We can associate to \( f \) a finite state automaton \( A_f \) with states \( \{ \alpha_{sra} \mu_f \alpha_{sb} : s \in R \} \). Transition and output functions are

\[
\delta(\alpha_{sra} \mu_f \alpha_{sb}, \tilde{s}) = \alpha_{-(sb+\tilde{s})ra} \mu_f \alpha_{(sb+\tilde{s})b}
\]

and

\[
\lambda(\alpha_{sra} \mu_f \alpha_{sb}, \tilde{s}) = r(\tilde{s} + (b - a)s)
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Theorem (S, Steinberg)

Let

\[ f(t) = r \left( \frac{1 - at}{1 - bt} \right) \]

where \( r \in R^\times \) and \( a, b \in R \). If \( a - b \in R^\times \), then

\[ \mathbb{G}(\mathcal{A}_f) = \langle \alpha_{-sra} \mu_f \alpha_{sb} : s \in R \rangle \cong R^+ \wr \mathbb{Z}. \]
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Proof.

(Sketch) \( \alpha_{-sra} \mu_f \alpha_{sb} = \alpha_{-sra + sbf} \mu_f = \alpha_{s(-ar + bf)} \mu_f \) so we can consider the generating set \( \{ \alpha_{s(-ar + bf)}, \mu_f : s \in \mathbb{R} \} \).
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If \( b - a \) a unit, then \( (-ar + bf)f^m \) is linearly independent over \( R \) (in fact if and only if). And so,

\[ N = \langle \alpha_{s(-ar + bf)} f^m : m \in \mathbb{Z} \rangle \cong \bigoplus_{\mathbb{Z}} R^+. \]
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\( N \) is normal and torsion and so intersects \( \langle \mu_f \rangle \cong \mathbb{Z} \) trivially and so \( \mathbb{G}(A_f) \cong R^+ \wr \mathbb{Z} \).
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where \( r \in R^\times \), \( a, b \in R \) and \( a - b \) is a unit. Then \( A_f \) has \( |R| \) states. Moreover,

1. \( A_f \) is reversible if and only if \( b \) is a unit.
2. \( (A_f)^{-1} \) is reversible if and only if \( a \) is a unit.
3. \( A_f \) is bireversible if and only if both \( a \) and \( b \) are units.
Theorem (S, Steinberg)

Let $A$ be a finite abelian group. Then there is a finite commutative ring $R$ with $R^+ \cong A$ and two elements $a, b \in R^\times$ with $a - b \in R^\times$ if and only if $A \cong A_1 \oplus A_2$ where $A_1$ has odd order and $A_2 \cong (\mathbb{Z}/2\mathbb{Z})^{a_1} \oplus (\mathbb{Z}/2^2\mathbb{Z})^{a_2} \oplus \cdots \oplus (\mathbb{Z}/2^t\mathbb{Z})^{a_t}$ with $a_i \neq 1$ for all $1 \leq i \leq t$. 
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Corollary

For any finite abelian group described in the last theorem, there exists a bireversible automaton generating $A \wr \mathbb{Z}$.
Examples

Figure: The bireversible automaton given by Bodarenko, D’Angeli, Rodaro for $\mathbb{Z}/3\mathbb{Z} \wr \mathbb{Z}$ and corresponding to $f(t) = 2 \left( \frac{1 - 2t}{1 - t} \right)$. (Observed by Bondarenko and Savchuk)
Figure: An automaton which generates $\mathbb{Z}/6\mathbb{Z} \wr \mathbb{Z}$ for $f = \frac{1 - 3t}{1 - 2t}$ that is not reversible and whose inverse is also not reversible.
Examples

Take $O = \mathbb{Z}_2[\zeta]$ with $\zeta$ a third root of unity and $R = O/4O \cong \mathbb{Z}/4\mathbb{Z}[\zeta]$. Taking $r = 1$, $a = 1$, and $b = 2 + \zeta$, we find that $a$, $b$, and $b - a$ are all units with inverses $1$, $3 + \zeta$, and $1 + \zeta$ respectively. $R^+ \cong (\mathbb{Z}/4\mathbb{Z})^2$.

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**Table:** The transition table for $f = r \left( \frac{1 - at}{1 - bt} \right)$ with $r = 1$, $a = 1$, and $b = 2 + \zeta$. 

Rachel Skipper joint with Ben Steinberg
Examples

Take $O = \mathbb{Z}_2[\zeta]$ with $\zeta$ a third root of unity and $R = O/4O \cong \mathbb{Z}/4\mathbb{Z}[\zeta]$. Taking $r = 1$, $a = 1$, and $b = 2 + \zeta$, we find that $a$, $b$, and $b - a$ are all units with inverses $1$, $3 + \zeta$, and $1 + \zeta$ respectively. $R^+ \cong (\mathbb{Z}/4\mathbb{Z})^2$.

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**Table:** The output table for $f = r \left( \frac{1 - at}{1 - bt} \right)$ with $r = 1$, $a = 1$, and $b = 2 + \zeta$. 

Rachel Skipper joint with Ben Steinberg
Thank you!

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