LUCA DI GRAVINA

11.05.2021

Homology & Cohomology with Coefficients - PART II

Cohomology of Groups

Oberseminar
Algebra und Geometrie

Recall some basic definitions and properties

Develop some ideas and techniques to compute and connect homology and cohomology

Eckmann - Shapiro lemma

Dimension Shifting

EXAMPLES

Transfer Map

Characterization via Functoriality of Homology & Cohomology

Homology & Cohomology with Coefficients - PART I

Let G be a group and F be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

$$\cdots \longrightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 \quad \text{exact}$$

Let M be a G-module.

$$\cdots \longrightarrow F_2 \otimes_{\mathbb{Z}G} M \longrightarrow F_1 \otimes_{\mathbb{Z}G} M \longrightarrow F_0 \otimes_{\mathbb{Z}G} M \longrightarrow \underbrace{\mathbb{Z} \otimes_{\mathbb{Z}G} M}_{\cong M_G} \longrightarrow 0$$

Homology of G with coefficients in M: $\boxed{H_*(G,M) := H_*(F \otimes_{\mathbb{Z}G} M)}$

$$\cdots \leftarrow \mathrm{Hom}_{\mathbb{Z}G}(F_2,M) \longleftarrow \mathrm{Hom}_{\mathbb{Z}G}(F_1,M) \longleftarrow \mathrm{Hom}_{\mathbb{Z}G}(F_0,M) \longleftarrow \underbrace{\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z},M)}_{\cong M^G} \longleftarrow 0$$

Cohomology of G with coefficients in M: $\boxed{H^*(G,M) := H^*(\mathrm{Hom}_{\mathbb{Z}G}(F,M))}$

Let $\alpha : R \to S$ be a ring homomorphism.

M : R-module

Regard S as a left R-module

$$\boxed{\mathrm{Hom}_R(S,M)}$$

S-mod.

CO-EXTENSION OF SCALARS

$$\begin{tikzcd} & \mathrm{Hom}_R(S,M) \arrow[d, "\pi"] \\ N \arrow[ur, dashed, "g"] \arrow[r, "f"'] & M \end{tikzcd}$$

N : S-mod.

$$\boxed{\mathrm{Hom}_S(N, \mathrm{Hom}_R(S,M)) \cong \mathrm{Hom}_R(N,M)}$$

There is also a canonical injective S-module map

$$N \hookrightarrow \mathrm{Hom}_R(S,N).$$

Regard S as a right R-module

$$\boxed{S \otimes_R M}$$

S-mod.

EXTENSION OF SCALARS

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d, "f"'] & S \otimes_R M \arrow[dl, dashed, "g"] \\ N & \end{tikzcd}$$

N : S-mod

$$\boxed{\mathrm{Hom}_S(S \otimes_R M, N) \cong \mathrm{Hom}_R(M,N)}$$

There is also a canonical surjective S-module map

$$S \otimes_R N \twoheadrightarrow N.$$

Eckmann-Shapiro lemma

G group, $H \leq G$ $\rightsquigarrow$ $\mathbb{Z}H \hookrightarrow \mathbb{Z}G$

Let M be an H-module.

$\mathrm{Ind}_H^G M := \mathbb{Z}G \otimes_{\mathbb{Z}H} M$, $\mathrm{Coind}_H^G M := \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M)$

Then there are canonical isomorphisms

$$H_*(H, M) \cong H_*(G, \mathrm{Ind}_H^G M)$$

and

$$H^*(H, M) \cong H^*(G, \mathrm{Coind}_H^G M).$$

PROOF

Take P : projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$

can be regarded as a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}H$ (by restriction of scalars)

$$\Rightarrow \quad H_*(H, M) = H_*(P \otimes_{\mathbb{Z}H} M)$$

$$\cong H_*((P \otimes_{\mathbb{Z}G} \mathbb{Z}G) \otimes_{\mathbb{Z}H} M)$$

$$\cong H_*(P \otimes_{\mathbb{Z}G} \underbrace{(\mathbb{Z}G \otimes_{\mathbb{Z}H} M)}_{\mathrm{Ind}_H^G M})$$

