# Spectral Sequences

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# Motivation

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**Graded Complex**
Suppose we have a chain complex $C$

$$\dots \xleftarrow{\partial} C_{d-1} \xleftarrow{\partial} C_d \xleftarrow{\partial} C_{d+1} \xleftarrow{\partial} \dots$$

that is *graded*, i.e. each $C_d$ splits into a direct sum

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Moreover, the boundary maps $\partial$ respect the grading, i.e.

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In that case we can calculate the homology by calculating the individual homologies of

$$\dots \xleftarrow{\partial} C_{d-1,p} \xleftarrow{\partial} C_{d,p} \xleftarrow{\partial} C_{d+1,p} \xleftarrow{\partial} \dots$$

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**Graded Complex**
To make it precise:

$$H_k(C) = \bigoplus_{p=1}^{n} H_k(C^p)$$

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**Filtered Complexes**
A filtered complex is a chain complex that comes with a filtration, i.e. each $C_d$ comes equipped with a nested sequence of submodules $(F_p C_d)_{i \in \mathbb{Z}}$

$$F_{p-1} C_d \subseteq F_p C_d$$

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**Warning**
This only prevents elements going *upwards* in the filtration degree, they can still go down. That fact will lead to the need for Spectral Sequences.

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**Associated Graded Module**
Given a filtration of a module $M$, the associated graded module is defined as

$$\bigoplus_{p\in\mathbb{Z}} F_p M_p / F_{p-1} M_{p-1}$$

with the canonical grading. This loses information, but not too much:

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**Lemma**
Let $f : M \to M'$ be a filtration-preserving morphism, where $M$ and $M'$ are modules with finite filtrations. If $\mathrm{Gr}\ f : \mathrm{Gr}\ M \to \mathrm{Gr}\ M'$ is an isomorphism, then $f$ is an isomorphism.

# Motivation

**Observation**
If we have a proper notion of rank for $R$-modules, such that it behaves well with short exact sequences, meaning:

$$\mathrm{rk} M = \mathrm{rk} M' + \mathrm{rk} M''$$

for every short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

then the rank of a finitely filtered module can be computed from the associated graded module:

$$\mathrm{rk}\ M = \sum_{p \in \mathbb{Z}} \mathrm{rk}\ \mathrm{Gr}_p\ M$$

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If the module $M$ is itself graded, then we can associate a bigraded module to $M$ with the convention that

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**Convention**
To simplify the notation we will suppress the second subscript and just write

$$\mathrm{Gr}_p\ M = F_p M/F_{p-1}M$$

# Motivation

Let $C = (C_n)_{n\in\mathbb{Z}}$ be a filtered chain complex with dimension-wise finite filtration i.e. each $\{F_p C_n\}_{p\in\mathbb{Z}}$ is a finite filtration for fixed $n$.

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**Induced Filtration**
There is an induced filtration on the homology $H(C)$ via $F_p H(C) = \mathrm{Im}\{H(F_p C) \to H(C)\} = (F_p C \cap Z)/(F_p C \cap B)$, this gives us an associated bigraded module $\mathrm{Gr} H(C)$:

$$\mathrm{Gr}_p H(C) = (F_p C \cap Z)/((F_p C \cap B) + (F_{p-1} C \cap Z))$$

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$$F_p C = Z^0_p \supseteq Z^1_p \supseteq \ldots \supseteq Z^\infty_p$$

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$$F_pC = Z^0_p \supseteq Z^1_p \supseteq \ldots \supseteq Z^\infty_p$$

Because of dimension-wise finiteness this stabilizes dimension-wise after finitely many steps to a series of equalities. That means for a fixed $(p, q)$ there is $r >> 0$ such that

$$Z^r_{p,q} = Z^{r+1}_{p,q} = \ldots = Z^\infty_{p,q}$$

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And define

$$E_p^r = Z_p^r/(B_p^r + Z_{p-1}^{r-1}) = Z_p^r/(B_p^r + (F_{p-1}C \cap Z_p^r))$$

and

$$E_p^\infty = Z_p^\infty = Z_p^\infty/(B_p^\infty + Z_{p-1}^\infty) = \mathrm{Gr}_pH(C)$$

And again for fixed $(p, q)$ this stabilizes after finitely many steps for $r >> 0$:

$$E_{p,q}^r = E_{p,q}^{r+1} = \ldots = E_{p,q}^\infty$$

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This sequence $\{E^r\}$ is the Spectral Sequence of $C$ and *converges* to $\mathrm{Gr}H(C)$.

