Cohomology of finite groups

29 June 2021

Let G be a finite group. The goal of this talk is to illustrate the similarities between H_* and H^*. Tate showed that there is a cohomology theory $\hat{H}^i$ ($i \in \mathbb{Z}$) such that

$$\begin{array}{cccccc} \cdots & H_2 & H_1 & H^1 & H^2 & \cdots \\ & \| & \| & \| & \| & \\ \cdots & \hat{H}^{-3} & \hat{H}^{-2} & \hat{H}^1 & \hat{H}^2 & \cdots \end{array}$$

and there is an exact sequence

$$0 \longrightarrow \hat{H}^{-1} \longrightarrow H_0 \xrightarrow{N} H^0 \longrightarrow \hat{H}^0 \longrightarrow 0$$

where N is the norm map. We will discuss the usefulness of *Tate cohomology theory* $\hat{H}^*$ by discussing

- periodic cohomology

Notation

1. Let R be a ring. If M a right R-module and N is a left R-module, we have the tensor product $M \otimes_R N$ given by the quotient of $M \otimes_{\mathbb{Z}} N$ obtained by introducing the relations $mr \otimes n = m \otimes rn$ $(m \in M, r \in R, n \in N)$.
2. In case $R = \mathbb{Z}G$, we can make a left $\mathbb{Z}G$-module M (also called G-module) into a right $\mathbb{Z}G$-module by setting $mg := g^{-1}m$. Thus, we can make sense out of the tensor product $M \otimes_{\mathbb{Z}G} N$ for M and N both left $\mathbb{Z}G$-modules.
3. Recall the"invariant" $M^G := \{m \in M : gm = m, \forall g \in G\}$ and the "co-invariant" $M_G := M/IM \cong \mathbb{Z} \otimes_{\mathbb{Z}G} M$ where I is the augmentation ideal of $\mathbb{Z}G$.
4. Note that $M \otimes_{\mathbb{Z}G} N = (M \otimes N)_G$ where G acts "diagonally" on $M \otimes N$, i.e. $g(m \otimes n) = gm \otimes gn$. Thus, $M \otimes_G N = N \otimes_G M$.
5. If M and N are both left $\mathbb{Z}G$-modules, we have a "diagonal" action of G on $Hom_{\mathbb{Z}}(M, N)$ by setting $(gu)(m) := g \cdot u(g^{-1}m)$ for $g \in G, u \in Hom_{\mathbb{Z}}(M, N), m \in M$. Note that

$$Hom_{\mathbb{Z}G}(M, N) = Hom_{\mathbb{Z}}(M, N)^G$$

since $gu = u$ if and only if u commutes with the action of g.

Remark (Warning)

If M is both a left and right $\mathbb{Z}G$-module, the tensor product $M \otimes_{\mathbb{Z}G} N$ should mean $M_{\mathbb{Z}G} \otimes_{\mathbb{Z}G} N$ (rather than ${}_{\mathbb{Z}G}M \otimes_{\mathbb{Z}G} N$.).

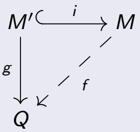
Definition

An injection $i : M' \hookrightarrow M$ of G-modules (i.e. *left* $\mathbb{Z}G$-modules) is called *admissible* if it is a split injection when regarded as a $\mathbb{Z}$-module, i.e. there is a $\mathbb{Z}$-module homomorphism $\pi : M \to M'$ such that $\pi i = \mathrm{id}$.

Definition

A G-module Q is called *relatively injective* if it satisfies the following equivalent conditions:

1. Every mapping problem (with i an admissible injection and g any map of G-modules)



   can be solved, i.e. there exists a G-module map f such that $f \circ i = g$.
2. The contravariant functor $Hom_G(-, Q)$ takes admissible injections of G-modules to surjections of $\mathbb{Z}$-modules.

Recall that the induced module $\mathrm{Ind}^G(N) := \mathbb{Z}G \otimes_{\mathbb{Z}} N$ and the co-induced module $\mathrm{Coind}^G N := Hom_{\mathbb{Z}}(\mathbb{Z}G, N)$ for any $\mathbb{Z}$-module N. Clearly, $\mathrm{Ind}^G(N)$ and $\mathrm{Coind}^G N$ are G-modules.

