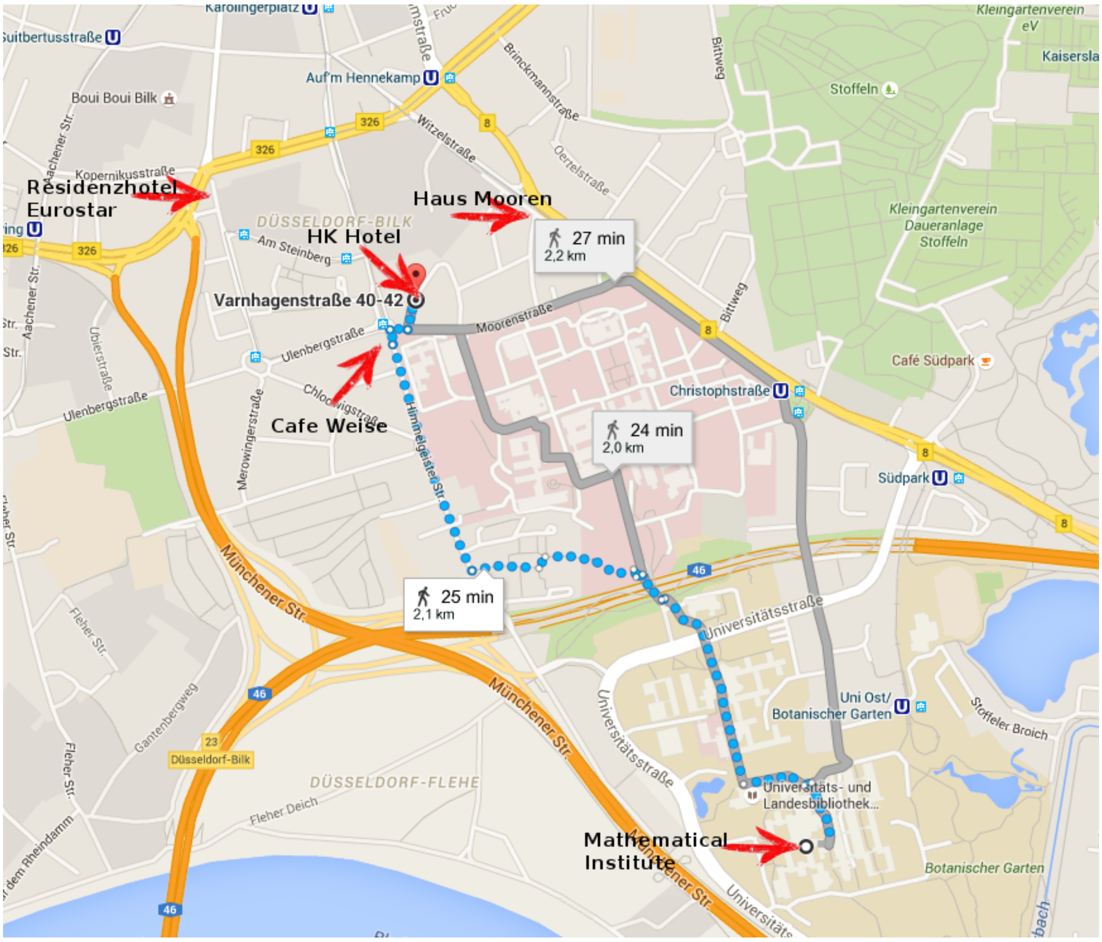
Zeta functions and motivic integration

Enumeration problems in Algebra and Geometry

Heinrich-Heine-Universität Düsseldorf

20 - 22 July 2016

http://reh.math.uni-duesseldorf.de/~internet/ZetaFunctions2016/



Getting here

To get to the university by public transport one can take **underground train U79** or **underground train U73** to "Universität Ost". Another possibility is to take the **tram 704** to "Christophstraße/Uni Nord". From there it is a 15 minutes walk to the mathematical institute. To get from the main station to the "HK Hotel" or "Haus Mooren" one can use the tram 704 to "Uni-Kliniken". Public transport within the city of Düsseldorf (e.g. university, main station, city centre, Heinrich-Heine-Allee, DUS airport) requires a ticket "Preisstufe A". A single ticket costs 2,60 Euro. One can buy at a slight discount a multi-ride ticket (4 independent rides). After validation a ticket is valid for 90 minutes for a trip in one direction (in tram, S-Bahn, bus, and even regional train to the airport).

