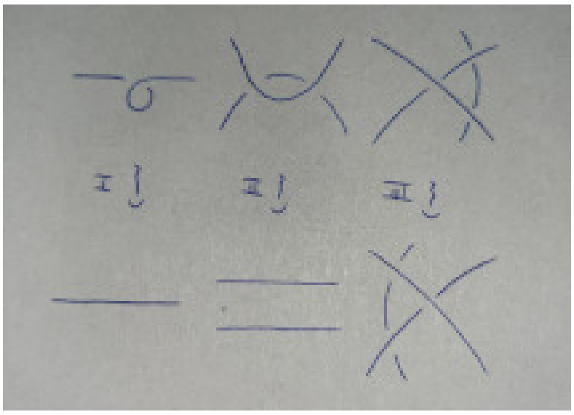
OBERSEMINAR ALGEBRA AND GEOMETRY - SUMMER TERM 2023

Knots are among the most studied objects in low-dimensional topology. Here a knot K is defined as an embedding of S^1 into $\mathbb{R}^3$ or into its one-point compactification S^3, where often some appropriate notion of tameness is implicitly assumed. Knots are studied up to ambient isotopies, which are called knot equivalences. It turns out that such equivalences can be understood in terms of so called Reidemeister moves:



Therefore, to define a knot invariant, it suffices to define an invariant which respects these three simple operations. Some knot invariants are the tricolorability, crossing numbers, various geometrically defined polynomials (such as the Jones, Alexander or the HOMFLY polynomial) and the fundamental group of the knot complement. The latter is called the knot group. While they serve some purpose, none of these invariants are complete, i.e. they cannot distinguish all knots. But the good news is that there is a complete invariant called knot quandle. But what exactly is a quandle?
From a knot-theoretic perspective, quandles are algebraic structures whose operations respect algebraic versions of the Reidemeister moves. The aforementioned knot quandles $Q(K)$ arise from the Wirtinger presentation of the knot group, a presentation with relations only given by conjugations of the generators. There are also Alexander quandles, which arise from a $\mathbb{Z}[t, t^{-1}]$-module structure on the abelianization of the knot group and which can be used to compute Alexander polynomials. From an algebraic perspective, quandles are groups who have forgotten their operation, the inversion of elements and their identity element, but still memorize the conjugations. Therefore, while mainly used in knot theory, quandles are interesting objects in their own right and can be studied similarly to groups or other algebraic objects.
There are plenty of books on basic knot theory. Many of them use quite

some vague/visual arguments, but the book [2] by Burde and Zieschang seems to provide rather precise arguments. For quandle theory, the thesis of Joyce [3] seems to be a good starting point.

List of talks

(1) 07.04.23 - **No Oberseminar.** (Good Friday)
(2) 14.04.23 - **Knots and knot equivalence.** Define knots (and links), discuss wild and tame knots, state that we restrict to the latter ones, introduce (ambient) isotopies, knot equivalences (leave out Δ-moves if you need to save time), knot projections, Reidemeister moves and discuss the characterisation of knot equivalences in terms of Reidemeister moves. [2, Chapter 1.A-C].
(3) 21.04.23 - **Geometric concepts.** Define Mirror images, invertible and amphicheiral knots, alternating knot projections, Seifert surfaces and genus, meridian and longitude, knot sums (sometimes called products), companion and satellite knots and discuss tricolorability as an elementary invariant of knots. [2, Chapter 2.A-C] and [1, Chapter 1.5] for tricolorability.
(4) 28.04.23 - **Knot groups and homology.** Discuss the homology of the knot complement, the Wirtinger Presentation of the knot group, define companion and satellite knots and discuss the knot groups of satellite and companion knots. [2, Def. 2.8 and Chapter 3.A and 3.B].
(5) 05.05.23 - **Peripheral systems.** Discuss peripheral systems, Waldhausen's Theorem and the characterisation of invertibility and amphicheirality. Finally, show the asphericity of the knot complement. [2, Chapter 3.C and 3.F].
(6) 12.05.23 - **The Alexander polynomial.** Discuss cyclic knot coverings and the Alexander module, band projections, Seifert matrices, Alexander matrices, Alexander polynomials and their properties. [2, Chapter 8.A-D].
(7) 19.05.23 - **Fox differential calculus.** Discuss the homology of covering spaces, Fox differential calculus and the calculation of Alexander matrices and polynomials. [2, Chapter 9.A-C].
(8) 26.05.23 - **Introduction to quandles** Introduce quandles, abelian quandles, $\mathrm{Conj}(G)$, Alexander quandles, involutary quandles (= Kei), $\mathrm{Inv}(G)$, $\mathrm{Core}(G)$, homomorphisms of quandles, free quandles, show that free quandles arise as subquandles of free groups, discuss admissible equivalence relations, presentations of quandles and the adjunction between Conj and Ad = Adconj. Stress the non-injectivity of the unit of the adjunction. [3, Sections 1 and 2.1] and [4] for Alexander quandles, free quandles, admissible equivalence relations and presentations.