Lemma

For any $\mathbb{Z}$-module N, the G-module $\mathrm{Coind}^G N$ is relatively injective.

Proof.

Let M be a G-module and N be a $\mathbb{Z}$-module. Recall that there is an isomorphism

$$Hom_G(M, Hom_{\mathbb{Z}}(\mathbb{Z}G, N)) \cong Hom_{\mathbb{Z}}(M, N)$$

which is natural in M and N. Thus, we obtain a natural isomorphism

$$Hom_G(-, \mathrm{Coind}^G N) \cong Hom_{\mathbb{Z}}(\mathrm{Res}_{\mathbb{Z}}^{\mathbb{Z}G}(-), N)$$

where $\mathrm{Res}_{\mathbb{Z}}^{\mathbb{Z}G}(-)$ is the functor of restriction of scalars. The functor $Hom_{\mathbb{Z}}(\mathrm{Res}_{\mathbb{Z}}^{\mathbb{Z}G}(-), N)$ takes admissible injections of G-modules to surjections of $\mathbb{Z}$-modules. □

We can always embed a G-module into a relatively injective one.

Construction

For any G-module M, there is a canonical admissible injection

$$i : M \hookrightarrow \overline{M}.$$

Define $\overline{M} := \mathrm{Coind}^G \mathrm{Res}^{\mathbb{Z}G}_{\mathbb{Z}}(M)$, i.e. the G-module $Hom_{\mathbb{Z}}(\mathbb{Z}G, M)$. The map i is defined by

$$i : M \hookrightarrow Hom_{\mathbb{Z}}(\mathbb{Z}G, M) : m \mapsto (g \mapsto gm)$$

which is split as a $\mathbb{Z}$-module. By the previous lemma, we know $\overline{M}$ is relatively injective.

Lemma

Assume that G is finite. Let M be a G-module. If M is finitely generated free as a $\mathbb{Z}$-module, then $\overline{M}$ is a finitely generated free as a G-module.

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Proof.

Write down a G-module isomorphism $\varphi : \mathbb{Z}G \otimes_{\mathbb{Z}} M \to Hom_{\mathbb{Z}}(\mathbb{Z}G, M) = \overline{M}$ by

$$\varphi(g \otimes m)(g') := \begin{cases} m & \text{if } g'g = 1 \\ 0 & \text{if } g'g \neq 1 \end{cases}$$

It is inverse is given by $\psi(f) = \sum_{g \in G} g \otimes f(g^{-1})$ which is well-defined as G is finite. If M is finitely generated free as $\mathbb{Z}$-module, then $\mathbb{Z}G \otimes_{\mathbb{Z}} M$ is clearly finitely generated free as a left $\mathbb{Z}G$-module. □

Remark

Note that $\text{Ind}^G M := \mathbb{Z}G \otimes_{\mathbb{Z}} M$. The proof above shows that

$$\text{Ind}^G M \cong \text{Coind}^G M$$

if G is finite.

Corollary

Suppose that G is finite. Any projective $\mathbb{Z}G$-module is relatively injective.

Proof.

By definition of relatively injective G-modules, a direct summand of a relatively injective is relatively injective. Thus, we are reduced to the case of free $\mathbb{Z}G$-modules. If F is a free $\mathbb{Z}G$-module, then we have

$$F \cong \mathbb{Z}G \otimes_{\mathbb{Z}} F' = \mathrm{Ind}^G F'$$

for some free $\mathbb{Z}$-module F'. We have seen that $\mathrm{Ind}^G F' \cong \mathrm{Coind}^G F'$. We also know that $\mathrm{Coind}^G F'$ is relatively injective. □

Definition

A *relatively injective resolution* of a G-module M is a non-negative cochain complex $Q^\bullet$ of relative injective G-modules, together with a quasi-isomorphism $\eta : M \to Q^\bullet$ such that the augmented complex

$$0 \to M \to Q^0 \to Q^1 \to \cdots$$

is acyclic and admissible.