$$\cong H_*(G, \mathrm{Ind}_H^G M).$$

Take P : projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$

$$
\begin{aligned}
H^*(H,M) &= H^*(\mathrm{Hom}_{\mathbb{Z}H}(P,M)) \\
&\cong H^*\Big(\mathrm{Hom}_{\mathbb{Z}G}\big(P, \underbrace{\mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G,M)}_{\mathrm{Coind}_H^G M}\big)\Big) \\
&\cong H^*(\mathrm{Hom}_{\mathbb{Z}G}(P, \mathrm{Coind}_H^G M)) \\
&\cong H^*(G, \mathrm{Coind}_H^G M)
\end{aligned}
$$

$$
\begin{tikzcd}
 & \mathrm{Hom}_R(S,M) \arrow[d, "\pi"] \\
N \arrow[ur, dashed, "g"] \arrow[r, "f"'] & M
\end{tikzcd}
$$

$$\boxed{\mathrm{Hom}_S(N,\mathrm{Hom}_R(S,M)) \cong \mathrm{Hom}_R(N,M)}$$

EXAMPLE $\mathbb{Z}$

$$H_*(H,M) \cong H_*(G, \operatorname{Ind}_H^G M)$$
$$H^*(H,M) \cong H^*(G, \operatorname{Coind}_H^G M)$$

Remind that

$\xrightarrow{\text{PROP. A1}}$ $\operatorname{Ind}_H^G M \cong \bigoplus_{g \in G/H} gM$

Let $M = \mathbb{Z}$. $\operatorname{Ind}_H^G \mathbb{Z} \cong \mathbb{Z}[G/H]$

$$\Rightarrow \boxed{H_*(H) := H_*(H,\mathbb{Z}) \cong H_*(G, \mathbb{Z}[G/H])}$$

$\xrightarrow{\text{PROP. E}}$ $|G:H| < \infty \Rightarrow \operatorname{Ind}_H^G M \cong \operatorname{Coind}_H^G M$

$$\boxed{\text{If } |G:H| < \infty, \text{ then } H^*(H,\mathbb{Z}) \cong H^*(G, \mathbb{Z}[G/H])}$$

EXAMPLE $\mathbb{Z}H$

$$H_*(H,M) \cong H_*(G, \mathrm{Ind}_H^G M)$$
$$H^*(H,M) \cong H^*(G, \mathrm{Coind}_H^G M)$$

Let $M = \mathbb{Z}H$.

We have

$$\mathrm{Ind}_H^G \mathbb{Z}H = \mathbb{Z}G \otimes_{\mathbb{Z}H} \mathbb{Z}H \cong \mathbb{Z}G$$

$$\Rightarrow \boxed{H_*(H, \mathbb{Z}H) \cong H_*(G, \mathbb{Z}G)}.$$

"trivial", in some sense

$\rightsquigarrow$ $\begin{matrix} H_n(G,\mathbb{Z}G) \\ H_n(H,\mathbb{Z}H) \end{matrix} = 0 \quad \forall n > 0$

And,

if $|G:H| < \infty$ $\left[\Rightarrow \mathrm{Coind}_H^G \mathbb{Z}H \cong \mathrm{Ind}_H^G \mathbb{Z}H \cong \mathbb{Z}G\right]$

then $H^*(H, \mathbb{Z}H) \cong H^*(G, \mathbb{Z}G)$.

EXAMPLE $H=1$

$$H_*(H,M) \cong H_*(G, \mathrm{Ind}_H^G M)$$
$$H^*(H,M) \cong H^*(G, \mathrm{Coind}_H^G M)$$

Let $H = \{1\} \leq G$.
We have $\mathbb{Z}H \cong \mathbb{Z}$.

