Computing zeta functions of groups, algebras, and modules

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Conditions for

$$\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ & x_5 & x_6 & x_7 \\ & & x_8 & x_9 \\ & & & x_{10} \end{bmatrix}:$$

$$V_p\big(\mathrm{U}_4(\mathbf{Z}_p) \curvearrowright \mathbf{Z}_p^4\big) = \Big\{ \boldsymbol{x} : x_{10} \mid x_6,\ x_{10} \mid x_8,\ x_5 \mid x_1,\ x_8 \mid x_5,\ x_{10} \mid x_3,\ x_8 \mid x_2,$$
$$x_5 x_8 \mid x_1 x_6,\ x_8 x_{10} \mid x_2 x_9,\ x_8 x_{10} \mid x_5 x_9,$$
$$x_5 x_8 x_{10} \mid x_1 x_7 x_8 - x_1 x_6 x_9 \Big\}$$

Recall: $\zeta_p(s) = \frac{1}{1-p^{-s}}$

Proposition

$$\begin{aligned}
\zeta_{\mathrm{U}_4(\mathbf{Z}_p)\curvearrowright\mathbf{Z}_p^4}(s) &= F_4(p, p^{-s}) \times \zeta_p(s)\zeta_p(2s-1)\zeta_p(3s-1)\zeta_p(4s-1)\zeta_p(4s-2) \\
&\quad \times \zeta_p(5s-2)\zeta_p(6s-2)\zeta_p(7s-3)\zeta_p(8s-4), \textit{ where} \\
F_4(X,Y) =& - X^{10}Y^{30} + X^9Y^{26} + X^9Y^{25} + X^9Y^{24} - X^9Y^{23} + 2X^8Y^{23} \\
&- X^8Y^{22} + 2X^7Y^{22} - 2X^7Y^{21} - 2X^7Y^{20} + X^6Y^{21} - 2X^7Y^{19} \\
&+ X^6Y^{20} - X^6Y^{18} - X^6Y^{17} - X^5Y^{18} - X^5Y^{17} + 2X^6Y^{15} \\
&- X^5Y^{16} + X^5Y^{14} - 2X^4Y^{15} + X^5Y^{13} + X^5Y^{12} + X^4Y^{13} \\
&+ X^4Y^{12} - X^4Y^{10} + 2X^3Y^{11} - X^4Y^9 + 2X^3Y^{10} + 2X^3Y^9 \\
&- 2X^3Y^8 + X^2Y^8 - 2X^2Y^7 + XY^7 - XY^6 - XY^5 - XY^4 + 1
\end{aligned}$$

Submodules: unipotent groups

$$
\begin{aligned}
V_p\big(\mathrm{U}_5(\mathbf{Z}_p) \curvearrowright \mathbf{Z}_p^5\big) = \Big\{ \boldsymbol{x} : x_{10}x_{13}x_{15} &\mid x_2x_{12}x_{13} - x_2x_{11}x_{14}, \\
x_{10}x_{13}x_{15} &\mid x_6x_{12}x_{13} - x_6x_{11}x_{14}, \\
x_6x_{10}x_{13} &\mid x_1x_8x_{10} - x_1x_7x_{11}, \\
x_6x_{10}x_{13}x_{15} &\mid x_1x_9x_{10}x_{13} - x_1x_7x_{12}x_{13} \\
&\qquad - x_1x_8x_{10}x_{14} + x_1x_7x_{11}x_{14}, \\
&+16 \text{ monomial conditions} \Big\}
\end{aligned}
$$

Submodules: unipotent groups

Theorem (R. 16)

Let $p \gg 0$. *Then:*

$$\begin{aligned}\zeta_{\mathrm{U}_5(\mathbf{Z}_p)\curvearrowright\mathbf{Z}_p^5}(s) = F_{\mathrm{U}_5}(p,p^{-s}) \times \zeta_p(13-6)\zeta_p(12s-6)\zeta_p(11s-4)\\ \times \zeta_p(10s-4)\zeta_p(10s-3)\zeta_p(9s-4)\zeta_p(9s-3)\\ \times \zeta_p(8s-4)\zeta_p(8s-3)\zeta_p(8s-2)\zeta_p(7s-3)\\ \times \zeta_p(7s-2)\zeta_p(6s-2)\zeta_p(5s-2)\zeta_p(5s-1)\\ \times \zeta_p(4s-2)\zeta_p(4s-1)\zeta_p(2s-1)\zeta_p(s),\end{aligned}$$

where $F_{\mathrm{U}_5} = 1 + \cdots (792 \textit{ terms}) \cdots + X^{43}Y^{124}$.