Construction (Relatively injective resolution)

Note that there exists a *canonical* relatively injective resolution. Take $Q^0 := \overline{M}$, and we have a canonical admissible injections $\eta : M \hookrightarrow Q^0$. Take the cokernel $\operatorname{coker}(\eta)$ of η. Set $Q^1 := \overline{\operatorname{coker}(\eta)}$ and take the canonical admissible injection $\operatorname{coker}(\eta) \hookrightarrow Q^1$... Do this inductively. We get a relatively injective resolution

$$0 \to M \to Q^0 \to Q^1 \to \cdots .$$

Proposition

Any two relatively injective resolutions of M are homotopy equivalent.

Proposition

Let G be a finite group. If M is a $\mathbb{Z}G$-module and finitely generated free as a $\mathbb{Z}$-module, then M admits a relatively injective resolution $\eta : M \to Q^\bullet$ such that each Q^n is finitely generated free $\mathbb{Z}G$-modules.

Proof.

This was done for Q^0. Note that $\eta : M \hookrightarrow Q^0$ is split as $\mathbb{Z}$-modules, and the cokernel coker(η) is also finite generated free as $\mathbb{Z}$-module. Thus, $Q^1 = \overline{\text{coker}(\eta)}$ is finitely generated free as a $\mathbb{Z}G$-module. Arguing this way, we see each Q^n is finitely generated free as $\mathbb{Z}G$-modules. □

Remark (Conclusion)

If G is finite, the canonical relatively injective resolution of the G-module $\mathbb{Z}$ provides a "backward" finitely generated free resolution (over $\mathbb{Z}G$)

$$0 \to \mathbb{Z} \to Q^0 \to Q^1 \to \cdots .$$

Definition

A *complete resolution* for the finite group G is an acyclic chain complex of projective G-modules

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_{-1} \longrightarrow P_{-2} \longrightarrow \cdots$$

such that the map $P_0 \to P_{-1}$ factors as a composition

$$P_0 \twoheadrightarrow \mathbb{Z} \rightarrowtail P_{-1}$$

where we regard $\mathbb{Z}$ as a G-module with trivial action. A complete resolution $P_\bullet$ is called of *finite type* if P_i are all finite type projective G-modules.

Proposition

Any two complete resolutions are homotopy equivalent.

Example

Suppose that G acts freely on a CW-complex X which is homeomorphic to the sphere S^{2k-1} (this action permutes cells of X). Recall that

$$H_i(X) \cong H_i(S^{2k-1}) = \begin{cases} \mathbb{Z} & i = 0 \text{ or } 2k-1 \\ 0 & \text{if otherwise} \end{cases}.$$

Then the cellular chain complex takes the form

$$0 \to \mathbb{Z} \to C_{2k-1}(X) \to \cdots \to C_0(X) \to \mathbb{Z} \to 0$$

which induces an infinite periodic chain complex of free G-modules

$$\cdots \to C_0(X) \to C_{2k-1}(X) \to \cdots \to C_0(X) \to C_{2k-1}(X) \to \cdots.$$

Here, the red arrows are precisely the composition

$$C_0(X) \to \mathbb{Z} \to C_{2k-1}(X).$$

(Note that $C_n(X)$ is a free G-module with one basis element for every G-orbit of cells.) This periodic chain complex gives a complete resolution for G.

Proposition

Let G be a finite group and M be a left $\mathbb{Z}G$-module, then there is a left $\mathbb{Z}G$-module isomorphism

$$\psi : Hom_{\mathbb{Z}}(M, \mathbb{Z}) \to M^* = Hom_{\mathbb{Z}G}(M, \mathbb{Z}G)$$

given by $\psi(u)(m) = \sum_{g \in G} u(g^{-1}m)g$ for $u \in Hom(M, \mathbb{Z}), m \in M$.

Proof.

