1 ~~Coho~~ Finiteness Condition III

Recall: G group, M a $\mathbb{Z}G$-module, $\mathcal{P}$ any proj res. of $\mathbb{Z}$ over $\mathbb{Z}G$

$$\rightsquigarrow H^*(G;M) := H^*(\mathcal{H}om_{\mathbb{Z}G}(\mathcal{P},M))$$

cochain complex with (non-degree-sep?)
$\mathcal{H}om_{\mathbb{Z}G}(\mathcal{P},M)^n = \mathcal{H}om_{\mathbb{Z}G}(\mathcal{P},M)_{-n} = \mathrm{Hom}_{\mathbb{Z}G}(P_n,M)$

$\rightsquigarrow$ $\underline{cd(G) \le n}$ if $\exists$ proj resolution $\mathcal{P}$ of $\mathbb{Z}$ over $\mathbb{Z}G$:

$$0 \to P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \to P_0 \to \mathbb{Z} \to 0$$

← (exact) ($\ker d_{n-1} = \mathrm{im}\, d_n$)

Projective $\mathbb{Z}G$-modules.

$\rightsquigarrow$ $\underline{G \text{ is FP}}$ if ~~also the P_k above are f.g.~~ $\mathcal{P}$ can be found with the P_k also $\underline{\text{fin. gen.}}$

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1 ~~G~~ FP gives rise to a duality

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Duality for Projective Modules.

Proposition: P a fin. gen. projective left R-mod.

Then: (a) $P^* := \mathrm{Hom}_R(P, R)$ is a right R-mod.

$[(f, r) \longmapsto (p \longmapsto f(p).r) \in \mathrm{Hom}_R(P,R).]$

(b) For any left Rmodule M

$$P^* \otimes_R M \xrightarrow{\cong} \mathrm{Hom}_R(P, M)$$

$$[\, f \otimes m \longmapsto (\underset{P}{\underset{\cap}{p}} \longmapsto f(x).m) \in \mathrm{Hom}_R(P,M)]$$

(2) **Set up:** P_* and $\mathbb{Z}G$ a left G-modules, (~~P_* a left $\mathbb{Z}G$-module~~)

Then ⇝ $\mathrm{Hom}_{\mathbb{Z}G}(P_*, \mathbb{Z}G)$ a right G-module

via $(f, g) \longmapsto \Big(\forall p \in P_*,\ \big(\underset{\in P_*}{p} \longmapsto \underset{\in \mathbb{Z}(G)}{f(g(p))}\big)\Big)$

Define $\varphi : H^*(G, \mathbb{Z}G) \otimes_{\mathbb{Z}G} M \to H^*(G, M)$

on cochains $u \in \mathrm{Hom}_{\mathbb{Z}G}(P, \mathbb{Z}G)$ and elements $m \in M$

via ~~canonical~~

$$u \otimes m \longmapsto (\underset{\in P}{x} \longmapsto u(x).m) \in \mathrm{Hom}_{\mathbb{Z}G}(P, M)$$

Prop: (universal coefficient formula). G type FP and $n = \mathrm{cd}\, G$. Then for any (left) G-modules M, for the map φ ~~above~~ is a (group) iso

$$\varphi : H^n(G, \mathbb{Z}G) \otimes_{\mathbb{Z}G} M \xrightarrow{\ \cong\ } H^n(G, M).$$

Proof: P finite proj. resolution of $\mathbb{Z}$ (length n).

Let $\bar{P}$ be dual complex $\mathrm{Hom}_{\mathbb{Z}G}(P, \mathbb{Z}G)$ of right G-mods.

so $H^*(G, \mathbb{Z}G) \cong H^*(\bar{P})$

→ s.e.s

$$\begin{tikzcd}
\bar{P}^{n-1} \arrow[r] \arrow[d] & \bar{P}^{n} \arrow[r] & H^n(G, \mathbb{Z}G) \arrow[r] & 0 \\
\bar{P}^{n-1} \otimes_G M \arrow[r] \arrow[d, "\cong \text{ Proj. Dual. (b)}"] & \bar{P}^{n} \otimes_G M \arrow[r] \arrow[d, "\cong"] & H^n(G, \mathbb{Z}G) \otimes_G M \arrow[r] \arrow[d, "\varphi"] & 0 \\
\mathrm{Hom}_G(P_{n-1}, M) \arrow[r] & \mathrm{Hom}_G(P_n, M) \arrow[r] & H^n(G, M) \arrow[r] & 0
\end{tikzcd}$$

$\boxed{\text{so } \varphi \text{ an iso}}$

③ Now let $D = H^n(G, \mathbb{Z}G)$, ~~we have~~ rewrite ⊛.

$$H^n(G, \mathbb{Z}G) \otimes_{\mathbb{Z}G} M = D \otimes_{\mathbb{Z}G} M = \cancel{(D \otimes M)_G} = H_0(G, D \otimes M)$$

where $D \otimes M = D \otimes_{\mathbb{Z}} M$ with diagonal G-action

[i.e. $\gamma \cdot (d \otimes m) = d\gamma^{-1} \otimes \gamma m$].

