The Homology of a Group I

Dominic Witt

Oberseminar Algebra und Geometrie

Heinrich-Heine-Universität Düsseldorf

20.04.2021

Contents

1. Homotopy

X, Y topological spaces, $f, g : X \to Y$ continuous maps

homotopy $H : X \times [0,1] \to Y$ continuous map with

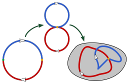
$H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$

We say: f and g are homotopic to each other.

Now consider continous maps $f : (S^n, a) \to (X, b)$.

homotopy group $\pi_n(X,b) =$ *group of the homotopy classes* $[f]$

where $[f_1] * [f_2] = (S^n \longrightarrow S^n \vee S^n \xrightarrow{f \vee g} X)$

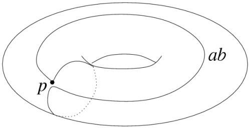


(Homotopy group - Wikipedia)

neutral element: $[n]$ where $n : S^n \to X, (x_1, \ldots, x_{n+1}) \mapsto b$

inverse element: $[\tilde{f}]$ where $\tilde{f}$ „goes the other way around"

Some examples:
(a) $\pi_n(\mathbb{R}^n \setminus \text{ pt.}) \cong \mathbb{Z}$
(b) $\pi_n(S^n) \cong \mathbb{Z}$
(c) $\pi_1(\mathbb{R}^2 \setminus \text{ 2 pts.}) \cong F_2$, free group with 2 generators
(d)$\pi_2(\mathbb{R}^3 \setminus \text{ 2 pts.}) \cong \mathbb{Z}^2$
(e) $\pi_n(X, b)$ abelian if $n \geq 2$
(f) $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$



(Fundamentalgruppe - Wikipedia)

The *first* homotopy group ($n = 1$) is called **fundamental group**.

$f : X \to Y$ continuous is called **homotopy equivalence** if there exists $g : Y \to X$ continous such that $[f \circ g] = [\mathrm{id}_Y]$ and $[g \circ f] = [\mathrm{id}_X]$.

So any homeomorphism is a homotopy equivalence.

For X, Y, homotopy equivalent, holds: $\pi_n(X, b) \cong \pi_n(Y, b')$ for all $n \in \mathbb{N}$.

2. Homology

A **chain complex** is a sequence of R-modules (ab. groups, ab. categories) $C = C_* = (C_n)_{n\in\mathbb{Z}}$ together with a sequence of R-module homomorphisms (group hom.s, morphisms) $d = (d_n : C_n \to C_{n-1})_{n\in\mathbb{Z}}$ with $d_n \circ d_{n+1} = 0, n \in \mathbb{Z}$.

The elements of C_n are called **n-chains**.
d is called **boundary operator** (or differential).

$$\underbrace{B_n(C,d)}_{\textbf{n}-\textbf{boundaries}} := \mathrm{im}(d_{n+1}) \subseteq \ker(d_n) =: \underbrace{Z_n(C,d)}_{\textbf{n}-\textbf{cycles}} \subseteq C_n$$

n-th homology group $H_n(C,d) := Z_n(C,d)/B_n(C,d)$

(quotient module, etc.)

homology $H = H_* = (H_n)_{n\in\mathbb{Z}}$

A **cochain complex** is a sequence of R-modules (ab. groups, ab. categories) $C = C^* = (C^n)_{n\in\mathbb{Z}}$ together with a sequence of R-module homomorphisms (group hom.s, morphisms) $d = (d^n : C^n \to C^{n+1})_{n\in\mathbb{Z}}$ with $d^n \circ d^{n-1} = 0$, $n \in \mathbb{Z}$.

The elements of C^n are called **n-cochains**.

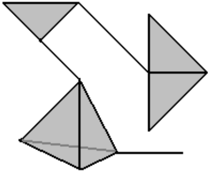
$$\underbrace{B^n(C,d)}_{\textbf{n}-\textbf{coboundaries}} := \mathrm{im}(d^{n-1}) \subseteq \ker(d^n) =: \underbrace{Z^n(C,d)}_{\textbf{n}-\textbf{cocycles}} \subseteq C^n$$

n-th cohomology group $H^n(C,d) := Z^n(C,d)/B^n(C,d)$

cohomology $H = H^* = (H^n)_{n\in\mathbb{Z}}$

3. Simplicial homology

X (geometrical) **simplicial complex** (union of simplices $\subseteq \mathbb{R}^d \dots$)



(Simplizialkomplex - Wikipedia)

A topological space is called **triangulable** if it is homeomorphic to a simplicial complex.