Recall that

- The action of G on $Hom_{\mathbb{Z}}(M, \mathbb{Z})$ is given by $(gu)(m) = u(g^{-1}m)$ for $g \in G, u \in Hom_{\mathbb{Z}}(M, \mathbb{Z}), m \in M$. In other words, $Hom_{\mathbb{Z}}(M, \mathbb{Z})$ is the left $\mathbb{Z}G$-module via the diagonal action, where G acts trivially on $\mathbb{Z}$.
- The action of G on M^* is described as follows. The left $\mathbb{Z}G$-module structure of M provides a canonical right $\mathbb{Z}G$-module structure on M^*, and we convert M^* to a left module by $g \mapsto g^{-1}$. The formula is given by $(gu)(m) = u(m)g^{-1}$ for $g \in G, u \in M^*, m \in M$

One check that ψ is a left $\mathbb{Z}G$-module homomorphism. Recall that $\mathbb{Z}G \cong Hom_{\mathbb{Z}}(\mathbb{Z}G, \mathbb{Z})$ and

$$Hom_{\mathbb{Z}G}(M, \mathbb{Z}G) \cong Hom_{\mathbb{Z}G}(M, Hom_{\mathbb{Z}}(\mathbb{Z}G, \mathbb{Z})) \cong Hom_{\mathbb{Z}}(M, \mathbb{Z}).$$

Note that ψ is the underlying map coming from Ind $\cong$ Coind. □

Proposition

Let G be a finite group. If $\epsilon : P \to \mathbb{Z}$ is a finite type projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, then $\epsilon^* : \mathbb{Z}^* = \mathbb{Z} \to P^*$ is a backward finite type projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Moreover, any finite type backward projective resolution is obtained in this way.

Proof.

The augmented chain complex of left $\mathbb{Z}G$-modules $\cdots \to P_1 \to P_0 \to \mathbb{Z} \to \cdots$ is contractible as a complex of $\mathbb{Z}$-modules, since any acyclic complex of free $\mathbb{Z}$-modules is contractible. It remains to be contractible when the duality functor $(-)^* = Hom_{\mathbb{Z}}(-, \mathbb{Z})$ is applied. Hence, the augmented complex of $\epsilon^* : \mathbb{Z}^* = \mathbb{Z} \to P^*$ is acyclic.
Note that if $\mathbb{Z} \to Q$ is a backward finite type projective resolution, its dual $Q^* \to \mathbb{Z}$ is a finite type projective resolution. Note also that Q^{**} is identified with Q. □

Remark

Finite type complete resolutions do exist by the canonical relatively injective resolution and its dual.

Definition

The *Tate cohomology* of a group G with coefficients in a G-module M is defined by

$$\hat{H}^i(G, M) := H^i(Hom_G(P, M))$$

for all $i \in \mathbb{Z}$, where P is a complete resolution.

Proposition

Let G be a finite group and M a G-module. Then, there is an identification

$$\hat{H}^i(G, M) = \begin{cases} H^i(G, M) & i > 0 \\ H_{-i-1}(G, M) & i < -1 \end{cases}$$

and there is an exact sequence

$$0 \to \hat{H}^{-1}(G, N) \to H_0(G, M) \xrightarrow{N} H^0(G, M) \to \hat{H}^0(G, M) \to 0$$

where N is the norm map $N : M_G \to M^G$.

Proposition

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where N is the norm map $N : M_G \to M^G$.

Remark

To understand the above statement, recall that

$$H_0(G,M) \cong M_G := M/IM \cong \mathbb{Z} \otimes_{\mathbb{Z}G} M \quad \text{and} \quad H^0(G,M) \cong M^G := Hom_G(\mathbb{Z}, M)$$

where I is the augmented ideal, i.e. the kernel of the map $\epsilon : \mathbb{Z}G \to \mathbb{Z} : g \mapsto 1$. The norm map

$$N : M_G \to M^G$$

is given by $\sum_{g \in G} g$.

Proof of the proposition.

