Finiteness Conditions II (01/06)

1 Geometric Dimension

Def: G = group.

x has no cells in dim > n

$$\mathrm{gd}\, G := \inf\{n\in\mathbb{N} \mid \exists X = \underbrace{K(G,1)\text{-complex}}_{\substack{\text{connected CW-complex with } \tilde{X} \text{ contractible} \\ \text{and } \pi_1(X)\simeq G.}} \text{ with } X = X^{(n)}\}$$

- Cellular chain complex of $\tilde{X}$ ⇝ finite length resolution for $\mathbb{Z}$ over $\mathbb{Z}G$.

$\Rightarrow$ $\mathrm{cd}\, G \leq \mathrm{gd}\, G$

Theorem: 1) If $\mathrm{cd}\, G \geq 3$, then $\mathrm{cd}\, G = \mathrm{gd}\, G$.

2) If $\mathrm{cd}\, G = 2$, then $\mathrm{gd}\, G \in \{2,3\}$.

Conjecture (Eilenberg-Ganea): $\mathrm{cd} = \mathrm{gd}$.
(1957)

② FP_n Conditions

Def: G is *of type FP_n* if there is a projective resolution:

$$P: \quad \cdots \to P_k \to P_{k-1} \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

of $\mathbb{Z}$ over $\mathbb{Z}G$ with P_k finitely generated $\forall k \leq n$.

- Immediate examples (of groups of type $FP_\infty = FP_n \ \forall n$)

1) Finite groups: Bar resolution! $\cdots \to \bigoplus_{G\times G} \mathbb{Z}G \to \bigoplus_{G} \mathbb{Z}G \to \mathbb{Z}G \to \mathbb{Z} \to 0$

2) Free groups of finite rank: $\cdots \to 0 \to \bigoplus_{j=1}^{n} \mathbb{Z}F_n \cdot e_j \to \mathbb{Z}F_n \to \mathbb{Z} \to 0$.

3) Free abelian groups: $G = \mathbb{Z}^n \Rightarrow \mathbb{Z}G = \underbrace{\mathbb{Z}[X_1, \dots, X_n, X_1^{-1}, \dots, X_n^{-1}]}_{\text{Noetherian ring!}}$

$$\begin{tikzcd}
\cdots \arrow[r] & F_4 \arrow[r, "d_4"] & F_3 \arrow[r, "d_3"] & \underset{fg}{F_2} \arrow[rr, "d_2"] \arrow[dr] & & \underset{fg}{F_1} \arrow[rr, "d_1"] \arrow[dr] & & \underset{F_0}{\mathbb{Z}G} \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[r] & 0 \\
& & & & \ker d_1 \ fg \arrow[ur] & & \ker \varepsilon = \mathrm{Aug}\,\mathbb{Z}G \arrow[ur] & & &
\end{tikzcd}$$

fg mod. of $\mathbb{Z}G$

4) More generally, if G is polycyclic-by-finite, then $\mathbb{Z}G$ is left and right Noetherian
($\Rightarrow G$ is of type FP_∞).

- FP_0 means nothing: $\mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ is always possible.

$$\sum a_g g \mapsto \sum a_g$$

Proposition: G is of type $FP_1 \iff G$ is a finitely generated group.

Proof: $\mathrm{Aug}\,\mathbb{Z}G = \langle g-1 ; g \in G \rangle_{\mathbb{Z}G}$

$(\Leftarrow)$ $G = \langle X \rangle \Rightarrow \mathrm{Aug}\,\mathbb{Z}G = \langle x-1 ; x \in X \rangle_{\mathbb{Z}G}$

$xy - 1 = (x-1)y + (y-1)$

$x, y \in X$

$x^{-1}y - 1 = -(x-1)x^{-1}y + (y-1)$

($\Leftarrow$) If $\mathrm{Aug}\,\mathbb{Z}G = \langle y-1 \,;\, y\in Y\subseteq G\rangle_{\mathbb{Z}G}$, let $H=\langle Y\rangle \overset{!}{\le} G$

Consider $0\to \mathrm{Aug}\,\mathbb{Z}H \to \mathbb{Z}H \xrightarrow{\varepsilon} \mathbb{Z}\to 0$ and take $-\otimes_{\mathbb{Z}H}\mathbb{Z}G$:

$$0\to \underbrace{\mathrm{Aug}\,\mathbb{Z}H\otimes_{\mathbb{Z}H}\mathbb{Z}G}_{\mathrm{Aug}\,\mathbb{Z}G} \to \underbrace{\mathbb{Z}H\otimes_{\mathbb{Z}H}\mathbb{Z}G}_{\mathbb{Z}G}\to \underbrace{\mathbb{Z}\otimes_{\mathbb{Z}H}\mathbb{Z}G}_{\mathbb{Z}[H\backslash G]} \to 0$$

$$\mathrm{Aug}\,\mathbb{Z}G \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z}[H\backslash G] \longrightarrow 0 \qquad \mathbb{Z}[H\backslash G]\cong\mathbb{Z} \Rightarrow G=H$$

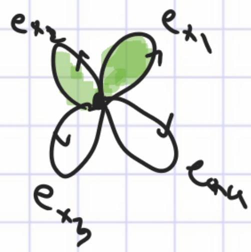
□

- Groups of type FP_2 are the "almost finitely presentable groups".

