Advanced Seminar on Group Theory

Bass-Serre Theory

Iker de las Heras and Luca Di Gravina

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The goal of this seminar is to give an introduction to Bass-Serre theory, a branch of modern geometric group theory developed by Hyman Bass and Jean-Pierre Serre in the '70s. The idea behind their work is to study some structural properties of groups by exploiting their action on trees. The relation between groups and such geometrical objects turned out to be a fruitful research topic not only in abstract group theory, but also in the context of profinite groups.

The programme is divided in two parts. In the first part (Talks 1-6), we will follow Chapter I of Serre's original book *Trees* [2], a concise but accessible source that contains all the necessary basic notions and culminates with the main structure theorems about groups acting on trees. The second part of the programme (Talks 7-10) is devoted to applications and generalisations. On the one hand, following Chapter II of [2], we will see some applications to the structure of $\mathbf{SL_2}$ over local fields, which constituted the main motivation for developing the theory (Talks $7, 8$). On the other hand, some recent generalisations to profinite groups will be studied (Talks $9, 10$). The main source for this will be Sections 1-4 of [3].

Additionally, for Talks 1-8, the notes [1] might be helpful. For Talks 9 and 10, a more recent and extensive exposition can be found in [5].

Talk 1. Amalgams ([2] Chapter I, §1). Introduce the notion of amalgams (§1.1), which constitutes one of the main group-theoretical ingredients of Bass-Serre theory. A normal form for the elements of amalgams and some related properties are given in Theorem 1 and Theorem 2 (§1.2). Sketch their proofs and give some details about the consequences (§1.3). Put special emphasis on the construction of HNN extensions in Proposition 5 and its corollary in §1.4. During the talk, when necessary, mention some of the basic examples in the end of §1.5, as well as the example constructed in Proposition 6 (§1.4).

Talk 2. Trees and free groups ([2] Chapter I, §2-3). Give an overview of §2, by briefly recalling the definition of a graph (à la Serre) and its main properties. The aim is not to give all details, but to make clear the necessary topological environment for the next talks. Trees will play a central role. In particular, try to collect what is useful to understand the proofs of Proposition 15 in §3.2 (about the Cayley graph of a free group) and Theorem 4 (or Theorem 4′) in §3.3, where free groups are characterised as groups that act freely on a tree. As a consequence, we get a proof of the Nielsen–Schreier theorem and the related Schreier index formula (see Theorem 5 and its corollary in §3.4).

Talk 3. Trees and amalgams ([2] Chapter I, §3-4). Give some more details about §3.1 and the proof of Proposition 14. Then start with §4.1 and introduce the notion of a fundamental domain for a graph under the action without inversion of a group: deduce a condition on the existence of such a fundamental domain when the graph is a tree (Proposition 17). Now consider a segment as a fundamental domain and obtain a characterisation of amalgams of two groups (Theorem 6 and Theorem 7). Give some examples and briefly discuss applications (see §4.2 and §4.3). Conclude by explaining that the case of amalgams of more than two factors is somehow analogous: define what a graph/tree of groups is, focus on the direct limit of a tree of groups, and sketch the proof of Theorem 9 and Theorem 10 (i.e., the generalisation of Theorem 6 and Theorem 7).

Talk 4. Structure of a group acting on a tree I ([2] Chapter I, §5.1-5.2). In Talk 3, graph of groups have been introduced. In this talk we will be concerned with the notion of the fundamental group of a graph of groups, which generalises both free groups and amalgams. Give the two definitions of this notion in §5.1 and compare them with the usual concept of a fundamental group. Provide some examples of graphs of groups and their fundamental groups, and state Proposition 20 and the remark after it. Define reduced words as in §5.2, sketch the proof of Theorem 11 (reduced words are not trivial), and give some of its consequences.

Talk 5. Structure of a group acting on a tree II ([2] Chapter I, §5.3-5.5). First, give an explicit construction of the universal covering of a graph of groups associated with a maximal tree, and state Theorem 12, according to which such a graph is always a tree (provide also the two examples in the end of §5.3). Use then this notion to state and prove Theorem 13 in §5.4, which provides the structure of a group G acting without inversion on a tree X. This generalises the cases where G acts freely on X and where the quotient graph $G\backslash X$ is a tree. Give the corollaries after it, and as an application, prove Kurosh's theorem in §5.5.