(9) 02.06.23 - **Coset quandles and augmented quandles.** Define coset quandles and homogeneous quandles and show that the latter are coset quandles. State the generalization to general quandles and discuss augmented quandles, their homomorphisms, limits and colimits and finally, their quotients arising from normal subgroups. [3, Sections 2.4, 2.10 and 2.11]
(10) 09.06.23 - **Fundamental quandle of a pair of spaces.** Introduce the fundamental quandle of a pair of spaces. Instead of copying the uninformative formulas, draw informative pictures (of nooses, the action of loops on nooses, ...). Follow the formal viewpoint as you find suitable. Show that the fundamental quandle of a disk with respect to the center point gives essentially the winding number. [3, Sections 4.1 and 4.2]
(11) 16.06.23 - **Seifert-van Kampen for quandles.** Show the Seifert–van Kampen theorem for quandles. To do so, it might be helpful to recall the categorical concept of a pushout and van Kampen's theorem for fundamental groups of spaces with open covers. As examples, consider punctured surfaces. [3, Sections 4.3 and 4.4]
(12) 23.06.23 - **Knot quandles and diagram quandles.** Give the geometric defintion of a knot quandle as the subquandle of the fundamental quandle of nooses with winding number one. Explain how to obtain a knot quandle from a Wirtinger presentation of the knot group. Explain briefly that the latter is a knot invariant using Reidemeister moves. Using van Kampen's theorem, show that the two definitions give isomorphic quandles. [3, Sections 4.5–4.8]
(13) 30.06.23 - **Completeness and abelian knot quandles as Alexander invariants.** Show that the knot quandle is a complete knot invariant by showing that it encodes the peripheral subgroup structure. Show that the Alexander invariant and the abelian knot quandle determine each other. If time permits, mention cyclic knot invariants and involutory knot quandles. [3, Sections 4.9–4.12]
(14) 07.07.23 - **Spare slot.**
(15) 14.07.23 - **Discussion on next term's topic.**

References

[1] C. C. Adams, *The knot book*, American Mathematical Society, Providence, RI, 2004. An elementary introduction to the mathematical theory of knots, Revised reprint of the 1994 original. MR2079925 ↑2
[2] G. Burde and H. Zieschang, *Knots*, 2nd ed., De Gruyter Studies in Mathematics, vol. 5, Walter de Gruyter & Co., Berlin, 2003. MR1959408 ↑2
[3] D. E. Joyce, *An algebraic approach to symmetry with applications to knot theory*, ProQuest LLC, Ann Arbor, MI, 1979. Thesis (Ph.D.)–University of Pennsylvania. MR2628474 ↑2, 3

[4] D. Nardin, *Fundamental quandle of knots and Alexander modules*, 2012. Available at `https://homepages.uni-regensburg.de/~nad22969/dispense/quandle.pdf`, version 1.6.0. ↑2