The map above shows how one can walk from the "HK Hotel" to the mathematical institute in about 20 – 25 minutes. It is also possible to walk from "Residenzhotel Eurostar" and "Haus Mooren".

Schedule

All talks, except for the colloquium talk of Wim Veys on Friday afternoon, will take place in room 01.81 in the building 25.22. The colloquium talk on Friday will take place in lecture hall 5H. The coffee breaks in the afternoon will take place in the meeting room 00.53.

Wednesday, 20 July 2016

8:30 - 9:00	Registration
9:00 - 10:30	Daniel Loughran (Hannover) *Height zeta functions* (1)
11:00 - 12:30	Immanuel Halupczok (Leeds) *Understanding Igusa zeta-functions using motivic integration* (1)
12:30 - 14:15	Lunch break
14:15 - 15:00	Matteo Vannacci (Düsseldorf) *Random relations and zeta functions of profinite groups*
15:00 - 15:45	Coffee break
15:45 - 16:15	Angela Carnevale (Bielefeld) *Orbit Dirichlet series*
16:30 - 17:15	Thomas Cauwbergs (Bielefeld) *Motivic zeta functions and splicing*

Thursday, 21 July 2016

9:00 - 10:30	Christopher Voll (Bielefeld) *Zeta functions of groups, algebras, and modules* (1)
11:00 - 12:30	Daniel Loughran (Hannover) *Height zeta functions* (2)
12:30 - 14:00	Lunch break
14:00 - 14:45	Alexander Stasinski (Durham) *Representation growth of* $GL_2(O)$ *and* $SL_2(O)$ *in residue characteristic* 2
15:00 - 15:30	Javier García Rodríguez (Madrid) *The Congruence Subgroup Property and representation growth*
15:30 - 16:10	Coffee break
16:10 - 16:55	Michael Schein (Bar-Ilan) *On the behavior of pro-isomorphic zeta functions under base extension*
17:00 - 17:30	Discussion session The discussion session is an opportunity to present informally problems questions, conjectures etc., with a short motivation.
19:00	Common dinner in "Cafe Weise" (see map)

Friday, 22 July 2016

9:00 - 10:30	Immanuel Halupczok (Leeds) *Understanding Igusa zeta-functions using motivic integration* (2)
11:00 - 12:30	Christopher Voll (Bielefeld) *Zeta functions of groups, algebras, and modules* (2)
12:30 - 14:15	Lunch break
14:15 - 15:00	Jan-Christoph Schlage-Puchta (Rostock) *Numerical evaluation of Witten's zeta function for symmetric groups*
15:15 - 16:00	Tobias Roßmann (Bielefeld) *Computing zeta functions of groups, algebras, and modules*
16:00 - 16:45	Coffee break
16:45 - 17:45	Wim Veys (Leuven) *Zeta functions and the monodromy conjecture* (Mathematical colloquium in HS 5H)

Minicourses

Height zeta functions

Daniel Loughran (Universität Hannover)

In this course we discuss the theory of height zeta functions. Such zeta functions naturally arise when one is interested in counting rational points of bounded height on algebraic varieties, and there is a conjecture of Manin concerning their analytic behaviour. After some basic examples we shall illustrate some of the range of behaviour which occurs via the cases of flag varieties and toric varieties, where the presence of a group structure allows one to address the zeta function using tools from harmonic analysis. If time permits, we shall also consider some related counting problems where the associated height zeta functions have branch point singularities.

Surveys:

[1] T.D. Browning, *Quantitative arithmetic of projective varieties*, Progress in Mathematics, 277. Birkhäuser Verlag, Basel, 2009.

[2] Y. Tschinkel,*Algebraic varieties with many rational points*, Clay Math. Proc., 8, Amer. Math. Soc., Providence, RI, 2009.

Articles:

[3] J. Franke, Y. Manin, Y. Tschinkel, *Rational points of bounded height on Fano varieties*, Invent. Math. 95 (1989), no. 2, 421–435.

[4] V. Batyrev, Y. Tschinkel - *Manin's conjecture for toric varieties*, J. Algebraic Geom. 7 (1998), no. 1, 15–53.

[5] D. Loughran, *The number of varieties in a family which contain a rational point*, arXiv:1310.6219.