Any differentiable manifold is triangulable.

$C_n, n \in \mathbb{N}_0$, free $\mathbb{Z}$-module generated by the n-dimensional simplices of X $\quad (C_n \cong \mathbb{Z}^m$, m possibly infinite)

$$d_n : C_n \to C_{n-1},$$

$$(c_0, c_1, \ldots, c_n) \mapsto \sum_{i=0}^{n} (-1)^i (c_0, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n)$$

$$\Rightarrow d_{n-1} \circ d_n = 0$$

So (C, d) is a chain complex.

X any topological space

$C_n = C_n(X), n \in \mathbb{N}_0$, free $\mathbb{Z}$-module generated by the continuous maps $\sigma : \Delta_n \to X$, where Δ_n is the n-dimensional unit simplex (singular n-simplices σ)

$$d_n : C_n(X) \to C_{n-1}(X),$$

$$\sigma \mapsto \sum_{i=0}^{n} (-1)^i \, \sigma_{| \, (c_0, \dots, c_{i-1}, c_{i+1}, \dots, c_n)}$$

$$\Rightarrow d_{n-1} \circ d_n = 0$$

n-th homology group of X $\quad H_n(X) := H_n(C, d)$

Example:
For $X = S^1 \times S^1$ we have
$H_0(X) = \mathbb{Z},\ H_1(X) = \mathbb{Z}^2,\ H_2(X) = \mathbb{Z},\ H_i(X) = 0, i > 2.$

For X, Y, homotopy equivalent, holds: $H_n(X) \cong H_n(Y)$
for all $n \in \mathbb{N}_0$.

$C_n(X), n \in \mathbb{N}_0$, $\mathbb{Q}$-vector space generated by the maps $\sigma : \Delta_n \to X$
$\to$ **betti numbers** $b_i(X) = \dim_{\mathbb{Q}} H_i(X, \mathbb{Q}), i \in \mathbb{N}_0$

$b_0(X)$: number of path connected components
$b_1(X)$: number of *2-dimensional holes*
$b_2(X)$: number of *3-dimensional holes*
(example $X = S^1 \times S^1$ above ...)

- 1935/1936: *Hurewicz* introduced $\pi_n(X), n > 1$, and **aspherical spaces** X, i.e., spaces X with $\pi_n(X) = 0$ for $n > 1$
- he proved: any aspherical CW-complex X is determined up to homotopy equivalence by $\pi := \pi_1(X)$; thus homotopy invariants of such an X can be thought of as invariants of the group π; so for all $n \in \mathbb{N}_0$ we define $H_n(\pi) := H_n(X)$
$\rightarrow$ *homology group of a group*
- we have $H_0(\pi) = \mathbb{Z}$ for all π
- $H_1(\pi) = \pi/[\pi, \pi]$, the factor commutator group of π

Example:
$X = S^1 \times S^1$

- 1942: *Hopf* showed for an arbitrary path-connected (not necessarily aspherical) space X that the *Hurewicz map* $h_2 : \pi_2(X) \to H_2(X)$ is an isomorphism if $\pi := \pi_1(X) = 0$ (i.e., if X is simply connected)
- he showed that in general there is an exact sequence

$$\pi_2(X) \xrightarrow{h_2} H_2(X) \longrightarrow H_2(\pi) \longrightarrow 0$$

- a second result of *Hopf's* paper is a calculation of $H_2(\pi)$ if $\pi = F/R$, quotient of a free group:

$$H_2(\pi) \cong (R \cap [F, F])/[R, F]$$

- at the end of the 1940s there was a purely algebraic definition of the homology and cohomology of a group
- one could then make connections with algebra going back to the early 1900s; for example, H^1 is a group of equivalence classes of „derivations" (also called „crossed homomorphisms") and H_2 coincides with the „Schur multiplier" (representation theory)

For an R-module M with n generators (n possibly infinite) we have an exact sequence

$$R^n \longrightarrow M \longrightarrow 0\,.$$

$R^n \twoheadrightarrow M$ has a kernel K, whose elements represent relations among the given generators.

If K admits m generators as an R-module then we have an exact sequence

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0\,.$$

$R^m \twoheadrightarrow K$ has a kernel $L \ldots$

So we obtain an exact sequence

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0\,,$$

where each F_i is a free R-module. Such an exact sequence is called a **free resolution** of M.

Now consider $R = \mathbb{Z}G$ (integral group ring) for a group G, and $M = \mathbb{Z}$, with trivial G-action.