Start with a complete resolution (or may be the canonical complete resolution)

$$\cdots \to P_1 \to P_0 \longrightarrow P_{-1} \to P_{-2} \to \cdots$$
$$P_0 \twoheadrightarrow \mathbb{Z} \rightarrowtail P_{-1}$$

Taking the dual $Hom_G(-, M)$, we get a complex

$$\cdots \to C^{-2} \xrightarrow{\delta^{-2}} C^{-1} \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \to \cdots$$

where $C^i = Hom_G(P_i, M)$. Note that the map δ^{-1} induces a commutative diagram

$$\begin{array}{ccc} C^{-1} & \xrightarrow{\delta^{-1}} & C^0 \\ \downarrow & & \uparrow \\ \operatorname{coker}(\delta^{-2}) & \xrightarrow{\alpha} & \ker(\delta^0) \end{array}$$

where the map α fits into an exact sequence

$$0 \to H^{-1}(C^\bullet) \to \operatorname{coker}(\delta^{-2}) \xrightarrow{\alpha} \ker(\delta^0) \to H^0(C^\bullet) \to 0.$$

By definition, we have $\hat{H}^i(G, M) := H^i(C^\bullet)$.

Proof of the proposition (continue).

Moreover, $H^i(G,M) = H^i(C^\bullet) = \hat{H}^i(G,M)$ for $i > 0$ and $H^0(G,M) = \ker(\delta^0)$.
Next, we will show that

$$H^i(C^\bullet) = \hat{H}^i(G,M) \cong H_{-i-1}(G,M)$$

for $i < -1$ and $\mathrm{coker}(\delta^{-2}) \cong H_0(G,M)$.
Recall that the backward projective resolution $\mathbb{Z} \to P_{-1} \to P_{-2} \to \cdots$ can be identified with the dual $(\Sigma F_\bullet)^*$ for some projective resolution $\cdots \to F_1 \to F_0 \to \mathbb{Z}$ of finite type if G is finite and P is of finite type. Therefore, the double dual identification gives

$$C^{<0} = Hom_G(P_{<0}, M) = Hom_G((\Sigma F_\bullet)^*, M) \cong \Sigma F \otimes_G M$$

where $C^{<0}$ is the truncation $\cdots \to C^{-2} \to C^{-1} \to 0 \to \cdots$. It follows that

$$H^i(C^\bullet) \cong H_{-i}(\Sigma F_\bullet \otimes_G M) = H_{-i-1}(F_\bullet \otimes_G M) = H_{-i-1}(G,M)$$

for $i < -1$ and $\mathrm{coker}(\delta^{-2}) \cong H_0(G,M)$. Summarizing the above, we have an exact sequence

$$0 \to \hat{H}^{-1}(G,N) \to H_0(G,M) \xrightarrow{\alpha} H^0(G,M) \to \hat{H}^0(G,M) \to 0.$$

It remains to show that α is the norm map $N : M_G \to M^G$.

Proof of the proposition (continue).

It remains to show that $\alpha : H_0(G,M) \to H^0(G,M)$ is the norm map $N : M_G \to M^G$. We may assume $F_0 = P_0 = \mathbb{Z}G$ (since we can take P and F to be arbitrary finite type projective resolutions and $\mathbb{Z}G$ is projective). The diagram

$$\begin{array}{ccc} C^{-1} & \xrightarrow{\delta^{-1}} & C^0 \\ \big\downarrow\!\!\downarrow & & \big\uparrow \\ \mathrm{coker}(\delta^{-2}) & \xrightarrow{\alpha} & \ker(\delta^0) \end{array}$$

becomes (isomorphic to)

$$\begin{array}{ccc} \mathbb{Z}G \otimes_G M & \xrightarrow{N} & Hom_G(\mathbb{Z}G, M) \\ \big\downarrow\!\!\downarrow & & \big\uparrow \\ \mathbb{Z} \otimes_G M & \xrightarrow{N} & Hom_G(\mathbb{Z}, M) \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{N} & M \\ \big\downarrow\!\!\downarrow & & \big\uparrow \\ M_G & \xrightarrow{N} & M^G \end{array}$$

where on the left square the upper arrow is induced by $\mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \xrightarrow{\eta} \mathbb{Z}G$ and $\mathbb{Z}G \cong Hom_G(\mathbb{Z}G, \mathbb{Z}G)$. Here, the map $\eta : \mathbb{Z} \to \mathbb{Z}G$ is given by $\eta(1) = \Sigma_{g \in G} g$. □

