Orbit Dirichlet series and multiset permutations

Angela Carnevale

Universität Bielefeld

(joint work with C. Voll)

Orbit Dirichlet series

Let X be a space and $T : X \to X$ a map. For $n \in \mathbb{N}$

$$\{x, T(x), T^2(x), \dots, T^n(x) = x\} = \text{closed orbit of length } n$$

$$\mathsf{O}_T(n) = \text{number of closed orbits of length } n \text{ under } T.$$

The **orbit Dirichlet series** of T is the Dirichlet generating series

$$\mathsf{d}_T(s) = \sum_{n=1}^{\infty} \mathsf{O}_T(n) n^{-s},$$

where s is a complex variable.

- If $\mathsf{O}_T(n) = 1$ for all $n \rightsquigarrow \mathsf{d}_T(s) = \zeta(s)$
- For $r \in \mathbb{N}$, if $\mathsf{O}_{T_r}(n) = a_n(\mathbb{Z}^r) = \text{number of index } n \text{ subgroups of } \mathbb{Z}^r$

  $\rightsquigarrow \mathsf{d}_{T_r}(s) = \prod_{i=0}^{r-1} \zeta(s-i)$

Products and periodic points

$n \mapsto \mathsf{O}_T(n)$ is multiplicative $\rightsquigarrow$ Orbit Dirichlet series satisfy an Euler product

$$\mathsf{d}_T(s) = \prod_{p \text{ prime}} \mathsf{d}_{T,p}(s) = \prod_{p \text{ prime}} \sum_{k=0}^{\infty} \mathsf{O}_T(p^k) p^{-ks}$$

To find the orbit series of a product of maps, we first look at another sequence:

$$\mathsf{F}_T(n) = \text{number of points of period } n = \sum_{d|n} d\mathsf{O}_T(d)$$

Möbius inversion gives

$$\mathsf{O}_T(n) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) \mathsf{F}_T(d)$$

For any finite collection of maps $T_1, \ldots, T_r$

$$\mathsf{F}_{T_1 \times \ldots \times T_r}(n) = \mathsf{F}_{T_1}(n) \cdots \mathsf{F}_{T_r}(n)$$

Orbit series of products of maps

Goal. For a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$, compute

$$\mathsf{d}_{T_\lambda}(s) = \mathsf{d}_{T_{\lambda_1} \times \cdots \times T_{\lambda_m}}(s) = \prod_{p \text{ prime}} \mathsf{d}_{T_\lambda, p}(s),$$

where $\mathsf{O}_{T_{\lambda_i}}(n)$ =number of index n subgroups of $\mathbb{Z}^{\lambda_i}$.
For $i = 1, \ldots, m$

$$\mathsf{O}_{T_{\lambda_i}}(p^k) = \binom{\lambda_i - 1 + k}{k}_p \quad \text{and} \quad \mathsf{F}_{T_{\lambda_i}}(p^k) = \binom{\lambda_i + k}{k}_p$$

$$\rightsquigarrow \mathsf{d}_{T_\lambda}(s) = \prod_p \sum_{k=0}^{\infty} \left(\prod_{i=1}^{m} \binom{\lambda_i + k}{k}_p \right) p^{-k-ks}.$$

Multiset permutations

Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a partition of $N = \sum_{i=1}^{m} \lambda_i$.

$$S_\lambda = \text{ set of all multiset permutations on } \{\underbrace{\mathbf{1}, \dots, \mathbf{1}}_{\lambda_1}, \underbrace{\mathbf{2}, \dots, \mathbf{2}}_{\lambda_2}, \dots, \underbrace{\mathbf{m}, \dots, \mathbf{m}}_{\lambda_m}\}.$$

- $\lambda = (1, \dots, 1) = (1^m) \rightsquigarrow S_m =$ permutations of the set $\{\mathbf{1}, \mathbf{2}, \dots, \mathbf{m}\}$,
- $\lambda = (2, 1) \rightsquigarrow S_\lambda = \{\mathbf{112}, \mathbf{121}, \mathbf{211}\}$