Understanding Igusa zeta-functions using motivic integration

Immanuel Halupczok (University of Leeds)

On $\mathbb{Q}_p$, the field of p-adic numbers, one has a natural Lebesgue measure, giving a meaning to the integrals of functions from $\mathbb{Q}_p^n$ to $\mathbb{C}$. If we are given one such function $f_p : \mathbb{Q}_p^n \to \mathbb{C}$ for each prime p in a suitable uniform way, then motivic integration provides a way to also compute the integrals uniformly in p; I will explain how this works.

As an example, I will show how this can be applied to Igusa Zeta functions, which fall into this framework. As a result, one obtains that a rationality result about Igusa Zeta functions can be made uniform in $\mathbb{Q}_p$. (All this will also be explained.)

The following article contains an introduction to motivic integration and its application to the p-adics:

[1] R. Cluckers, J. Gordon, I. Halupczok: *Motivic functions, integrability, and applications to harminic analysis on p-adic groups* Electron. Res. Announc. Math. Sci., 21 (2014), 137-152.

Zeta functions of groups, algebras, and modules

Christopher Voll (Universität Bielefeld)

I will give a very concise introduction to various zeta functions used in asymptotic group and ring theory, concentrating on those enumerating subobjects of various kinds. A prototype of such a function is the zeta function enumerating the normal subgroups of finite index in a finitely generated nilpotent group. The zeta functions in question typically satisfy Euler product decompositions, with ("local") factors indexed by places of number fields.

In the first talk I will explain a number of intringuing features these Euler factors tend to exhibit (such as rationality and functional equations), talk about methods from p-adic integration to compute them, and introduce their reduced and topological counterparts.

In the second talk I will concentrate on zeta functions enumerating submodules invariant under a nilpotent algebra of endomorphisms. These generalize the normal subgroup zeta functions mentioned above. I will discuss a sufficient condition for these zeta functions to satisfy local functional equations, shedding new light on a conjecture of du Sautoy and Woodward.

[1] C. Voll, *Zeta functions of groups and rings - recent developments*, in C. M. Campbell, M. R. Quick, E. F. Robertson, C. M. Roney-Dougal (eds), Groups St Andrews 2013, London Mathematical Society Lecture Note Series 422.
http://arxiv.org/abs/1409.7955

[2] C. Voll, *A newcomer's guide to zeta functions of groups and rings*, in D. Segal (ed), Lectures on Profinite Topics in Group Theory, London Mathematical Society Student Texts 77.
http://arxiv.org/abs/0906.1832

[3] C. Voll, *Local functional equations for submodule zeta functions associated to nilpotent algebras of endomorphisms*, preprint 2016.
http://arxiv.org/abs/1602.07025

Research talks

Random relations and zeta functions of profinite groups

Matteo Vannacci (Universität Düsseldorf)

We introduce the concept of "positive finitely related" (PFR) profinite group, simply put: a profinite group G is PFR if, for some integer k, there is a positive probability that a randomly picked k-tuple of relations for G (normally) generates all relations. This is a natural generalisation of the concept of positively finitely generated groups, see [Man]. In this talk we will show that a profinite group G is PFR if and only if the growth rate of minimal normal extensions of a given size is polynomial. Even more interestingly, we show that a profinite group is PFR if and only if it has at most exponential subgroup growth and "few"representations over the field with p elements for every prime p. Finally we will discuss the possibility of defining a "PFR zeta function"for PFR groups. This is joint work with S. Kionke.

[Man] Mann, Avinoam. Positively finitely generated groups. Forum Math. 8 (1996), no. 4, 429–459.

Orbit Dirichlet series

Angela Carnevale (Universität Bielefeld)

We study Dirichlet series enumerating orbits of products of maps whose orbit distributions are modelled on the distributions of finite index subgroups of free abelian groups. We interpret Euler factors of such Dirichlet series in terms of generating polynomials for statistics on multiset permutations. As applications, we establish local functional equations, determine the (global) abscissae of convergence and exhibit natural boundaries. This is joint work with Christopher Voll.

[1] A. Carnevale, C. Voll, *Orbit Dirichlet series and multiset permutations.*
http://arxiv.org/abs/1607.01292.