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**Warning**
If $R$ is not a field, then even if $E^r$ converges it does not necessarily converge to the Homology as $\mathrm{Gr}H(C) \ncong H(C)$ in general, we still get a structural result:

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**Proposition**
Let $\tau : C \to C'$ be a filtration-preserving chain map, where $C$ and $C'$ have dimension-wise finite filtrations.
If the induced map

$$E^r(\tau) : E^r(C) \to E^r(C')$$

is an isomorphism for *some* $r$, then so is

$$H(\tau) : H(C) \to H(C')$$

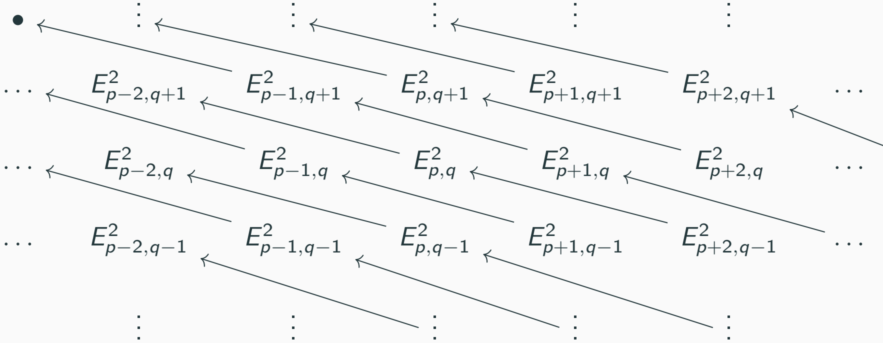
# Proof of the Proposition

**Proof.**
If $E^r$ is an isomorphism, then so is $E^s(\tau)$ for every $r > s$, since $E^s = H(E^{s-1})$. Thus $E^\infty = \mathrm{Gr}H(\tau)$ is an isomorphism. □

# $E^1$ visualized

$$\begin{array}{ccccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
\cdots & E^1_{p-2,q+1} & \longleftarrow & E^1_{p-1,q+1} & \longleftarrow & E^1_{p,q+1} & \longleftarrow & E^1_{p+1,q+1} & \longleftarrow & E^1_{p+2,q+1} & \cdots \\
\cdots & E^1_{p-2,q} & \longleftarrow & E^1_{p-1,q} & \longleftarrow & E^1_{p,q} & \longleftarrow & E^1_{p+1,q} & \longleftarrow & E^1_{p+2,q} & \cdots \\
\cdots & E^1_{p-2,q-1} & \longleftarrow & E^1_{p-1,q-1} & \longleftarrow & E^1_{p,q-1} & \longleftarrow & E^1_{p+1,q-1} & \longleftarrow & E^1_{p+2,q-1} & \cdots \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}$$

# $E^2$ visualized



# Double Complexes

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**Definition**
A double complex is a bigraded module $C = (C_{p,q})_{p,q\in\mathbb{Z}}$ with a horizontal differential and a vertical differential:

$$\begin{array}{ccc} C_{p-1,q} & \xleftarrow{\partial'} & C_{p,q} \\ \downarrow{\scriptstyle \partial'} & & \downarrow{\scriptstyle \partial''} \\ C_{p-1,q-1} & \xleftarrow{\partial'} & C_{p,q-1} \end{array}$$

# Double Complexes

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$$\partial'' C_{*,q} \to C_{*,q-1}$$

The vertical chain complexes $C_{p,*}$ with differential $\partial''$ and chain maps

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$$\partial'' C_{*,q} \to C_{*,q-1}$$

The vertical chain complexes $C_{p,*}$ with differential $\partial''$ and chain maps

$$\partial' : C_{p,*} \to C_{p-1,*}$$

# Picture of Double Complex with Horizontal and Vertical Chain Complexes

$$
\begin{array}{ccccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & C_{p-2,q+1} & \longleftarrow & C_{p-1,q+1} & \longleftarrow & C_{p,q+1} & \longleftarrow & C_{p+1,q+1} & \longleftarrow & C_{p+2,q+1} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & C_{p-2,q} & \longleftarrow & C_{p-1,q} & \longleftarrow & C_{p,q} & \longleftarrow & C_{p+1,q} & \longleftarrow & C_{p+2,q} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
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 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

# The Total Complex

**Definition**
Get back an ordinary chain complex by setting

$$(TC)_n = \bigoplus_{p+q=n} C_{p,q}$$

with differentials

$$\partial_{|C_{p,q}} = \partial' + (-1)^p \partial''$$

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**Observation**
This is dimension-wise finite, if $C$ has only finitely many non-zero modules in any total degree $p + q$

# Again Spectral Sequences

This gives us the first of two emerging spectral sequences $E^r$ converging to $H_*(TC)$.