For $w \in S_\lambda$, $w = w_1 \dots w_N$

$$\mathrm{Des}(w) = \{i \in [N-1] \mid w_i > w_{i+1}\}, \quad \text{descent set of } w$$

$$\mathrm{des}(w) = |\,\mathrm{Des}(w)|, \quad \text{number of descents}$$

$$\mathrm{maj}(w) = \sum_{i \in \mathrm{Des}(w)} i, \quad \text{major index}$$

- $\lambda = (3, 2, 1)$, $w = \mathbf{121231} \in S_\lambda \rightsquigarrow \mathrm{Des}(w) = \{2, 5\}$, $\mathrm{des}(w) = 2$ and $\mathrm{maj}(w) = 7$.

Euler-Mahonian distribution and orbit series

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of $N = \sum \lambda_i$

$$C_\lambda(x, q) = \sum_{w \in S_\lambda} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)} \in \mathbb{Z}[x, q]$$

Theorem (MacMahon 1916)

$$\sum_{k=0}^{\infty} \left(\prod_{i=1}^{m} \binom{\lambda_i + k}{k}_q \right) x^k = \frac{C_\lambda(x, q)}{\prod_{i=0}^{N} (1 - xq^i)}.$$

Theorem (C.-Voll 2016)

$$\mathsf{d}_{T_\lambda}(s) = \prod_{p \text{ prime}} \frac{C_\lambda(p^{-1-s}, p)}{\prod_{i=1}^{N} (1 - p^{i-1-s})} = \prod_{p \text{ prime}} \frac{\sum_{w \in S_\lambda} p^{(-1-s)\,\mathrm{des}(w) + \mathrm{maj}(w)}}{\prod_{i=1}^{N} (1 - p^{i-1-s})}.$$

Example: $\lambda = (1^m)$

$$S_{(1^m)} = S_m = \text{ symmetric group on } n \text{ letters},$$

$$C_{(1^m)}(x,q) = \text{ Carlitz's } q\text{-Eulerian polynomial},$$

$$\begin{aligned} \mathsf{d}_{T_{(1^m)},p}(s) &= \frac{C_{(1^m)}(p^{-1-s},p)}{\prod_{i=1}^m(1-p^{i-1-s})} = \frac{\sum_{w\in S_m}\prod_{j\in \mathrm{Des}(w)} p^{j-1-s}}{\prod_{i=1}^m(1-p^{i-1-s})} \\ &= \frac{1}{1-p^{m-1-s}} \sum_{I\subseteq[m-1]} \binom{m}{I} \prod_{i\in I} \frac{p^{i-1-s}}{1-p^{i-1-s}}. \end{aligned}$$

Is an istance of an "**Igusa function**"

$$\rightsquigarrow \mathsf{d}_{T_{(1^m)},p}(s)|_{p\to p^{-1}} = (-1)^m p^{m-1-s}\mathsf{d}_{T_{(1^m)},p}(s).$$

Local functional equations

$\lambda = (\lambda_1, \ldots, \lambda_m)$ is a **rectangle** if $\lambda_1 = \cdots = \lambda_m$.

Theorem (C.-Voll)

1. Let p be a prime. For all $r, m \in \mathbb{N}$,

$$\mathsf{d}_{T_{(r^m)},p}(s)|_{p\to p^{-1}} = (-1)^{rm} p^{m\binom{r+1}{2}-r-rs} \mathsf{d}_{T_{(r^m)},p}(s).$$

2. If λ is not a rectangle, then $\mathsf{d}_{T_\lambda,p}(s)$ does not satisfy a functional equation of the form

$$\mathsf{d}_{T_\lambda,p}(s)|_{p\to p^{-1}} = \pm p^{d_1-d_2 s} \mathsf{d}_{T_\lambda,p}(s)$$

for $d_1, d_2 \in \mathbb{N}_0$.

Proof

1. Symmetry of $C_{(r^m)}(x, q)$ + involution on $S_{(r^m)}$
2. $C_\lambda(x, 1)$ has constant term 1. It is monic if and only if λ is a rectangle.

