Advanced Seminar on Group Theory
The Congruence Subgroup Problem

WS 2021-2022

The aim of this seminar is to give an elementary introduction to the congruence subgroup problem (CSP). We will mainly deal with the congruence subgroup problem for SL_n and with some group-theoretic applications of it, following [S]. At the end we will survey some results and techniques concerning the congruence subgroup problem for more general linear algebraic groups.

Other sources for learning about the subject are [H] and [PR].

TALK 1: The congruence subgroup problem for $\mathrm{SL}_2(\mathbb{Z})$

Introduce the problem (Sect. 3.1). Describe the structure of $\mathrm{SL}_2(\mathbb{Z})$ as presented in Sect. 3.3 (in particular Lemma 3-3.1). Prove that the congruence subgroup problem has a negative solution for $\mathrm{SL}_2(\mathbb{Z})$ (Sect. 3.4, Theorem 3-4.1) and give an example of a non-congruence subgroup of finite index (Example 3-4.3). If time permits, explain the criterion in Section 3.5.

Main source: [S], Chapter 3, Sections 3.1, 3.3, 3.4, 3.5

TALK 2: The congruence subgroup problem for $\mathrm{SL}_2(\mathcal{O})$

State the modern formulation of the congruence subgroup problem in terms of the congruence kernel (Sect. 3.7). Prove Theorem 3-8.3 and Proposition 3-8.4 in Section 3.8. Then show that the congruence subgroup problem has a negative answer for $\mathrm{SL}_2(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of an imaginary quadratic field, following Sections 3.12 and 3.13. Throughout the talk recall definitions and results from Chapter 1 as needed.

Main source: [S], Chapter 1; Chapter 3, Sections 3.7, 3.8, 3.12, 3.13

TALK 3: The congruence subgroup problem for $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$

Describe the strategy that we will use to show that the congruence kernel in this case is trivial. Define the Steinberg group; state Theorem 4-1.7 and give an idea of its proof. Then go through Sections 4.2 and 4.3 outlining the proofs and skipping the details. Finally prove Theorem 4-4.2.

Main source: [S], Chapter 4, Sections 4.1, 4.2, 4.3, 4.4

TALK 4: The congruence subgroup problem for $\mathrm{SL}_n(\mathcal{O}_S)$

Given a number field K and a finite set of places S containing all the archimedean ones, we denote by $\mathcal{O}_S$ the ring of S-integers of K. Explain the strategy used to deal with the congruence subgroup problem for $\mathrm{SL}_n(\mathcal{O}_S)$ (Sect. 4.5) and outline the proof of the centrality of the congruence kernel following Section 4.6. Outline Section 4.7 and, if time permits, Section 4.8, presenting as many details as you wish. Throughout the talk recall definitions and results from Chapter 1 as needed.

Main source: [S], Chapter 1; Chapter 4, Sections 4.5, 4.6, 4.7, 4.8

TALK 5: The metaplectic kernel I

Define the metaplectic kernel and describe its role in solving the CSP (Sect. 4.9); then explain Sections 4.10 and 4.11, which deal with Steinberg cocycles. See also Chapter 1, Section 1.15.

Main source: [S], Chapter 1, Section 1.15; Chapter 4, Sections 4.9, 4.10, 4.11

TALK 6: The metaplectic kernel II

Conclude the computation of the congruence kernel for SL_n ($n \geq 3$) following Sections 4.12 and 4.13.

Main source: [S], Chapter 4, Sections 4.12, 4.13

TALK 7: Some group-theoretic applications

Pick some group-theoretic applications from Chapter 5 according to your preferences.

A possible suggestion:

1) *A criterion for linearity*: After proving Theorem 5-3.1 explain Section 5.4 and, in particular, prove Theorem 5-4.3

2) *Bounded generation*: Explain Sections 5.9 and 5.11

3) *Free*: If you have time left choose one (or more) other application(s) from Chapter 5.

Main source: [S], Chapter 5, Sections 5.3, 5.4, 5.9, 5.11,...

TALK 8: The congruence subgroup problem in linear algebraic groups: a survey

The main aim of this talk is to explain Section 6.7; recall the definitions and results that are needed from earlier sections of Chapter 6. You can take for granted the material that we already treated in the Oberseminar on Superrigidity. Other surveys can be found for example in [PR] (Chapter 9), [PrR], [R] and [Ra].

Main source: [S], Chapter 6

References

[H] Humphreys, J. E., *Arithmetic Groups*, Lecture Notes in Mathematics **789**, Springer, Berlin, 1980

[PR] Platonov V., Rapinchuk A.S., *Algebraic groups and Number Theory*, Pure and Applied Mathematics **139**, Academic Press, San Diego, 1994

[PrR] Prasad G., Rapinchuk A. S., *Developments on the congruence subgroup problem after the work of Bass, Milnor and Serre*, ArXiv preprint, 2008

[R] Raghunathan, M. S., *The congruence subgroup problem*, Proc. Indian Acad. Sci. (Math. Sci.) Vol. **114**, No. 4, pp. 299–309, November 2003

[Ra] Rapinchuk, A. S., *Congruence subgroup problem for algebraic groups: old and new*, Astérisque, tome **209** (1992), p. 73-84

[S] Sury, B., *The congruence subgroup problem. An elementary approach aimed at applications*, Texts and Readings in Mathematics **24**, Hindustan Book Agency, New Delhi, 2003