$E^0_{p,q} = C_{p,q}$ with $d^0 = \pm\partial$

$E^1_{p,q} = H_q(C_{p.*})$ with the differential $d^1 : E^1_{p,q} \to E^1_{p-1,q}$ being induced by the chain map $\partial' C_{p,*} \to C_{p-1,*}$

# The other spectral sequence

**Filtration**
Filter $TC$ again by setting $F_p(TC)_n = \bigoplus_{j \leq p} C_{n-j,}$. We obtain a second spectral sequence $E^r$ with

$$E^0_{p,q} = C_{q,p}$$
$$E^1_{p,q} = H_q(C_{*,p}) \text{ with the differential induced by } \partial''$$

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**Warning**
Both spectral sequences have the same abutment $H_*(TC)$, but they do not generally have the same $E^\infty$-term, because the filtration differs.

# Homology of a Union

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**Known Situation**
If $J = \{1, 2\}$, then there is a short exact sequence

$$0 \to C(X_1 \cap X_2) \to C(X_1) \oplus C(X_2) \to C(X) \to 0$$

from which one obtains the Mayer-Vietoris sequence.

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In the general case this sequence is replaced by:

$$\ldots \to \bigoplus_{\alpha<\beta<\gamma} (X_\alpha \cap X_\beta \cap X_\gamma) \to \bigoplus_{\alpha<\beta} C(X_\alpha \cap X_\beta) \to \bigoplus_{\alpha} C(X_\alpha) \to C(X) \to 0$$

and the Mayer-Vietoris sequence is replaced by a spectral sequence.

# Homology of a Union

**Definition**
Let $K$ be the abstract simplicial complex with vertices $J$ and simplices non-empty finite subsets $\sigma \subseteq J$ such that

$$X_\sigma = \bigcap_{\alpha \in \sigma} X_\alpha \neq \emptyset$$

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Let $K$ be the abstract simplicial complex with vertices $J$ and simplices non-empty finite subsets $\sigma \subseteq J$ such that

$$X_\sigma = \bigcap_{\alpha \in \sigma} X_\alpha \neq \emptyset$$

$K$ is called Nerve of the covering $\{X_\alpha\}$ For $p \geq 0$ let $C_p := \bigoplus_{\sigma \in K^{(p)}} C(X_\alpha)$.

# Homology of a Union

**Appearance of the Double Complex**
If $\sigma$ has vertices $\alpha_0 < \ldots < \alpha_p$, we denote for $0 \leq i \leq p$ by $\partial\sigma$ the $(p-1)$-simplex $\{\alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_p\}$.

$$C(X_\sigma) \hookrightarrow C(X_{\partial_i\sigma})$$

induces a chain map $\partial_i : C_p \to C_{p-1}$ for $p \geq 1$.

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Similarly

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induces a chain map $\epsilon : C_0 \to C(X)$ We get the chain complex in the category of chain complexes:

$$\ldots \to C_p \to C_{p-1} \to \ldots \to C_0 \to C(X) \to 0 \ (*)$$

thus a double complex $C$ with $C_{p,q} = \bigoplus_{\sigma \in K^{(p)}} C_q(X_\sigma)$

This sequence is exact. We omit the proof.

# Homology of a Union

**Second Spectral Sequence**
Exactness of $(*)$ gives

$$E^1_{p,q} = H_q(C_{*,p}) = \begin{cases} 0 & \text{if } q \neq 0 \\ C_p(X) & \text{if } q = 0 \end{cases}$$

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Taking the homology to go over to the next sheet gives us:

$$E^2_{p,q} = \begin{cases} 0 & \text{if } q \neq 0 \\ H_p(X) & \text{if } q = 0 \end{cases}$$