$\leadsto$ Let M be a $\mathbb{Z}$-module. So

- $H_*(1,M) \cong H_*(G, \mathbb{Z}G \otimes_{\mathbb{Z}} M)$
- $H^*(1,M) \cong H^*(G, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G, M))$

$$\begin{cases} H_n(1,M) = H^n(1,M) = 0 & \forall n>0 \\ H_0(1,M) = M_1 = M \\ H^0(1,M) = M^1 = M \end{cases}$$

$$\boxed{\begin{cases} H_n(G, \mathbb{Z}G \otimes_{\mathbb{Z}} M) = 0 & \forall n > 0 \\ H^n(G, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G, M)) = 0 & \forall n > 0 \end{cases}}$$

$\leadsto$ induced modules $\mathbb{Z}G \otimes_{\mathbb{Z}} M$ are H_*-acyclic;

$\leadsto$ co-induced modules $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G, M)$ are H^*-acyclic.

$\leadsto \begin{cases} H_n(G,-) \text{ is effaceable } \forall n>0 \\ H^n(G,-) \text{ is coeffaceable } \forall n>0 \end{cases}$

PROJECTIVE MODULE

$\Updownarrow$

$$\begin{tikzcd} & P \arrow[dl, dashed, "\bar{\psi}"'] \arrow[d, "\bar{\varphi}"] \\ M \arrow[r, two heads, "\pi"'] & \bar{M} \end{tikzcd}$$

- $\boxed{\mathrm{Hom}_R(P,-) \text{ is exact}}$

- Every free R-module is projective
- Every R-module is quotient of a projective R-module

- Every projective R-module is flat, i.e. $- \otimes_R P$ is exact.

INJECTIVE MODULE

$\Updownarrow$

$$\begin{tikzcd} & Q \\ M \arrow[ur, dashed, "\tilde{\psi}"] & \tilde{M} \arrow[l, hook', "i"] \arrow[u, "\tilde{\varphi}"'] \end{tikzcd}$$

- $\boxed{\mathrm{Hom}_R(-,Q) \text{ is exact}}$

- Every R-module can be embedded in an injective R-module

Def. Let G be a group, A be a $\mathbb{Z}G$-module.
We say that A is H_*-acyclic (resp. H^*-acyclic)
if $H_n(G,A) = 0 \quad \forall n>0$
(resp. if $H^n(G,A) = 0 \quad \forall n>0$).

$\boxed{P \text{ projective } \mathbb{Z}G\text{-module} \Rightarrow P \text{ is } H_*\text{-acyclic}}$

$\Leftrightarrow$ $H_n(G,-)$ is <u>effaceable</u> for $n>0$

$-\otimes_G P$ is exact

$\boxed{Q \text{ injective } \mathbb{Z}G\text{-module} \Rightarrow Q \text{ is } H^*\text{-acyclic}}$

$\Leftrightarrow$ $H_n(G,-)$ is <u>co-effaceable</u> for $n>0$

$\mathrm{Hom}_G(-,Q)$ is exact

H_* and H^* as functors of the coefficient module

Let F be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

$$\cdots \to F_2 \to F_1 \to F_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

$H_*(G,-)$ and $H^*(G,-)$ are covariant functors of the coefficient modules

$F \otimes_{\mathbb{Z}G} -$ and $\mathrm{Hom}_{\mathbb{Z}G}(F,-)$ are covariant

Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of G-modules.

$F \otimes_{\mathbb{Z}G} -$ and $\mathrm{Hom}_{\mathbb{Z}G}(F,-)$ are exact (because F is projective)

$$0 \to M' \otimes_G F \to M \otimes_G F \to M'' \otimes_G F \to 0$$

$$0 \to \mathrm{Hom}_G(F,M') \to \mathrm{Hom}_G(F,M) \to \mathrm{Hom}_G(F,M'') \to 0$$

are exact sequences of chain complexes.

$\Rightarrow$ For any exact sequence of G-modules

$$0 \to M' \to M \to M'' \to 0$$

and $\forall n$ there is a natural map

$$\partial : H_n(G, M'') \to H_{n-1}(G, M')$$

such that the long sequence

$$\cdots \to H_1(G,M) \to H_1(G,M'') \xrightarrow{\partial} H_0(G,M') \to H_0(G,M) \to H_0(G,M'') \to 0$$

is exact.