Example

Let G be a finite group with $|G| = p$ and consider $M = \mathbb{Z}$ a G-module (G acts trivially on $\mathbb{Z}$). Then, $H^0(G, \mathbb{Z}) = M^G = \mathbb{Z}$ and $H_0(G, \mathbb{Z}) = M_G = \mathbb{Z}$. Now the exact sequence

$$0 \to \hat{H}^{-1}(G, M) \to H_0(G, M) \xrightarrow{N} H^0(G, M) \to \hat{H}^0(G, M) \to 0$$

becomes

$$0 \to 0 \to \mathbb{Z} \xrightarrow{N} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$$

Here, we compute $N(m) = \sum_{g \in G} gm = p \cdot m$ (since $gm = m$ for any $g \in G$). Therefore,

$$\hat{H}^{-1}(G, M) = 0 \text{ and } \hat{H}^0(G, M) \cong \mathbb{Z}/p\mathbb{Z}.$$

Let G be a finite group with $|G| = p$.

Theorem

Let M, N be G-modules. Then, there is a cup product

$$\cup : \hat{H}^p(G, M) \otimes \hat{H}^q(G, N) \to \hat{H}^{p+q}(G, M \otimes N)$$

with formal properties analogous to the cup product of the cohomology $H^*(G, N)$ discussed before.

Proposition (Duality theorem)

The cup product

$$\cup : \hat{H}^i(G, \mathbb{Z}) \otimes \hat{H}^{-i}(G, \mathbb{Z}) \to \hat{H}^0(G, \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$$

is a duality pairing, i.e. the induced map

$$\hat{H}^i(G, \mathbb{Z}) \to Hom(\hat{H}^{-i}(G, \mathbb{Z}), \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism.

Remark

The cup product makes $\hat{H}^*(G, \mathbb{Z})$ into an anti-commutative graded ring with identity $1 \in \hat{H}^0(G, \mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$.

Definition

A finite group G is said to have *periodic cohomology* if for some $d \neq 0$ there is an element $u \in \hat{H}^d(G, \mathbb{Z})$ which is invertible in the ring $\hat{H}^*(G, \mathbb{Z})$.

Proposition

Let G be a finite group with $p = |G|$. The following are equivalent:

1. *G has periodic cohomology*
2. *For some $d \neq 0$ there is an element $u \in H^d(G, \mathbb{Z})$ such that multiplication by u gives an isomorphism $u \cup - : \hat{H}^n(G, M) \xrightarrow{\cong} \hat{H}^{n+d}(G, M)$ for all $n \in \mathbb{Z}$ and all G-modules M.*
3. *There are integers n and d, with $d \neq 0$, such that $\hat{H}^n(G, M) \cong \hat{H}^{n+d}(G, M)$ for all G-modules M.*
4. *For some $d \neq 0$, $\hat{H}^d(G, \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$.*
5. *For some $d \neq 0$, $\hat{H}^d(G, \mathbb{Z})$ contains an element u of order p.*

Example

Suppose that G acts freely on a CW-complex X which is homeomorphic to the sphere S^{2k-1} (this action permutes cells of X). By the example above, we see

$$\hat{H}^n(G, M) \cong \hat{H}^{n+2k}(G, M)$$

for all G-modules M and all integers n. Thus, G has periodic cohomology.

Groups with periodic cohomology have been completely classified by the following theorem.

Theorem

The following are equivalent:

1. *G has periodic cohomology.*
2. *Every abelian subgroup of G is cyclic.*
3. *For every prime p, every elementary abelian p-subgroup of G has rank at most 1.*
4. *The Sylow subgroups of G are cyclic or generalized quaternion groups.*

- Recall that an *elementary abelian p-group* of rank $r \geq 0$ is a group isomorphic to $(\mathbb{Z}/p\mathbb{Z})^r = \mathbb{Z}/p\mathbb{Z} \times \cdots \times \mathbb{Z}/p\mathbb{Z}$ (r factors).
- Recall also that the *generalized quaternion group Q_{4m}* is defined to be the subgroup of the multiplicative group $\mathbb{H}^*$ generated by $x = e^{\pi i/m}$ and $y = j$ where $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ is the quaternion algebra. This group has a presentation $Q_{4m} = \langle x, y : x^m = y^2, x^{2m} = 1, yxy^{-1} = x^{-1} \rangle$.