Abscissae of convergence and growth

Fact. For an Euler product

$$\prod_p W(p, p^{-s}) = \prod_p \sum_{(k,j)\in I} c_{kj} p^{k-js}, c_{kj} \neq 0$$

- $\alpha =$ abscissa of convergence $= \max\left\{\frac{a+1}{b} \mid (a,b) \in I\right\}$
- Meromorphic continuation to $\{\mathrm{Re}(s) > \beta\}$, $\beta = \max\left\{\frac{a}{b} \mid (a,b) \in I\right\}$

Theorem (C.-Voll)

$\lambda = (\lambda_1, \ldots, \lambda_m)$, $N = \sum_i \lambda_i$

1. $\alpha_\lambda =$ abs. of conv. of $\mathsf{d}_{T_\lambda}(s) = N$, meromorphic continuation to $\{\mathrm{Re}(s) > N - 2\}$
2. There exists a constant $K_\lambda \in \mathbb{R}_{>0}$ such that

$$\sum_{\nu \leqslant n} \mathsf{O}_{T_\lambda}(\nu) \sim K_\lambda n^N \quad \text{as } n \to \infty.$$

Abscissae of convergence and growth

In our case

$$\prod_p C_\lambda(p^{-1-s}, p) = \prod_p \sum_{(k,j)\in I_\lambda} c_{kj} p^{k-js} = \prod_p \sum_{w\in S_\lambda} p^{\mathrm{maj}(w)-(1+s)\,\mathrm{des}(w)}$$

- $\alpha = \max\left\{ \frac{\mathrm{maj}(w)-\mathrm{des}(w)+1}{\mathrm{des}(w)} \mid w \in S_\lambda \right\}$
- $\beta = \max\left\{ \frac{\mathrm{maj}(w)-\mathrm{des}(w)}{\mathrm{des}(w)} \mid w \in S_\lambda \right\}$

Theorem (C.-Voll)

$\lambda = (\lambda_1, \dots, \lambda_m)$, $N = \sum_i \lambda_i$

1. $\alpha_\lambda =$ abs. of conv. of $\mathsf{d}_{T_\lambda}(s) = N$, meromorphic continuation to $\{\mathrm{Re}(s) > N-2\}$
2. There exists a constant $K_\lambda \in \mathbb{R}_{>0}$ such that

$$\sum_{\nu \leqslant n} \mathsf{O}_{T_\lambda}(\nu) \sim K_\lambda n^N \quad \text{as } n \to \infty.$$

Abscissae of convergence and growth

In our case

$$\prod_p C_\lambda(p^{-1-s},p) = \prod_p \sum_{(k,j)\in I_\lambda} c_{kj} p^{k-js} = \prod_p \sum_{w\in S_\lambda} p^{\mathrm{maj}(w)-(1+s)\,\mathrm{des}(w)}$$

- $\alpha = \max\left\{ \frac{\mathrm{maj}(w)-\mathrm{des}(w)+1}{\mathrm{des}(w)} \mid w \in S_\lambda \right\} = N-1$
- $\beta = \max\left\{ \frac{\mathrm{maj}(w)-\mathrm{des}(w)}{\mathrm{des}(w)} \mid w \in S_\lambda \right\} = N-2$

Proof

$\lambda = (\lambda_1, \ldots, \lambda_m)$, $N = \sum_i \lambda_i$

1. $\alpha_\lambda = \max\left\{ N-1, \text{ abscissa of convergence of } \frac{1}{\prod_{i=1}^{N}(1-p^{i-1-s})} \right\} = N.$
2. There exists a constant $K_\lambda \in \mathbb{R}_{>0}$ such that

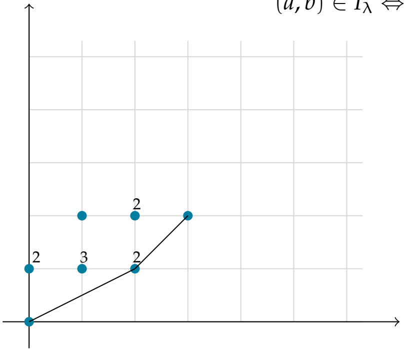
$$\sum_{\nu\leqslant n} \mathsf{O}_{T_\lambda}(\nu) \sim K_\lambda n^N \quad \text{as } n \to \infty \quad (\text{Tauberian theorem}).$$

