# Finiteness Conditions I (25/05)

## ① Cohomological dimension.

$G$ group, $A = \mathbb{Z}G$-module $\rightsquigarrow H^*(G;A) = H^*(\mathrm{Hom}_{\mathbb{Z}G}(P, A))$

$\uparrow$ ANY projective resol. of $\mathbb{Z}$ over $\mathbb{Z}G$

Ex: Bar resolution : $\cdots \to \mathbb{Z}G\otimes G^{``2"} \to \mathbb{Z}G\otimes G^{``1"} \to \mathbb{Z}G \to \mathbb{Z} \to 0$

Ex: $F = F(X)$ group $\cdots \to 0 \to 0 \to \bigoplus_{x\in X} \mathbb{Z}F\cdot e_x \to \mathbb{Z}F \to \mathbb{Z} \to 0$

Def: $G$ has finite cohomological dimension if $\exists$ projective resol.

$$0 \to P_n \to P_{n-1} \to \dots \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

of finite length of the $\underset{\text{trivial}}{\hat{}}\mathbb{Z}G$-module $\mathbb{Z}$ $\rightsquigarrow$ $\mathrm{cd}\,G = \inf\{n \mid -\}$

or $\mathrm{cd}\,G = \infty$ otherwise

Proposition 1: For any group, the following are equivalent:

(1) $\mathrm{cd}\,G \leq n$;

(2) $H^i(G;A) = 0 \quad \forall i > n, \ \forall A = \mathbb{Z}G$-module;

(3) $\forall\, 0 \to K \to P_{n-1} \to \dots \to P_1 \to P_0 \to \mathbb{Z} \to 0$ exact complex of $\mathbb{Z}G$-mod, with $P_j$ projective $\forall j$, $K$ is projective too.

Proof: 2⇒3

$$\begin{tikzcd}
\dots \arrow[r] & P_{n+2} \arrow[r, "d"] & P_{n+1} \arrow[rr, "d"] \arrow[dr, two heads] & & P_n \arrow[rr, "d"] \arrow[dr] & & P_{n-1} \arrow[r, "d"] & \dots \arrow[r] & P_0 \arrow[r, "d"] & \mathbb{Z} \arrow[r] & 0 \\
& & & L \arrow[ur, hook] & & K \arrow[ur, hook] & & & & &
\end{tikzcd}$$

find ↓ free over $P_{n+1}$; free over $P_n$

$\rightsquigarrow \quad \dots \to \mathrm{Hom}_{\mathbb{Z}G}(P_n, A) \xrightarrow{d^*} \mathrm{Hom}_{\mathbb{Z}G}(P_{n+1}, A) \xrightarrow{d^*} \mathrm{Hom}_{\mathbb{Z}G}(P_{n+2}, A) \to \dots$

$H^{n+1}(G,A) = 0 \Rightarrow \underbrace{\forall \varphi : P_{n+1} \to A \text{ st } \varphi\circ d = d^*\varphi = 0}_{*}$, $\exists \psi : P_n \to A$ st $\varphi = d^*\psi = \psi\circ d$..

$(*) \Leftrightarrow \varphi$ induces $\bar{\varphi} : L \to A$.

$\Rightarrow \forall \varphi : L \to A$ lifts to $\tilde{\varphi} : P_n \to A$.

$\Rightarrow A = L$: $\mathrm{id} : L \to L$ lifts to $\tilde{\mathrm{id}} : P_n \to L \Rightarrow P_n \simeq L \oplus \frac{P_n}{L} \simeq L \oplus K$.

$\Uparrow$ K is projective

$$\begin{tikzcd}
P_{n+2} \arrow[r, "d"] \arrow[dr, "0"'] & P_{n+1} \arrow[r, "d"] \arrow[d, "\varphi"] & P_n \arrow[dl, dashed, "\exists\psi"] \\
& A &
\end{tikzcd}$$

## Examples

① $\operatorname{cd} 1 = 0$

② $\operatorname{cd} \underset{n>1}{\mathbb{Z}/n\mathbb{Z}} = \infty$ : $\cdots \to \mathbb{Z}G \to \mathbb{Z}G \to \mathbb{Z}G \to \mathbb{Z} \to 0$ ; $H^{2k}(\mathbb{Z}/_{n\mathbb{Z}};\mathbb{Z}) = \mathbb{Z}/_{n\mathbb{Z}} \neq 0$