Similar calculations: subalgebras of $\mathfrak{gl}_2(\mathbf{Z}_p)$, $\mathbf{Z}_p[X]/X^4$

$\mathfrak{sl}_2$ and $\mathfrak{gl}_2$

Theorem (Ilani '99; du Sautoy '00; White '00; du Sautoy & Taylor '00)

Let $p \neq 2$. *Then* $\zeta^{\leqslant}_{\mathfrak{sl}_2(\mathbf{Z}_p)}(s) = W(p, p^{-s})$, *where*

$$W(X, Y) = \frac{1 - XY^3}{(1 - X^2Y^2)(1 - XY^2)(1 - XY)(1 - Y)}.$$

Theorem

Let $p \gg 0$. *Then* $\zeta^{\leqslant}_{\mathfrak{gl}_2(\mathbf{Z}_p)}(s) = W(p, p^{-s})$, *where*

$$\begin{aligned} W(X, Y) = \big(&-X^8Y^{10} - X^8Y^9 - X^7Y^9 - 2X^7Y^8 + X^7Y^7 - X^6Y^8 - X^6Y^7 \\ &+ 2X^6Y^6 - 2X^5Y^7 + 2X^5Y^5 - 3X^4Y^6 + 3X^4Y^4 - 2X^3Y^5 \\ &+ 2X^3Y^3 - 2X^2Y^4 + X^2Y^3 + X^2Y^2 - XY^3 + 2XY^2 + XY \\ &+ Y + 1\big)/\big((1 - X^7Y^6)(1 - X^3Y^3)(1 - X^2Y^2)^2(1 - Y)\big). \end{aligned}$$

Strategy

Goal ("uniformity problem")

Let $\Omega \subset \mathrm{M}_n(\mathbf{Z})$.
If possible, find $W(X, Y)$ s.t. $\zeta_{\Omega \curvearrowright \mathbf{Z}_p^n}(s) = W(p, p^{-s})$ for $p \gg 0$.

- Previous computations (ad hoc, partially manual):
  Taylor '01, Woodward '05, …
- Here: fully automated but restricted by genericity assumptions
- Software: Zeta (R. '14–present)
  `http://www.math.uni-bielefeld.de/~rossmann/Zeta`

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If possible, find $W(X, Y)$ s.t. $\zeta_{\Omega \curvearrowright \mathbf{Z}_p^n}(s) = W(p, p^{-s})$ for $p \gg 0$.

- Compute a "Denef formula"

$$\zeta_{\Omega \curvearrowright \mathbf{Z}_p^n}(s) = \sum_i \#\bar{V}_i(\mathbf{F}_p) \cdot W_i(p, p^{-s}).$$

  - Theory (du Sautoy, Grunewald '00): resolution of singularities
  - Here: combine "toric methods" and computational algebra
- Symbolically compute $\#\bar{V}_i(\mathbf{F}_p)$ as a polynomial in p.
- Compute each W_i as a sum of rational functions.
  $\rightsquigarrow$ Algorithms of Barvinok et al.
- Final summation.

First task

Compute a Denef formula for

$$\zeta_{\Omega \curvearrowright \mathbf{Z}_p^n}(s) = (1 - p^{-1})^{-n} \int_{V_p} |x_{11}|_p^{s-1} \cdots |x_{nn}|_p^{s-n} \, \mathrm{d}\mu(\boldsymbol{x}),$$

where $V_p = \left\{ \boldsymbol{x} = [x_{ij}] \in \mathrm{Tr}_n(\mathbf{Z}_p) \;\middle|\; x_{11} \cdots x_{nn} \mid f_1(\boldsymbol{x}), \dots, f_r(\boldsymbol{x}) \right\}$.