Talk 6. Amalgams and fixed points ([2] Chapter I, §6). Define property (FA), namely G acts without fixed points in any tree, and prove Theorem 15, about the fact that having property (FA) is "almost" equivalent to not being an amalgam (§6.1). Briefly mention some of the consequences in §6.2 and focus on the examples in §6.3. The results in §6.4 and §6.5 lead to the proof in §6.6 that $\mathrm{SL}_3(\mathbb{Z})$ has property (FA) and therefore it is not an amalgam (Theorem 16 and its corollary). Prove this theorem by sketching the results in the previous sections. Make use of pictures and diagrams in order to shorten the computations.

Talk 7. The tree of $\mathbf{SL}_2$ over a local field I ([2] Chapter II, §1.1-1.3). Introduce the main concepts that will allow us to clarify some connections between trees, amalgams, and $\mathbf{SL_2}$, as presented by Serre in Chapter II. Let V be a two-dimensional vector space over a local field K. Recall the definition of a lattice of V and what the class of a lattice is. By using a notion of distance between two classes, associate with the set of lattice classes a graph structure and show that the obtained graph is a tree (*the tree of* V), as stated in Theorem 1. Sketch the remaining part of §1.1. The subgroups of $\mathrm{GL}(V)$ introduced in §1.2 will be relevant in Talk 8. Define them and present their connection with the different types of stabilisers of the action of $\mathrm{GL}(V)$ on the tree of V, as in §1.3.

Talk 8. The tree of SL_2 over a local field II ([2] Chapter II, §1.4-1.6). By using the same notation and the preliminary results given in Talk 7, prove Theorem 2, stating that $\mathrm{SL}(V)$ acts on the tree of V with a segment as a fundamental domain. Combining this theorem with the characterisation of amalgams from Ch. I §4, obtain Theorem 3 and deduce that $\mathrm{SL}(V)$ is an amalgam (see Corollary 1, Corollary 2, and other examples in §1.4). If time permits, go through the presentations of $\mathrm{SL}_2(\mathbb{Z}[1/2])$ and $\mathrm{SL}_2(\mathbb{Z}[1/3])$. Prove the version of Ihara's theorem on subgroups of $\mathrm{GL}(V)$ given in §1.5. Then, either address the locally compact case at the end of §1.5 or the proof of Nagao's theorem on the decomposition of $\mathrm{GL}_2(k[t])$ as an amalgamated product, as in §1.6.

Talk 9. Profinite graphs and pro-p trees ([3] Sections 1,2). The first section of [3] concerns abstract and profinite graphs. Focus on the latter, introducing the concept (they are inverse limits of finite graphs) and giving the several properties and examples in the section (the notation in this book is slightly different from that in Serre's book, so clarify the differences with respect to the previous notation). Following Section 2, give an overview of the necessary theory of profinite modules and homology theory in order to define pro-p trees. Finish the talk giving the properties of pro-p trees in the end of the section.

Talk 10. Pro-p groups acting on pro-p trees ([3] Sections 3,4). The guiding idea for the definition of a pro-p tree is that the Cayley graph of a free pro-p group should be the prototype of such an object. Define the action of a pro-p group on a profinite graph and show that, indeed, the Cayley graph of a free pro-p group is a pro-p tree (Theorem 3.3). Go through some of the main theorems in Section 3, pointing out the similarities with the abstract case: for instance, a result of special interest is Theorem 3.18, where the structure of pro-p groups acting on pro-p trees is described. In Section 4, the profinite versions of amalgamated products and HNN extensions are introduced. Give the definitions of these constructions (Chapter 9 of [4] may also be useful) and present the interpretation of Theorem 3.18 for free products, free products with amalgamation and HNN-extensions as three separate theorems (Theorems 4.6, 4.7 and 4.8, respectively). Since the material covered in this talk is quite large, feel free to skip anything that is not strictly necessary.

References

[1] A. Raghuram and B. Sury. Groups acting on trees, Notes of a course given at the Indian Institute of Technology in Guwahati, India, December 2002. http://www.isibang.ac.in/ sury/tree.pdf

[2] J.-P. Serre. *Trees*, Springer Verlag, 1980.

[3] L. Ribes and P. Zalesskii. Pro-p trees and applications, in *New Horizons in pro-p groups*, Springer, 2000.

[4] L. Ribes and P. Zalesskii. *Profinite groups*, Springer, 2010.

[5] L. Ribes. *Profinite graphs and groups*. Springer, 2017.