③ $\operatorname{cd} \underset{= G}{\mathbb{Z}^n} = n$ : $\mathbb{Z}^n \simeq \pi_1(S^1 \times \_\_ \times S^1) \rightsquigarrow$ resol. $\{\mathbb{Z}G \otimes \mathbb{Z}^{\binom{n}{k}}\}_k$

$$0 \to \underset{= \mathbb{Z}^{\binom{n}{n}}}{\mathbb{Z}G\otimes\mathbb{Z}} \to \cdots \to \mathbb{Z}G\otimes\mathbb{Z}^{\binom{n}{2}} \to \mathbb{Z}G\otimes\mathbb{Z}^n \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

$\Rightarrow \operatorname{cd} G = n$

and $H^n(\mathbb{Z}^n;\mathbb{Z}) \simeq \mathbb{Z}$.

④ $\operatorname{cd} \underset{\neq 1}{F(X)} = 1$ : $\operatorname{cd} F(X) \leq 1$ ; $H^1(F(X);\mathbb{Z}) \simeq \underset{\text{grps}}{\operatorname{Hom}}(F(X),\mathbb{Z}) \simeq \prod_{x\in X} \mathbb{Z} \neq 0$. $\Rightarrow \operatorname{cd} F \geq 1$.

**Theorem** (Stallings, Swan): $G$ is free $\Leftrightarrow$ $\operatorname{cd} G \leq 1$.

**Proposition 2**: (1) If $H \leq G$, then $\operatorname{cd} H \leq \operatorname{cd} G$.

(2) If $[G:H] < \infty$ and $\operatorname{cd} G < \infty$, then $\operatorname{cd} H = \operatorname{cd} G$.

(3) If $N \trianglelefteq G$, then $\operatorname{cd} G \leq \operatorname{cd} N + \operatorname{cd} G/N$. spectral sequence LHS

**Proof**: (1) Any projective resol. for $\mathbb{Z}|_{\mathbb{Z}G}$ is a proj. resol. for $\mathbb{Z}|_{\mathbb{Z}H}$.

(2) Let $\operatorname{cd} G = n < \infty$. $\Rightarrow \exists A$ - $\mathbb{Z}G$-mod st $H^n(G;A) \neq 0$. If $F$ is a free $\mathbb{Z}G$-mod st $F \overset{\varphi}{\twoheadrightarrow} A$, then $H^n(G;F) \neq 0$.

$$\cdots \to \underset{\neq 0}{H^n(G;F)} \to \underset{\neq 0}{H^n(G;A)} \to \underbrace{H^{n+1}(G;\ker\varphi)}_{0} \to \cdots$$

Let $F = \bigoplus_{i\in I} \mathbb{Z}G$. Then:

$$H^n(H;\bigoplus_{i\in I}\mathbb{Z}H) \underset{\substack{\uparrow \\ \text{Shapiro's} \\ \text{Lemma}}}{\cong} H^n(G;\operatorname{Coind}_{\mathbb{Z}H}^{\mathbb{Z}G}(\bigoplus_{i\in I}\mathbb{Z}H)) \overset{\substack{\text{Coind = ind because} \\ [G:H]<\infty \\ \downarrow}}{\cong} H^n(G;\operatorname{Ind}_{\mathbb{Z}H}^{\mathbb{Z}G}(\bigoplus_{i\in I}\mathbb{Z}H))$$

$$= H^n(G;\mathbb{Z}G\otimes_{\mathbb{Z}H}(\bigoplus_{i\in I}\mathbb{Z}H))$$

$$= H^n(G;\underbrace{\bigoplus_{i\in I}\mathbb{Z}G}_{F}) \neq 0.$$

$\Rightarrow \operatorname{cd} H \geq n$.

and $\operatorname{cd} H \leq \operatorname{cd} G = n$ by part (1).

$\square$

## More examples

Ex $\mathbb{Z}^k \underset{fi}{\leq} \mathbb{Z}^k \times \mathbb{Z}/2\mathbb{Z}$

cd $k$ cd $\infty$

① $\mathrm{cd}\, G < \infty \Rightarrow G$ is torsion-free !

② $G$ torsion-free nilpotent $\Rightarrow \mathrm{cd}\, G = h(G)$ Hirsch length.

③ Torsion-free groups of infinite cohomological dimension:

(3.1) $\mathrm{cd} \bigoplus_{n\in\mathbb{Z}} \mathbb{Z} \geq \mathrm{cd}(\mathbb{Z}^k) - k \quad \forall k$.