- dSG'00: "explicit" (but impractical) computation in terms of a resolution of singularities of $x_{11} \cdots x_{nn} \cdot f_1(\boldsymbol{x}) \cdots f_r(\boldsymbol{x}) = 0$.
- Here: combine "toric resolutions" for sufficiently generic f_i (Khovanskii et al., '70s) and techniques inspired by Gröbner bases.
- Inspired by "toric formulae" for Igusa's local zeta function (Denef et al. '92–01; Veys, Zúñiga-Galindo '08).

Definition

The **Newton polytope** $\mathrm{New}(f)$ of $f = \sum a_e X^e$: convex hull of $\{e : a_e \neq 0\}$.

Fact

Faces $\tau \subset \mathrm{New}(f_1 \cdots f_r)$ define canonical sub-polynomials $f_{i,\tau}$ of the f_i.

Write $\boldsymbol{f} = (f_1, \dots, f_r)$. For $J \subset \{1, \dots, r\}$, write $\boldsymbol{f}_{J,\tau} = \left(f_{j,\tau}\right)_{j \in J}$.

Definition

$\boldsymbol{f}$ is **non-degenerate** (w.r.t. $\mathrm{New}(f_1 \cdots f_r)$) if

$$\boldsymbol{f}_{J,\tau}(\boldsymbol{x}) = 0 \implies \mathrm{rk}(\boldsymbol{f}'_{J,\tau}(\boldsymbol{x})) = \#J$$

for all faces $\tau \subset \mathrm{New}(f_1 \cdots f_r)$, subsets $J \subset \{1, \dots, r\}$ and $\boldsymbol{x} \in (\mathbf{C}^\times)^d$.

Computing p-adic integrals

Theorem (R. ’15)

Suppose $\boldsymbol{f}$ is non-degenerate. Then there are explicit $W_{\tau,J} \in \mathbf{Q}(X,Y)$ indexed by faces $\tau \subset \mathrm{New}(f_1 \cdots f_r)$ and subsets $J \subset \{1,\dots,r\}$ s.t.

$$\zeta_{\Omega \curvearrowright \mathbf{Z}_p^n}(s) = \sum_{\tau,J} c_{\tau,J}(p) W_{\tau,J}(p, p^{-s})$$

for almost all p, where $c_{\tau,J}(p) = \#\left\{ u \in (\mathbf{F}_p^\times)^d \,\middle|\, f_{j,\tau}(u) = 0 \iff j \in J \right\}$.

Heuristic observations

- Typical forms of degeneracy can often be fixed using a “toric reduction” process inspired by Gröbner bases machinery.
- $c_{\tau,J}(p)$ can often be computed by combining three steps:
  - split off torus factors
  - solve for variables
  - reduction (again!)

Computing p-adic integrals 4/4

- $\zeta_{\mathfrak{sl}_2(\mathbf{Z}_p)}(s) = (1-p^{-1})^{-3} \int\limits_{V_p} |a|^{s-1}|x|^{s-2}|z|^{s-3}\,\mathrm{d}\mu(a,\dots,z)$, where

$$V_p = \left\{ \begin{bmatrix} a & b & c \\ & x & y \\ & & z \end{bmatrix} \in \mathrm{Tr}_3(\mathbf{Z}_p) : xz \;\Big|\; ax^2 + 4bxy - 4cy^2, cy, cz \right\}.$$

- Newton polytope = $\triangle$, 7 faces.
- One easily confirms non-degeneracy.
- Zeta computes $\zeta_{\mathfrak{sl}_2(\mathbf{Z}_p)}(s)$ as a sum of 414 rational functions.

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Behaviour at zero

"Semi-simplification conjecture"

Let $\mathcal{A} \subset \mathrm{M}_n(\mathbf{Q})$ be a unital associative subalgebra. Then, for $p \gg 0$,

$$\left.\frac{\zeta_{\mathcal{A} \curvearrowright \mathbf{Z}_p^n}(s)}{\zeta_{\mathcal{A}/\mathrm{rad}(\mathcal{A}) \curvearrowright \mathbf{Z}_p^n}(s)}\right|_{s=0} = 1.$$

- Solomon '77: explicit formula for $\zeta_{\mathcal{A}/\mathrm{rad}(\mathcal{A}) \curvearrowright \mathbf{Z}_p^n}(s)$
- Confirmed for numerous specific examples and $\mathcal{A} = \mathbf{Q}[A]$ (R. '16).