This situation arises naturally in topology:

X **CW-complex** (generalization of simplicial complex ...) with a G-action that permutes the **cells** (i.e., for each $g \in G$ and cell v of X gv is a cell of X) is called **G-complex**.

If X is a G-complex then the action of G on X induces an action of G on the **cellular chain complex** $C_*(X)$.

So $C_*(X)$ is a chain complex of $\mathbb{Z}G$-modules. (Each $C_n(X)$ is a $\mathbb{Z}G$-module, G permutes the n-cells.)

Moreover, the **augmentation map** $\epsilon : C_0(X) \to \mathbb{Z}$ with $\epsilon(v) = 1$ for every 0-cell v is a $\mathbb{Z}G$-module homomorphism.

X is called a **free G-complex** if for all cells $v : \; gv = v \;\Rightarrow\; g = 1\,.$

If X is a free G-complex then each $C_n(X)$ is a free $\mathbb{Z}G$-module with one basis element for every G-orbit of cells.

Finally, if X is contractible (i.e., X is homotopy equivalent to a point of X) then $H_n(X) \cong H_n(\text{pt.}) = 0$ for all $n > 0$ and so the sequence

$$\cdots \longrightarrow C_n(X) \xrightarrow{d_n} C_{n-1}(X) \longrightarrow \cdots \longrightarrow C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

is exact.

Proposition. *Let X be a contractible free G-complex. Then the augmented cellular chain complex of X is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.*

$Y := G \setminus X$ quotient complex ($Y = X/\sim$ quotient space)

If G acts freely on X (as above) then we can view X as a covering space of Y with G as the group of deck transformations (homeomorphisms $g : X \to X$ with $pg = p$ where $p : X \to Y$ is the covering map). Here we have $G \cong \pi_1(Y)/\pi_1(X)$.

If X is simply connected then X is the universal cover of Y (so X is unique up to homeomorphy) and G can be identified with $\pi_1(Y)$.

If in addition X is contractible then $\pi_i(X) = 0$ and so $\pi_i(Y) = 0$ for all $i > 1$. Thus Y is an aspherical space.

Conversely, if we start with an aspherical CW-complex Y then its universal cover X has trivial homotopy groups in all dimensions and hence is contractible by a theorem of *Whitehead*.

So we have:
If Y is an aspherical CW-complex with fundamental group G then its universal cover X is a contractible, free G-complex. The proposition gives us a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

Definition. Let Y be a CW-complex with fundamental group G. We say that Y is an *Eilenberg-MacLane complex* of type $K(G,1)$ if it is aspherical or, equivalently, if its universal cover is contractible.

Further, every group G admits a $K(G,1)$-complex Y.

With the theorem of *Hurewicz* (see (12)) Y is unique up to homotopy equivalence.

This fact has an algebraic analogue,
the **fundamental lemma of homological algebra**:

Given any module M, free resolutions of M exist and are unique up to chain homotopy equivalence.

(We already proved existence above.)

The **coinvariants** functor $M \mapsto M_G$ is the algebraic analogue of forming the quotient by a G-action. Here M is a $\mathbb{Z}G$-module and

$$M_G := M/\langle gm - m \mid g \in G, m \in M \rangle \,.$$

Thus M_G is the largest quotient of M on which G acts trivially. It holds: $M_G \cong \mathbb{Z} \bigotimes_{\mathbb{Z}G} M$. Note: $1 \otimes gm = 1 \cdot g \otimes m = 1 \otimes m$.

Definition. Given a group G, choose a *projective resolution* (f.i. a free resolution) $P = (P_n)_{n \in \mathbb{N}_0}$ of $\mathbb{Z}$ over $\mathbb{Z}G$ and set $H_*(G) := H_*(P_G)\,.$

The fundamental lemma of homological algebra guarantees that $H_*(G)$ is well-defined (independent of the choice of P) up to canonical isomorphism.

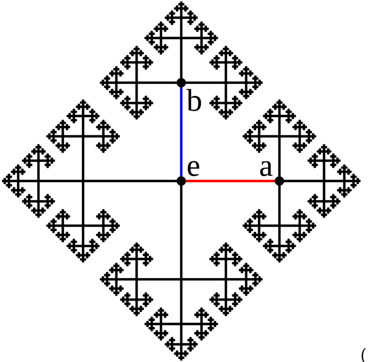
We also immediately get $H_*(G) = H_*(Y)$ if Y is a $K(G,1)$-complex. If X is the universal cover of Y then we can choose $P = C_*(X)$ what implies $P_G = C_*(X)_G \cong C_*(Y)$ (as $\mathbb{Z}$-modules).