We say that $H_*(G,-)$ is homological

(We could define, in general, homological functors)

"Natural" means that

for any commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N'' \arrow[r] & N \arrow[r] & N'' \arrow[r] & 0
\end{tikzcd}$$

with exact rows, the square

$$\begin{tikzcd}
H_n(G, M'') \arrow[r, "\partial"] \arrow[d] & H_{n-1}(G, M') \arrow[d] \\
H_n(G, N'') \arrow[r, "\partial"] & H_{n-1}(G, N')
\end{tikzcd}$$

is commutative.

Similarly,

For any exact sequence of G-modules
$$0 \to M' \to M \to M'' \to 0$$
and $\forall n$ there is a natural map
$$\delta : H^n(G, M'') \longrightarrow H^{n+1}(G, M')$$
such that
$$\cdots \leftarrow H^1(G,M) \leftarrow H^1(G,M') \xleftarrow{\delta} H^0(G,M'') \leftarrow H^0(G,M) \leftarrow H^0(G,M') \leftarrow 0$$
is exact.

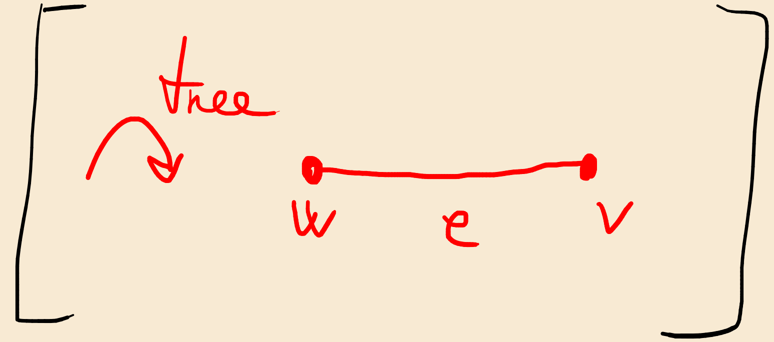
We say that $H^*(G,-)$ is cohomological

(We could define, in general, cohomological functors)

EXAMPLE (Back to the homology for amalgamated free products)

$$\begin{tikzcd} A \arrow[r, hook, "\alpha_2"] \arrow[d, hook, "\alpha_1"'] & G_2 \arrow[d, "\beta_2"] \arrow[ddr, bend left, "\gamma_2"] & \\ G_1 \arrow[r, "\beta_1"] \arrow[drr, bend right, "\gamma_1"'] & G \arrow[dr, dashed, "\varphi"] & \\ & & H \end{tikzcd}$$

$$G = G_1 *_A G_2$$



$\Rightarrow$ There exists an exact sequence of permutation modules

$$0 \to \mathbb{Z}[G/A] \to \mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2] \to \mathbb{Z} \to 0$$

Then we can obtain a long exact sequence

$$\cdots \to H_n(G, \mathbb{Z}[G/A]) \to H_n(G, \mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2]) \to H_n(G, \mathbb{Z}) \to \cdots$$

$$H_n(G, \mathbb{Z}[G/A]) \cong H_n(A) \qquad H_n(G, \mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2]) \cong H_n(G_1) \oplus H_n(G_2) \qquad H_n(G, \mathbb{Z}) \cong H_n(G)$$

by example $\mathbb{Z}$

using $H_*(G,-) \cong \mathrm{Tor}^G_*(\mathbb{Z},-)$

Theorem

As homological functor on the category of G-modules which is effaceable for all $n>0$, $H_*(G,-)$ is uniquely determined by the isomorphism

$$H_0(G,M) \cong M_G \quad \forall \text{ G-module } M.$$

Cotheorem

As cohomological functor on the category of G-modules which is coeffaceable for all $n>0$, $H^*(G,-)$ is uniquely determined by the isomorphism

$$H^0(G,M) \cong M^G \quad \forall \text{ G-module } M.$$

These are immediate consequences of the following general theorems

Theorem (7.3, Brown)
Let H be a homological functor such that H_n is effaceable $\forall n>0$.
If T is an arbitrary ∂-functor and $\varphi_0 : T_0 \to H_0$ is a natural transformation, then φ_0 extends uniquely to a map $\varphi: T \to H$ of ∂-functors. This map φ is an isomorphism if and only if the following three conditions hold:

(i) φ_0 is an isomorphism
(ii) T is homological
(iii) T_n is effaceable $\forall n>0$

Brown, thm. 7.5 $\rightsquigarrow$ the same, but for cohomological functors.