~~So ⊛ case then~~ For G FP with $\mathrm{cd}(G) = n$

$$⊛ \rightsquigarrow \quad H^n(G, M) \cong H_0(G, D \otimes M).$$

cohomology — Homology

← DUALITY →

Theorem / ~~Definition~~ (BIERI - Eckmann)

G FP and $n = \mathrm{cd}\, G$.

(i) writing $D = H^n(G, \mathbb{Z}G)$ (right) G-module, have for any G-module M and $k \in \mathbb{Z}$. in

$$H^k(G, M) \cong H_{n-k}(\Gamma, D \otimes M).$$

(ii) $H^k(G, \mathbb{Z}G \otimes A) = 0$ for all $k \neq n$ and all abelian A

(iii) For all $k \neq n$ $H^k(G, \mathbb{Z}G) = 0$ and $D^n(G, \mathbb{Z}G)$ is torsion-free

[(iv)]

Definition: G ~~as~~ satisfying conditions is called a ~~dual~~ (Bieri-Eckmann) DUALITY GROUP.

$D = H^n(G, \mathbb{Z}G)$ the DUALIZING MODULE.

④ <u>Prop</u> G <u>torsion-free</u>, $H \underset{\text{F.I.}}{\leq} G$ with $[G:H]$ finite.

Then G is duality group $\Leftrightarrow$ H duality gp.

<u>Sketch</u>: G is FP $\Leftrightarrow$ H is FP. (<u>Serre</u> $cd(G) = cd(H)$)

Shapiro $\rightsquigarrow$ $H^*(G, \mathbb{Z}G) \simeq H^*(H, \mathbb{Z}H)$.

So (iii) holds for G $\Leftrightarrow$ (iii) — " — H. $\square$

(5)

Virtual Notions

recall: G not torsion-free $\Rightarrow cd(G) = \infty$.

Defn: G is virtually torsion-free if $\exists H \leq G$ f. ind. with H torsion-free.

Note: G ~~torsion~~ v. t.f $\xrightarrow{\text{Serre}}$ $cd(H) = cd(H')$ for all fin. index ~~subgps~~ t.f. subgps H, H'.

[Pf. $H \cap H'$ has ~~fin. index in~~ is t. free and f. i. index in H, H'].

Defn:

- $vcd(G) = cd(H)$ for H f.i. torsion-free $\leq G$.
- G has VFP if ~~fin~~ $H \underset{f.i}{\leq} G$ is FP.

Prop: G is ~~VFP~~ a virtual duality group $\Leftrightarrow$

(1) G is VFP

(2) $\exists n$ s.t. $H^i(G, \mathbb{Z}G) = 0$ for $i \neq n$ and $H^n(G, \mathbb{Z}G)$ is torsion-free.

(6) ## Examples: (1) $\mathrm{vcd}\ G = 0 \iff G$ finite

(2) ~~If~~ ~~If~~ G_1, G_2 finite then

$$G_1 *_A G_2 \text{ is } VFP \text{ (even } WF\ldots).$$

and ~~vcd~~ $\mathrm{vcd}(G_1 *_A G_2) \leq 1$.

Via Bass-Serre theory, note for $\Gamma = \pi_0$(of finite g.o. groups).

~~Converse~~ Even Fact: $\mathrm{vcd}(\Gamma) \leq 1 \iff \Gamma = \pi_0$(Finite graph of groups).

($\Leftarrow$) Bass-Serre Theory

($\Rightarrow$) ~~Stall~~ Stallings-Swan(Free f.i. subgps)

& Karrass-Pietrowski-Solitar

(3) Arithmetic groups (Borel-Serre) e.g. $\boxed{SL_n(\mathbb{Z})}$

$n \geq 3$ $SL_n(\mathbb{Z})$ has torsion

~~e.g.~~ But ~~&~~ $\mathrm{vcd}\ SL_n(\mathbb{Z}) = \dfrac{n(n-1)}{2}$. ~~, ... although~~

~~....~~

via Principal Congruence Subgroups (of fin. index).

$$\Gamma_N := \ker\left(GL_n(\mathbb{Z}) \longrightarrow GL_n(\mathbb{Z}/N\mathbb{Z})\right)$$
$$= \{ g \in GL_n(\mathbb{Z}) : g \equiv I_n \pmod N \}.$$

$\Gamma^{SL}_N = \Gamma_N \cap SL_n(\mathbb{Z})$ ~~is~~ torsion free of finite index in $SL_n(\mathbb{Z})$

$\boxed{\text{~~...~~ } U_n := \{ \text{upper triangular gps} \} \leq SL_n(\mathbb{Z}).}$

Then (hard) ~~...~~ ~~$U_n \cap \Gamma$~~ $\mathrm{cd}(\Gamma^{SL}_N) = \dfrac{n(n-1)}{2}$

$SL_n(\mathbb{Z})$ is FP

TOPOLOGY

Also ~~(4) $SL_n(\mathbb{Z})$ is a duality gp~~

6+ Also ⊛ Γ_N^{SC} is FP

⊛ Γ_N^{SC} is duality group.

So $SL_n(\mathbb{Z})$ is VFP, virtual duality gp.