Topological zeta functions

"Definition"

$$\zeta_{\Omega \curvearrowright \mathbf{Z}^n, \mathrm{top}}(s) = \lim_{p \to 1} \zeta_{\Omega \curvearrowright \mathbf{Z}_p^n}(s).$$

- Arithmetic approach (Denef, Loeser '92):

$$(1 - p^{-1})^n \zeta_{\Omega \curvearrowright \mathbf{Z}_p^n}(s) = \sum_i \#\overline{V}_i(\mathbf{F}_p) \cdot W_i(p, p^{-s})$$

$$\downarrow$$

$$\zeta_{\Omega \curvearrowright \mathbf{Z}^n, \mathrm{top}}(s) = \sum_i \chi(V_i(\mathbf{C})) \cdot \lim_{p \to 1} W_i(p, p^{-s}) \in \mathbf{Q}(s),$$

where powers p^z are expanded binomially via

$$p^z = \sum_{k=0}^{\infty} \binom{z}{k} (p-1)^k.$$

- Geometric approach: motivic integration (Denef, Loeser '98; du Sautoy, Loeser '04).

- Informally:
  "$\zeta_{\Omega \curvearrowright \mathbf{Z}^n, \mathrm{top}}(s) =$ constant term of $(1-p^{-1})^n \zeta_{\Omega \curvearrowright \mathbf{Z}_p^n}(s)$ in $p-1$"

Example

$$\zeta_{\mathbf{Z}_p^n}(s) = \zeta_p(s)\zeta_p(s-1)\cdots\zeta_p(s-(n-1))$$

$$\downarrow$$

$$\zeta_{\mathbf{Z}^n,\mathrm{top}}(s) = \frac{1}{s(s-1)\cdots(s-(n-1))}$$

Topological zeta functions 3/3

Topological zeta functions ...

- preserve analytic properties of their local ancestors
- can be studied and computed more easily
  Example:
  - $\chi(V_i(\mathbf{C}))$ via Bernstein-Khovanskii-Kushnirenko '70s
  - "half-open cones" become "cones"
  - final summation via polynomial interpolation
- exhibit interesting new features

Behaviour at infinity

"Degree conjecture"

Let $\mathcal{A} \subset \mathrm{M}_n(\mathbf{Q})$ be a nilpotent subalgebra. Then

$$\deg(\zeta_{\mathcal{A} \curvearrowright \mathbf{Z}^n,\mathrm{top}}(s)) = -n.$$

Example

Recall: $\zeta_{\mathrm{U}_3(\mathbf{Z}) \curvearrowright \mathbf{Z}^3}(s) = \frac{\zeta(s)\zeta(2s-1)\zeta(3s-1)\zeta(4s-2)}{\zeta(4s-1)}$.
Thus,

$$\zeta_{\mathrm{U}_3(\mathbf{Z}),\mathrm{top}}(s) = \frac{4s-1}{s(2s-1)(3s-1)(4s-2)}$$

has degree -3.

Behaviour at infinity

"Degree conjecture"

Let $\mathcal{A} \subset \mathrm{M}_n(\mathbf{Q})$ be a nilpotent subalgebra. Then

$$\deg(\zeta_{\mathcal{A} \curvearrowright \mathbf{Z}^n, \mathrm{top}}(s)) = -n.$$

Example

$$\zeta_{\mathrm{U}_4(\mathbf{Z}) \curvearrowright \mathbf{Z}^4, \mathrm{top}}(s) = \frac{3360s^5 - 5192s^4 + 3139s^3 - 930s^2 + 136s - 8}{8(7s-3)(5s-2)(4s-1)(3s-1)^2(2s-1)^3 s}$$

$$\zeta_{\mathrm{U}_5(\mathbf{Z}) \curvearrowright \mathbf{Z}^5, \mathrm{top}}(s) = \frac{12108096000s^{14} - 54378038400s^{13} \pm \cdots\cdots - 40104s + 864}{(13s-6)(11s-4)(10s-3)(9s-4)(8s-3)(7s-2)(7s-3)(5s-1)(5s-2)^2(4s-1)^2(3s-1)^2(2s-1)^4 s}$$

The End