1. Free Groups

$G = F(S)$ free group generated by S

X Cayley graph of G with respect to S

- tree with vertex set G, with an edge from g to gs for each $g \in G$ and $s \in S$; f.i. $G = F_2$:



(Cayleygraph - Wikipedia)

- contractible free G-complex with 0-simplices g and 1-simplices (g, gs)
- one G-orbit of vertices, one G-orbit of edges for each $s \in S$
- $Y = G \setminus X \approx \bigvee_{s \in S} S^1_s$ bouquet of circles indexed by S, $\pi_1(Y) \cong G$, thus X is $K(G,1)$
- resulting free resolution

$$0 \longrightarrow \mathbb{Z}G^{(S)} \xrightarrow{\partial} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0\,,$$

where $\mathbb{Z}G^{(S)}$ is a free $\mathbb{Z}G$-module with a basis $((1,s))_{s \in S}$ and $\partial(1,s) = s - 1$ for $s \in S$ (standard boundary operator).

We apply the coinvariants functor: // $(F_i)_G = F_i / \langle gm - m \mid g \in G, m \in F_i \rangle$

$(\mathbb{Z}G)_G \cong \mathbb{Z}$ $\quad (\mathbb{Z}G^{(S)})_G \cong \mathbb{Z}^{(S)}$ $\quad (0)_G \cong 0$

$\overline{\partial}(\overline{(1,s)}) = \overline{s-1} = \overline{0}$

resulting chain complex:

$$0 \longrightarrow \mathbb{Z}^{(S)} \xrightarrow{\overline{\partial}} \mathbb{Z}$$

thus:

$$H_0(G) \cong \mathbb{Z}\,, \quad H_1(G) \cong \mathbb{Z}^{(S)}\,, \quad H_i(G) \cong 0 \;\; (i > 1)$$

2. Cyclic Groups

(a) $G = F(S)$ where $|S| = 1$, hence $G = \langle t \rangle \cong \mathbb{Z}$

free resolution

$$0 \longrightarrow \mathbb{Z}[t, t^{-1}] \xrightarrow{\partial} \mathbb{Z}[t, t^{-1}] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0\,,$$

where $\partial(1, t) = t - 1$

Now it is easy to verify the exactness by pure algebra.

($\mathrm{im}(\partial) = (t-1)\mathbb{Z}[t, t^{-1}] = \ker(\epsilon)$, because $f = \sum a_i t^i \in \ker(\epsilon) \Rightarrow 0 = \epsilon(f) = f(1)$, hence
$f = f - f(1) = \sum_{i \in \mathbb{Z}} a_i(t^i - 1) = \sum a_{-i}(t^{-i} - 1) + \sum a_i(t^i - 1)$ is divisible by $t - 1$ in $\mathbb{Z}[t, t^{-1}]$)

we have:

$$H_0(G) \cong \mathbb{Z}\,, \quad H_1(G) \cong \mathbb{Z}\,, \quad H_i(G) \cong 0 \ \ (i > 1)$$

(b) $G = \langle t \mid t^n = 1 \rangle \cong \mathbb{Z}/n\mathbb{Z}$

X regular n-gon

- vertex set G, edges from t^i to t^{i+1}
- non-contractible free G-complex with 0-simplices t^i and 1-simplices (t^i, t^{i+1})
- exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\eta} \mathbb{Z}G \xrightarrow{\partial} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0\,,$$

where $\eta(1) = (\sum_{i=0}^{n-1} t^i)(1, t)$ (note: $\sum_{i=0}^{n-1} t^i(t-1) = t^n - 1 = 0$)

- we obtain the free resolution:

$$\ldots \xrightarrow{\eta} \mathbb{Z}G \xrightarrow{\partial} \mathbb{Z}G \xrightarrow{\eta} \mathbb{Z}G \xrightarrow{\partial} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

resulting chain complex:

$$\dots \xrightarrow{\overline{\eta}} \mathbb{Z} \xrightarrow{\overline{\partial}} \mathbb{Z} \xrightarrow{\overline{\eta}} \mathbb{Z} \xrightarrow{\overline{\partial}} \mathbb{Z}$$

$$\overline{\eta}(\overline{1}) = \overline{n} \qquad \overline{\partial}(\overline{(1,t)}) = \overline{0}$$

so we have:

$$H_0(G) \cong \mathbb{Z}\,, \quad H_i(G) \cong \mathbb{Z}/n\mathbb{Z}\,, i \text{ odd}, \quad H_i(G) \cong 0\,, i \text{ even}, i > 0$$

Thanks for listening!