$R$ , $RG$-mod

(3.2) $\mathbb{Z} \wr \mathbb{Z} = \left(\bigoplus_{n\in\mathbb{Z}} \mathbb{Z}\right) \rtimes \mathbb{Z}$. finitely generated

(3.3) Thompson's group $F \geq \bigoplus_{n\in\mathbb{Z}} \mathbb{Z}$, finitely presented, $FP_\infty$.

④ $\mathrm{cd}\, G \ast_K H \leq \max\{\mathrm{cd}\, G, \mathrm{cd}\, H, 1+\mathrm{cd}\, K\}$

Mayer-Vietoris sequence

**Theorem (Serre):** If $G$ is torsion-free and $H \leq G$ is a subgroup of finite index, then $\mathrm{cd}\, H = \mathrm{cd}\, G$.

**Proof:** It is enough to show that $\mathrm{cd}\, H < \infty \Rightarrow \mathrm{cd}\, G < \infty$.

**Idea:** If $[G:H] = n$, combine $n$ copies of a projective resol.

$$P: \quad 0 \to P_m \xrightarrow{\partial} P_{m-1} \to \dots \to P_1 \xrightarrow{\partial} P_0 \xrightarrow{\partial} \mathbb{Z} \to 0$$

of $\mathbb{Z}$ over $\mathbb{Z}H$ into a $\underset{\text{(but big)}}{\text{finite}}$ length projective resol. for $\mathbb{Z}|_{\mathbb{Z}G}$.

- $Q = \underbrace{P \otimes_{\mathbb{Z}} \dots \otimes_{\mathbb{Z}} P}_{n \text{ copies}}$ $\nearrow$ $Q_k = \bigoplus_{i_1+\dots+i_n=k} P_{i_1} \otimes \dots \otimes P_{i_n}$ $\searrow$ differential is $d = \sum (-1)^{\square}\, id \otimes \dots \otimes \partial \otimes \dots \otimes id$

$$\Rightarrow \quad 0 \to Q_{nm} \to Q_{nm-1} \to \dots \to Q_1 \to Q_0 \to \mathbb{Z} \to 0$$

- $Q$ is a complex of $\mathbb{Z}$-mod, of finite length. ✓
- $Q$ is exact by the Künneth formula ✓

- $Q$ is a complex of $\mathbb{Z}G$-modules.

$G = \bigsqcup_{i=1}^{n} x_i H$ ; $g \in G \Rightarrow g x_i = x_{a_i} h_{a_i}$   $h_{a_i} \in H$ , $a_i \in \{1, \dots, n\}$.

$$g \cdot (\underbrace{p_1 \otimes \dots \otimes p_n}_{\overset{\uparrow}{Q_k < Q}}) = (h_{a_1} p_{a_1}) \otimes \dots \otimes (h_{a_n} p_{a_n}) \qquad (*)$$

It can be checked that this defines an action of $\mathbb{Z}G$ on $Q$ (it commutes with the differential)

- $Q$ is projective : treat $Q$ as a module $Q = \dot{\bigoplus_k} Q_k$.

$P$ is projective over $\mathbb{Z}H \Rightarrow P = \bigoplus_k P_k$ is a free summand of some free $\mathbb{Z}H$-module $F$

$\Rightarrow Q = P \otimes \dots \otimes P$ is a direct summand of $F \otimes \dots \otimes F$.

With the structure given by $(*)$, $F \otimes \dots \otimes F$ is actually FREE over $\mathbb{Z}G$:

If $\{b_\alpha\}_\alpha$ is a free basis of $F$ as $\mathbb{Z}H$-mod, then

$$X = \{h_1 b_{\alpha_1} \otimes \dots \otimes h_n b_{\alpha_n}\}_{\alpha_i}$$

is a $\mathbb{Z}$-basis for $F \otimes \dots \otimes F$.

- $G$ permutes $X$, but actually the stabilizers are trivial:

$$x \in X \ , \ G_x \cap \Big(\underbrace{\ker\big(G \to \mathrm{Sym}(G/H)\big)}_{N}\Big) = 1$$

$$\Rightarrow G_x \hookrightarrow G/N \Rightarrow G_x \text{ is finite} \Rightarrow \underset{\substack{\uparrow \\ G \text{ torsion-free.}}}{G_x = 1}$$

$$\Rightarrow F \cong \bigoplus_{G \backslash X} \mathbb{Z}G$$

□