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This gives

$$H_*(TC) \cong H_*(X)$$

**First Spectral Sequence**

$$E^1_{p,q} = H_q(C) = \bigoplus_{\sigma \in K^{(p)}} H_q(X_\sigma)$$

# Homology of a Union

**Definition**
A coefficient system on a simplicial complex $K$ is a family $\mathcal{A} = \{A_\sigma\}$ of abelian groups, where $\sigma$ ranges over the simplices of $K$, together with a map $f_{\sigma\tau} : A_\sigma \to A_\tau$ whenever $\tau$ is a face of $\sigma$ such that

$$f_{\tau\rho} f_{\sigma\tau} = f_{\sigma\rho}$$

if $\rho \subseteq \tau \subseteq \sigma$. We get a chain complex $C(K, \mathcal{A})$ with

$$C_p(K, \mathcal{A}) = \bigoplus_{\sigma \in K^{(p)}} A_\sigma$$

thus we get homology groups

$$H_*(K, \mathcal{A})$$

# Homology of a Union

**Coefficient System of the Nerve**
If $K$ is the nerve of the $\{X_\alpha\}$ we have for each $q \geq 0$ a coefficient system $\mathcal{H}_q = \{H_q(X_\sigma)\}$ on $K$, where $f_{\sigma\tau} : H_q(X_\sigma) \to H_q(X_\tau)$ is induced by $X_\sigma \hookrightarrow X_\tau$.

**The second sheet**
From the definition we get that:

$$E^1_{p,q} = C_p(K, \mathcal{H}_q)$$

hence

$$E^2_{p,q} = H_p(K, \mathcal{H}_q)$$

If $X_\sigma$ is acyclic for all $\sigma$ we have $\mathcal{H}_0 = \mathbb{Z}$ and $\mathcal{H}_q = 0$ for $q \neq 0$. The spectral sequence collapses at $E^2$ and we get:

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**Theorem**
Suppose $X$ is the union of subcomplexes $X_\alpha$ such that every non-empty intersection

$$X_{\alpha_0} \cap \ldots \cap X_{\alpha_p}$$

is acyclic. Then

$$H_*(X) \cong H_*(K)$$

where $K$ is the nerve of the cover. So we can calculate the Homology on the nerve!

# Homology of a Group with Coefficients in a Chain Complex

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**Generalization**
Let $C = (C_n)_{n \geq 0}$ be a non-negative chain complex. We set

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**Generalization**
Let $C = (C_n)_{n \geq 0}$ be a non-negative chain complex. We set

$$H_*(G, C) = H_*(F \otimes_G C)$$

If $C$ consists of one module $M$ concentrated in dimension 0, then we get the usual $H_*(G, M)$.

# Homology of a Group with Coefficients in a Chain Complex

The complex $F \otimes_g C$ is the total complex of the double complex of abelian groups $(F_p \otimes_G C_q)$. Again, we get two spectral sequences converging to $H_*(G, C)$.

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Both spectral sequences give approximations of $H_*(G, C)$ in terms of *ordinary* homology groups $H_*(G, M)$.

# Homology of a Group with Coefficients in a Chain Complex

Now assume that $C_p$ is an $H_*$-acyclic $G$-module. Then $E^1$ is concentrated on the line $q = 0$ and $E^1_{p,0} = (C_p)_G$. The spectral sequence thus collapses at $E^2$ and we get

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**Proposition**
Let $C$ be a non-negative chain complex of $G$-modules such that each $C_n$ is $H_*$-acyclic. Then there is a spectral sequence

$$E^2_{p,q} = H_p(G, H_q C)$$

converging to $H_{p+q}(C_G)$.

# Hochschild-Serre Spectral Sequence

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We consider a group extension $1 \to H \to G \to Q \to 1$. Let $F$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ and $M$ a $G$-module, then $(F \otimes M)_G$ can be computed by:

First dividing out by the action of $H$ on $F \otimes M$

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Thus

$$H_*(G, M) = H_*(C_Q)$$

with $C = F \otimes_H M$

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We get a $Q$-module isomorphism:

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**Theorem**
For any group extension $1 \to H \to G \to Q \to 1$ and any $G$-module $M$, there is a spectral sequence of the form

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**Corollary**
Under the same hypotheses we have an exact sequence

$$H_2(G, M) \to H_2(G, M_H) \to H_1(H, M)_Q \to H_1(G, M) \to H_1(Q, M_H) \to 0$$

# Hochschild-Serre Spectral Sequence

**Proof**
$E^\infty = \mathrm{Gr}H(G,M)$, so we have a short sequence

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Which yields

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And $E^{2}_{p,q} = H_p(Q, H_q(H, M))$ and $E^{\infty}_{2,0}$ is a quotient of $H_2(G, M)$. □