Motivic zeta functions and splicing

Thomas Cauwbergs (Universität Bielefeld)

The motivic zeta function forms a generalization of the p-adic and topological zeta function. It provides more information about the polynomial, to which it is associated. We will focus on the case of plane curve singularities. A lot of information on a plane curve singularity is contained in the so-called splice diagram. This is a graph associated to this polynomial and comes with supplementary information. By incorporating a differential form into the splice diagram, Némethi and Veys proved a splicing formula connecting the topological zeta function of subdiagrams of this splice diagram. An interesting question is then what happens if we look at these more general zeta functions such as the motivic zeta function and the monodromic motivic zeta function. I discuss these (splice) diagrams and give another proof of the splicing formula. The advantage of this proof is that it also is valid for these other zeta functions. However I will also discuss some problems arising from considering these other zeta functions.

[1] T. Cauwbergs, *Splicing motivic zeta functions*, Revista Matemática Complutense, 29 (2016), no.2, 455–483.

Representation growth of $\mathrm{GL}_2(O)$ and $\mathrm{SL}_2(O)$ in residue characteristic 2

Alexander Stasinski (Durham University)

For a group G, let $R_N(G)$ denote the number of irreducible representations of G of dimension at most N (if finite). By representation growth, we mean the growth of the sequence $R_N(G)$, $N = 1, 2, ...$ If the representation growth of G is at most polynomial in N, the growth rate is given by the abscissa of convergence of the representation zeta function of G. An interesting family of groups to study is $\mathrm{SL}_n(O)$, where O is the ring of integers in a local field with finite residue field k. When the characteristic $char(O)$ is zero and $char(k)$ is large enough, we know the abscissa of convergence for SL_2, SL_3 and SL_4, by work of Jaikin-Zapirain, Avni–Klopsch–Onn–Voll, and Zordan. For SL_2 we actually know everything as long as $char(k)$ is not 2. An annoying gap in our knowledge is SL_2, where $char(k) = 2$. This deceptively simple group has a very rich representation theory, and when $char(O) = 2$ it has not even been established that it has polynomial representation growth (because this case is not covered by results of Martin and Lubotzky). We will present a part of joint work with Jokke Häsä, in which we show that the abscissa of $\mathrm{SL}_2(O)$ is 1 if $char(O) = 0$ and $char(k) = 2$, and describe how to obtain bounds on the abscissa of $\mathrm{SL}_2(k[[t]])$, where k has an even number of elements. In the latter case, there is no reason to expect that the abscissa is 1, although this is still work in progress. We will also discuss the twist representation zeta function of $\mathrm{GL}_n(O)$, explain that it is well-defined and that its abscissa often, but possibly not always, equals that of the zeta function of $\mathrm{SL}_n(O)$. In particular, we show that the abscissa of the twist zeta function of $\mathrm{GL}_2(O)$ is always 1.

The Congruence Subgroup Property and representation growth

Javier García Rodríguez (Universidad Autonoma de Madrid-ICMAT)

Consider the group $\mathrm{SL}_n(\mathbb{Z})$. For every integer $m \in \mathbb{Z}$ we have a homomorphism $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}/m\mathbb{Z})$. The kernel of such an homomorphism is called the principal congruence subgroup of level m. Clearly, any such subgroup has finite index in $\mathrm{SL}_n(\mathbb{Z})$. The Congruence Subgroup Problem for $\mathrm{SL}_n(\mathbb{Z})$ asks whether every finite index subgroup of $\mathrm{SL}_n(\mathbb{Z})$ contains some congruence subgroup.

We will present the Congruence Subgroup Problem in full generality, namely, if k is a global field and $\mathcal{O}_S$ its ring of integers with respect to a finite set of places S, does every finite index subgroup of $G(\mathcal{O}_S)$ contain a group of the form $G(\mathfrak{p})$ for some ideal $\mathfrak{p} \subset \mathcal{O}_S$? If the answer is positive, we say that $G(\mathcal{O}_S)$ has the Congruence Subgroup Property (CSP).

Lubotzky and Martin showed in [1] that for a semisimple group G and char $k > 2$ if $G(\mathcal{O}_S)$ has the Congruence Subgroup Property, then $G(\mathcal{O}_S)$ has Polynomial Representation Growth, i.e., the number of irreducible n-dimensional representations of $G(\mathcal{O}_S)$ is bounded by a polynomial in n.

Using a new approach we show that this result holds for every global field k. Moreover, we explain how to use this to obtain the corresponding result for the subgroup growth of the groups $G(\mathcal{O}_S)$.

[1] A. Lubotzky and B. Martin,, *Polynomial representation growth and the Congruence Subgroup Problem*, Israel Journal of Mathematics **144** (2004), 293-316.