Natural boundaries: an example

$$\lambda = (2,1,1) \rightsquigarrow m = 3, N = 4, \beta = 2$$

$$C_\lambda(X,Y) = 1 + 2Y + 3XY + 2X^2Y + XY^2 + 2X^2Y^2 + X^3Y^2$$

$$(a,b) \in I_\lambda \Leftrightarrow \exists w \in S_\lambda \mid \mathrm{des}(w) = b \text{ and } \mathrm{maj}(w) = a + b$$



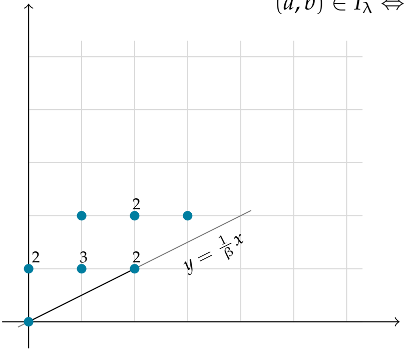
$\bullet = I_\lambda$

Natural boundaries: an example

$$\lambda = (2,1,1) \rightsquigarrow m = 3, N = 4, \beta = 2$$

$$C_\lambda(X,Y) = 1 + 2Y + 3XY + 2X^2Y + XY^2 + 2X^2Y^2 + X^3Y^2$$

$$(a,b) \in I_\lambda \Leftrightarrow \exists w \in S_\lambda \mid \mathrm{des}(w) = b \text{ and } \mathrm{maj}(w) = a + b$$



$\bullet = I_\lambda$

$$\widetilde{C_\lambda^1}(X,Y) = 1 + 2X^2Y, \text{ not "cyclotomic"}$$

$$\Downarrow$$

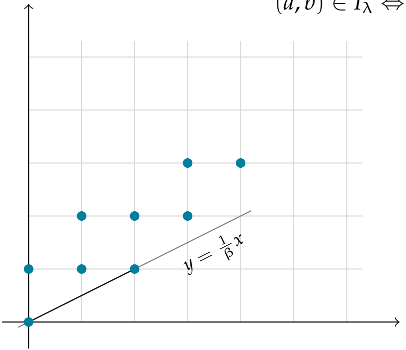
$$\mathrm{Re}(s) = \beta \text{ is a natural boundary}$$

Natural boundaries: an example

$$\lambda = (\lambda_1, \dots, \lambda_m) \rightsquigarrow N = \sum_i \lambda_i, \beta = N - 2$$

$$C_\lambda(X, Y) = \sum_{w \in S_\lambda} X^{\mathrm{maj}(w) - \mathrm{des}(w)} Y^{\mathrm{des}(w)}$$

$$(a, b) \in I_\lambda \Leftrightarrow \exists w \in S_\lambda \mid \mathrm{des}(w) = b \text{ and } \mathrm{maj}(w) = a + b$$



$\bullet = I_\lambda$

$$\widetilde{C^1_\lambda}(X, Y) = 1 + (\mathbf{m} - \mathbf{1}) X^\beta Y$$

$$\rightsquigarrow$$

$\mathrm{Re}(s) = \beta$ is a natural boundary

Natural boundaries

Theorem (C.-Voll)

Assume that $m > 2$. Then the orbit Dirichlet series $\mathsf{d}_{T_\lambda}(s)$ has a natural boundary at

$$\{\mathrm{Re}(s) = N - 2\}.$$

For $m = 2$ and $\lambda \neq (1,1)$ we conjecture that the same holds. We prove it subject to:

Conjecture 1 For $\lambda_1 > \lambda_2$

$$C_{(\lambda_1,\lambda_2)}(-1,1) = \sum_{i=0}^{\lambda_2} (-1)^i \binom{\lambda_1}{i} \binom{\lambda_2}{i} \neq 0$$

Conjecture 2 For $\lambda = (\lambda_1, \lambda_1)$, $\lambda_1 \equiv 1 \pmod{2}$

$$C_\lambda(x,q) = (1 + xq^{\lambda_1}) C'_\lambda(x,q),$$

where $C'_\lambda(-1,1) \neq 0$.