Proposition: 1) G is finitely presentable $\Longrightarrow$ G is of type FP_2.

2) If $G=F/R$, then G is FP_2 $\Leftrightarrow$ $\frac{R}{[R,R]}$ is a fg $\mathbb{Z}G$-module and G is fg
(F free)

Proof: 1) $G=\langle X\mid R\rangle$ finite presentation $\Longrightarrow \exists K(G,1)$ complex with one 0-cell, $\#X$ 1-cells and $\#R$ 2-cells:

X  $\Rightarrow \pi_1|X|\cong G$

Cellular chain complex for $\tilde{X}$: $\cdots\to \overset{fg}{\bigoplus_{r\in R}\mathbb{Z}G\cdot f_r} \to \overset{fg}{\bigoplus_{x\in X}\mathbb{Z}G\cdot e_x} \to \overset{fg}{\mathbb{Z}G}\to\mathbb{Z}\to 0$

□

Remark: • $\mathbb{Z}\otimes_{\mathbb{Z}G}-$ commutes with direct limits $\underset{\longrightarrow}{\lim}$ (sums $\bigoplus$) $\Longrightarrow$ so does $H_*(G;-)$

• $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z};-)$ commutes with inverse limits $\underset{\longleftarrow}{\lim}$, $\prod$ $\Longrightarrow$ so does $H^*(G;-)$

$\{A_i\}_{i\in I}$ direct system (I directed) $\qquad \underset{\longrightarrow}{\lim}\left(H_k(G;A_i)\right) \xrightarrow{\cong} H_k(G;\underset{\longrightarrow}{\lim}A_i)$

Theorem (Bieri–Eckmann): TFAE for a finitely generated group:

1) G is of type FP_∞, FP_n

× 2) $H_k(G;-)$ commutes with exact inverse limits $\forall k$

× 3) $H^k(G;-)$ commutes with exact direct limits $\forall k$.

4) $H_k(G; \prod_{\lambda \in \Lambda} \mathbb{Z}G) = 0$ $\forall k, \forall \Lambda$. $k<n$

(4') $H^k(G; \varinjlim A_i) = 0$ whenever $\varinjlim A_i = 0$

• **Idea**: Use that a finitely generated free $\mathbb{Z}G$-module is at the same time a direct sum and a direct product.

Corollary: 1) $H \trianglelefteq G$: H and G/H FP_n $\Rightarrow$ G FP_n. LHS spectral sequence

2) $H \leq G$, $[G:H] < \infty$: G $FP_n \Leftrightarrow H$ FP_n. Shapiro, Ind = Coind

Non examples: If G is FP_n, then $H_k(G;\mathbb{Z})$ is fg abelian group. $\forall k \leq n$

subquotient of $\mathbb{Z} \otimes_{\mathbb{Z}G} (\bigoplus_{i=1}^{n} \mathbb{Z}G) \cong \mathbb{Z}^n$

$\mathbb{Z} \wr \mathbb{Z}$ not FP_2

1) Bieri–Stallings groups (~30's): $G_n = \ker(\overbrace{F_2 \times \dots \times F_2}^{n \text{ copies}} \to \mathbb{Z})$, all gens $\mapsto 1$, is FP_{n-1}, not FP_n,

or $G = \{(x,y,z) \in F_2 \times F_2 \times F_2 \mid xy^{-1}z \in [F,F]\}$ fp, not FP_3.

2) Bestvina–Brady (~1997): Anything can happen for $\ker(A_\Gamma \to \mathbb{Z})$, gens $\mapsto 1$, including non-finitely presentable, type FP_∞ groups.

right angled Artin group

$FP_2 \not\Rightarrow fp$

3) Leary (2018): There exist uncountably many groups of type FP_∞!

fp —— FP_2 – fg FP

4) Types FP and FL

Def: G is of type FP (resp. FL) if there is a projective (resp. free) resolution $0 \to P_n \to \dots \to P_1 \to P_0 \to \mathbb{Z} \to 0$ of $\mathbb{Z}$ over $\mathbb{Z}G$ with all P_i finitely generated.

Proposition: G is of type FP $\Leftrightarrow$ G is of type FP_∞ and $\mathrm{cd}\, G < \infty$.

Proof: $0 \to K \to \underbrace{P_{n-1} \to \dots \to P_1 \to P_0}_{\text{fg projective}} \to \mathbb{Z} \to 0 \Rightarrow K$ is fg and projective.

$\mathrm{cd}\, G = n$

$\square$

Examples: 1) $\mathbb{Z}^n$, F_n, A_Γ are of type FP.

2) Thompson's group F is of type FP_∞, but not FP!

(Brown-Geoghegan 1984, first example)

Question: Is FP = FL ?