EXAMPLE

We can use thm 7.3 to prove

PROPOSITION 2

Let M and N be G-modules. If M is $\mathbb{Z}$-torsion-free, then

$$\mathrm{Tor}_*^G(M,N) \cong H_*(G, M \otimes N).$$

M $\mathbb{Z}$-torsion free $\Rightarrow$ M is $\mathbb{Z}$-flat $\Rightarrow$ $M \otimes -$ is exact

(i)

$$\Rightarrow H_*(G, M\otimes -) = F \otimes_{\mathbb{Z}G} (M \otimes -)$$

is homological

projective resolution of $\mathbb{Z}$

$$\mathrm{Tor}_*^G(M,-) = H_*(F' \otimes_{\mathbb{Z}G} -)$$ is homological

projective resolution of M

(ii) Both functors are effaceable

diagonal action of G on $M \otimes N$

(iii) $H_0(G, M\otimes N) \underset{\text{def.}}{\cong} (M \otimes N)_G \cong M \otimes_G N \underset{\text{def.}}{=} \mathrm{Tor}_0^G(M,N)$.

Dimension Shifting

Given a $\mathbb{Z}G$-module M, choose an H_*-acyclic module $\bar{M}$ s.t.

$$\bar{M} \twoheadrightarrow M.$$

e.g. $\bar{M} = \mathbb{Z}G \otimes_{\mathbb{Z}} M$

cf example $H=1$

Let $K = \ker(\bar{M} \twoheadrightarrow M)$.

Then we have an exact sequence $0 \to K \to \bar{M} \to M \to 0$.

But $H_*(G,-)$ is homological,

$$\cdots \to H_n(G,K) \to \underbrace{H_n(G,\bar{M})}_{=0\ \forall n>0} \to H_n(G,M) \xrightarrow{\partial} H_{n-1}(G,K) \to \underbrace{H_{n-1}(G,\bar{M})}_{=0\ \forall n>0} \to \cdots$$

$$\Rightarrow \quad H_n(G,M) \cong \begin{cases} H_{n-1}(G,K) & \text{if } n \geq 2 \\ \ker\big(H_0(G,K) \to H_0(G,\bar{M})\big) & \text{if } n=1 \end{cases}$$

Similarly for cohomology:

Given a $\mathbb{Z}G$-module M, choose an H^*-acyclic module $\tilde{M}$

$$M \hookrightarrow \tilde{M}$$

e.g. $\tilde{M} = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G, M)$

Let $C = \operatorname{coker}(M \hookrightarrow \tilde{M})$

cf example $H=1$

Then we have an exact sequence $0 \to M \to \tilde{M} \to C \to 0$.

But $H_*(G,-)$ is cohomological

$$\Rightarrow \quad H^n(G,M) \cong \begin{cases} H^{n-1}(G,C) & \text{if } n \geq 2 \\ \operatorname{coker}(H^0(G,\tilde{M}) \to H^0(G,C)) & \text{if } n = 1 \end{cases}$$

« Heuristically homology / cohomology theory is determined by H_0 / H^0 »

H_* and H^* as functors of two variables

↳ Back to Eckmann-Shapiro lemma

¿ Can we give explicitly the canonical isomorphisms in the lemma ?

Let $\mathcal{C}$ be the following category:

$ob(\mathcal{C})$: pairs (G, M), G group, M G-module

$hom(\mathcal{C})$: pairs $(\alpha, f) : (G, M) \longrightarrow (G', M')$

where $\alpha : G \to G'$

$f : M \to M'$

such that $f(gm) = \alpha(g) f(m)$ $\forall\, g \in G,\ m \in M$.

F projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$

F' " " " $\mathbb{Z}$ over $\mathbb{Z}G'$

$\tau : F \to F'$ chain map s.t. $\tau(gx) = \alpha(g)\tau(x)$ $\forall g \in G$, $x \in F$

$\Rightarrow (\tau, f) : F \otimes_G M \longrightarrow F' \otimes_G M'$

$\rightsquigarrow (\tau, f)$ induces a well-defined map

$$\boxed{(\alpha, f)_* : H_*(G, M) \longrightarrow H_*(G', M')}$$

$\Rightarrow H_*$ becomes a covariant functor on $\mathcal{C}$.

In case $M = M'$ & $f = id_M$, we can write

$$\boxed{\alpha_* : H_*(G, M) \longrightarrow H_*(G', M) .}$$

Similar situation for cohomology.

Let $\mathcal{D}$ be the category with the same objects (G,M) as $\mathcal{C}$, but where a map $(\alpha, f) : (G,M) \to (G',M')$ is now given by

$\alpha : G \to G'$

$f : M' \to M$ such that $f(\alpha(g)m') = g \cdot f(m') \quad \forall g \in G, m' \in M'$

We have a chain map

$$\mathrm{Hom}(\tau, f) : \mathrm{Hom}_{G'}(F', M') \longrightarrow \mathrm{Hom}_G(F, M)$$

$$\rightsquigarrow \quad \boxed{(\alpha, f)^* : H^*(G', M') \longrightarrow H^*(G, M)}$$

↳ H^* is a contravariant functor on $\mathcal{D}$

For $M = M'$ & $f = \mathrm{id}$ $\quad \boxed{\alpha^* : H^*(G', M) \longrightarrow H^*(G, M)}$.

Canonical isomorphisms in Eckmann-Shapiro lemma

Take $H \le G$, let $\alpha : H \hookrightarrow G$.

Let M be an H-module.

We have canonical H-maps

[use the same project. res. F of $\mathbb{Z}$ over $\mathbb{Z}G$, and choose $\tau : F \to F$ identity]

$$i : M \longrightarrow \mathrm{Ind}_H^G M = \mathbb{Z}G \otimes_H M, \qquad i(m) = 1 \otimes m$$

$$\pi : \mathrm{Coind}_H^G M = \mathrm{Hom}_H(\mathbb{Z}G, M) \longrightarrow M, \qquad \pi(f) = f(1)$$

Then

$$\boxed{\begin{aligned} (\alpha, i)_* &: H_*(H, M) \xrightarrow{\cong} H_*(G, \mathrm{Ind}_H^G M) \\ (\alpha, \pi)^* &: H^*(G, \mathrm{Coind}_H^G M) \xrightarrow{\cong} H^*(H, M). \end{aligned}}$$

Another example - CONJUGATION

$H \leq G$, let M be a G-module. Fix $g \in G$:

$$c(g) : (H, M) \longrightarrow (gHg^{-1}, M)$$
$$(h, m) \longmapsto (ghg^{-1}, gm)$$

Choose a projective resolution F of $\mathbb{Z}$ over $\mathbb{Z}G$ and use F to compute the homology of H and gHg^{-1}.

$\hookrightarrow$ $\tau(x) = gx \quad \forall x \in F \Rightarrow \tau(hx) = ghx = (ghg^{-1})gx = (ghg^{-1})\tau(x)$

$\rightsquigarrow$ $c(g)_* : H_*(H, M) \longrightarrow H_*(gHg^{-1}, M)$ is induced by the chain map $F \otimes_H M \longrightarrow F \otimes_{gHg^{-1}} M$ given by $x \otimes m \mapsto gx \otimes gm$

$$\boxed{\text{If } h \in H, \text{ then } c(h)_* z = z \quad \forall z \in H_*(H, M).}$$

$\hookrightarrow$ if $H \trianglelefteq G$, the conj. action of G on (H, M) induces an action of G/H on $H_*(H, M)$ [and the same for cohomology].