On the behavior of pro-isomorphic zeta functions under base extension

Michael Schein (Bar-Ilan University)

The pro-isomorphic zeta function of a nilpotent group is a Dirichlet series the finite-index subgroups whose profinite completion is isomorphic to that of the original group. These functions were first investigated by Grunewald, Segal, and Smith, and later by du Sautoy and Lubotzky, among others. It is well-known that their computation is equivalent to the computation of some p-adic integrals over algebraic groups; integrals of this type have been studied extensively since the 1960's.

This talk will discuss joint work with Mark Berman on the behavior of pro-isomorphic zeta functions under base extension; for a group scheme G defined over $\mathbb{Z}$, we study the pro-isomorphic zeta functions of $G(\mathcal{O})$, as $\mathcal{O}$ varies over rings of integers of number fields. We consider analytic properties such as functional equations satisfied by the local zeta factors and abscissae of convergence.

The key tool is an extension of a theorem of D. Segal controlling the group of k-automorphisms of $L \otimes_k K$ in certain situations, where L is a k-Lie algebra and K/k is an extension of fields. In cases where these results hold, the corresponding pro-isomorphic zeta function has a fine Euler decomposition into factors parametrized by the prime ideals of $\mathcal{O}$, and not just over the rational primes. This property does not generally hold for normal subgroup zeta functions. Based on the local symmetry factors of examples that we have computed, we make general speculations.

Numerical computation of the Witten Zetafunction for the symmetric groups.

Jan-Christoph Schlage-Puchta (Universität Rostock)

For a finite group G, define $Z_G(s) = \sum \chi(1)^{-s}$, where the sum runs over all irreducible characters of G. If $G = S_n$ and $n \to \infty$, then the behaviour of $Z_n = Z_{S_n}$ is well understood. For every fixed $s > 0$ there exists an asymptotic series $Z_n(s) \approx 2 + \sum_{k\geq 1} a_k n^{-s}$, and $Z_n(\frac{t}{\log n})$ converges uniformly to $2\sum_{n\geq 1} p(n)n^{-t}$ in every half-plane $\Re s > \epsilon$, where $p(n)$ counts the partitions of n.

For applications one sometime needs estimates which are valid for all n. By enumerating all characters, Z_n can be computed up to $n \approx 120$, but the asymptotic estimates can be proven only for much larger n. To close this gap I describe an algorithm to approximate $Z_n(s)$ numerically. It turns out that the large values of n required in the asymptotic proofs are not due to wasteful techniques, but Z_n really converges only after increasing for a rather long time. For example, $Z_n(0.1)$ increases and reaches $Z_{60}(0.1) = 517$, and then decreases, but even $Z_{1000}(0.1) = 9.202 \pm 0.002$.

As application we give a quantitative version of Fenchel's conjecture: The minimal index of a torsion free subgroup of a fuchsian group Γ is $\delta_\Gamma m_\Gamma$, where m_Γ is the least common multiple of the orders of elliptic elements in Γ, and $\delta_\Gamma \in \{1, 2\}$ is a certain parity obstruction.

Computing zeta functions of groups, algebras, and modules

Tobias Roßmann (Universität Bielefeld)

I will describe practical methods for computing local and topological zeta functions associated with a range of algebraic counting problems. In addition to giving a number of examples of interest, I will also discuss conjectures which arose from extensive computer calculations.

[1] T. Roßmann, *Computing topological zeta functions of groups, algebras, and modules I*, Proc. London Math. Soc. (2015) 110 (5): 1099-1134.
http://dx.doi.org/10.1112/plms/pdv012

[2] T. Roßmann, *Computing topological zeta functions of groups, algebras, and modules, II*, J. Algebra 444 (2015), 567–605.
http://dx.doi.org/10.1016/j.jalgebra.2015.07.039

[3] T. Roßmann, *Topological representation zeta functions of unipotent groups*, J. Algebra 448 (2016), 210–237.
http://dx.doi.org/10.1016/j.jalgebra.2015.09.050

[4] T. Roßmann, *Computing local zeta functions of groups, algebras, and modules.*
http://arxiv.org/abs/1602.00919

Zeta functions and the monodromy conjecture

Wim Veys (KU Leuven)

The monodromy conjecture is a mysterious open problem in singularity theory. It relates arithmetic and topological/geometric properties of a multivariate polynomial over the integers. The case of interest is when the zero set of the polynomial has singular points. We will present some history, motivation, and partial results.