The Transfer Map

$H \leq G$, $\alpha : H \hookrightarrow G$, M: G-module.

So there are maps

$$\alpha^* : H^*(G,M) \longrightarrow H^*(H,M)$$

$$\alpha_* : H_*(H,M) \longrightarrow H_*(G,M)$$

" given by the functorial properties of H^* and H_* "

We will write $\alpha^* = \mathrm{res}^G_H$, $\alpha_* = \mathrm{cor}^G_H$.

The purpose is to show that, if $|G:H| < \infty$, then there are maps "going in the other direction".

↳ $\boxed{\text{REMARK: these maps are not induced by maps in the categories } \mathcal{C} \text{ and } \mathcal{D} \text{ as before.}}$

So $H \leq G$, $|G:H| < \infty$ $\Rightarrow$ $\mathrm{Coind}_H^G M \cong \mathrm{Ind}_H^G M$

There is a canonical surjection

① $\mathrm{Ind}_H^G M = \mathbb{Z}G \otimes_H M \twoheadrightarrow M$, $\quad r \otimes m \mapsto rm$

and a canonical injection

② $M \longrightarrow \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M) = \mathrm{Coind}_H^G M$, $\quad m \mapsto (s \mapsto sm)$

Applying $H_*(G,-)$ and using Shapiro's lemma:

① $\begin{cases} H_*(G, \mathbb{Z}G \otimes_{\mathbb{Z}H} M) \cong H_*(H,M) \\ H_*(G,M) \end{cases}$ $\Rightarrow$ we obtain a map $H_*(H,M) \to H_*(G,M)$ (this is α_*)

② $\begin{cases} H_*(G,M) \\ H_*(G, \overbrace{\mathrm{Hom}_H(\mathbb{Z}G,M)}^{\cong\, \mathrm{Ind}_H^G M}) \cong H_*(H,M) \end{cases}$ $\Rightarrow$ we obtain a map $H_*(G,M) \to H_*(H,M)$ (this is the transfer map)

Similarly for cohomology:

$\boxed{H \leq G,\ |G:H| < \infty \quad \Rightarrow \quad \mathrm{Coind}_H^G M \cong \mathrm{Ind}_H^G M}$

There is a canonical surjection

① $\mathrm{Ind}_H^G M = \mathbb{Z}G \otimes_H M \twoheadrightarrow M$, $\quad r \otimes m \mapsto rm$

and a canonical injection

② $M \longrightarrow \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M) = \mathrm{Coind}_H^G M$, $\quad m \mapsto (s \mapsto sm)$

Applying $H^*(G,-)$ and using Shapiro's lemma:

② $\begin{cases} H^*(G,M) \\ H^*(G, \mathrm{Coind}_H^G M) \cong H^*(H,M) \end{cases}$ $\Rightarrow$ we obtain a map $H^*(G,M) \longrightarrow H^*(H,M)$ (this is our α^*)

① $\begin{cases} H^*(G, \overbrace{\mathrm{Ind}_H^G M}^{\cong \mathrm{Coind}_H^G M}) \cong H^*(H,M) \\ H^*(G,M) \end{cases}$ $\Rightarrow$ we obtain a map $H^*(H,M) \longrightarrow H^*(G,M)$ (this is the transfer map)

NOTE. There are also other ways of explaining the existence of the transfer map.

HISTORICAL NOTE. The name "transfer" comes from the special case $H_1(G) \to H_1(H)$, which is a map on the abelianizations $G_{ab} \to H_{ab}$ that goes back to Schur

$$g[G,G] \mapsto \prod_{g' \in E} g'g\,\overline{g'g}^{-1}\,[H,H]$$

E set of representatives for the right cosets Hg

$\bar{x}$ = representative for Hx

NOTATION. The transfer map is still denoted by:

$$\mathrm{cor}_H^G : H^*(H,M) \to H^*(G,M)$$

$$\mathrm{res}_H^G : H_*(G,M) \to H_*(H,M) .$$

Properties

i) $K \leq H \leq G$ with $|G:K| < \infty$. Then

$$\mathrm{cor}_K^G = \mathrm{cor}_H^G \circ \mathrm{cor}_K^H \quad \text{and} \quad \mathrm{res}_K^G = \mathrm{res}_K^H \circ \mathrm{res}_H^G .$$

ii) $|G:H| < \infty$ and $z \in H(G,M)$:

$$\mathrm{cor}_H^G \, \mathrm{res}_H^G \, z = |G:H| \cdot z .$$

iii) $H \leq G$, $K \leq G$: $|G:H| < \infty$ if we consider cohomology
$|G:K| < \infty$ if we consider homology

Then

notation for conjugacy action

$$\mathrm{res}_K^G \, \mathrm{cor}_H^G \, z = \sum_{g \in E} \mathrm{cor}^K_{K \cap gHg^{-1}} \, \mathrm{res}^{gHg^{-1}}_{K \cap gHg^{-1}} \, gz$$

$\forall z \in H(H,M)$, where E is a set of representatives for the double cosets KgH.

In particular, if $H \triangleleft G$ and $|G:H| < \infty$, then

$$\mathrm{res}_H^G \, \mathrm{cor}_H^G \, z = \sum_{g \in G/H} gz .$$

PROOF OF

$$\boxed{\begin{array}{l} \text{ii) } |G:H| < \infty \text{ and } z \in H(G,M): \\ \qquad\qquad \mathrm{cor}_H^G \, \mathrm{res}_H^G z = |G:H| \cdot z \end{array}}$$

For instance, let $z \in H^*(G,M)$.
We want to apply

$$H^*(G,M) \xrightarrow{\mathrm{res}_H^G} H^*(H,M) \xrightarrow{\mathrm{cor}_H^G} H^*(G,M) .$$

But remember that

$$M \longrightarrow \mathrm{Ind}_H^G M \longrightarrow M$$

turns out to be given by $m \longmapsto \sum_{g \in G/H} g \otimes g^{-1} m \longmapsto |G:H| \cdot m$.

Thus also $\mathrm{cor}_H^G \, \mathrm{res}_H^G z = |G:H| \cdot z$.

#

Proposition

Let M be a G-module and $H \leq G$ subgroup of finite index such that
$$H^n(H,M) = 0 \quad \text{for some } n.$$
Then $H^n(G,M)$ is annihilated by $|G:H|$.

In particular, if $|G:H|$ is invertible in M (i.e., if the multiplication by $|G:H|$ is an isomorphism)

then $H^n(G,M) = 0$.

$\rightsquigarrow$ Take $n>0$, $H=\{1\}$

$$\left(\begin{array}{l} H^n(1,M) = 0 \quad \forall n>0 \\ |G:1| = |G| \end{array} \right)$$

Corollary

- If G is finite, then $H^n(G,M)$ is annihilated by $|G|$ $\forall n>0$.
- If $|G|$ is invertible in M (e.g., if M is a $\mathbb{Q}G$-module) then $H^n(G,M) = 0$ $\forall n>0$.

It follows that, for $|G|<\infty$, $H^n(G,M)$ admits a primary decomposition:

$$\boxed{H^n(G,M) = \bigoplus_p H^n(G,M)_{(p)}}$$

where p ranges over the primes dividing $|G|$ and $H^n(G,M)_{(p)}$ is the p-primary component of $H^n(G,M)$.

Let p be a prime. If H is p-Sylow subgroup, then

$$H^n(G,M)_{(p)} \hookrightarrow H^n(H,M)$$

[in this case, $|G:H|$ is coprime with p $\rightsquigarrow$ $\mathrm{cor}_H^G \mathrm{res}_H^G$ induces iso on $H^n(G,M)_{(p)}$]

Moreover, $\underset{\uparrow \, p\text{-Sylow}}{H} \triangleleft G \implies \boxed{H^n(G,M)_{(p)} \cong H^n(H,M)^{G/H}}$.

the invariants w.r.t. the action induced by conjugation $G/H \curvearrowright H^*(H,M)$.