OBERSEMINAR ALGEBRA & GEOMETRY
DÜSSELDORF

RATIONAL VARIETIES.

Facts

Where and when: Friday 10:30-12:00 in 25.22.02.81.

Website: `http://www.math.uni-duesseldorf.de/~perrin/oberseminar.html`

The Book: *Rational and Nearly Rational Varieties*
J. Kollár, K.E. Smith and A. Corti.
Cambridge Studies in Advanced Mathematics, 92. Cambridge University Press, Cambridge, 2004.

Program

Date	Talk	Speaker
17.04.2015	1	Nicolas Perrin
24.04.2015	2	Richard Gonzales
08.05.2015	3	Anitha Thillaisundaram
15.05.2015	4	Steffen Kionke
22.05.2015	5	Benedikt Schilson
29.05.2015	6	Andre Schell
05.06.2015	7	Alexander Samokhin
12.06.2015	8	Leif Zimmermann
19.06.2015	9	Saša Novaković
26.06.2015	10	Christoph Bärligea
03.07.2015	11	
10.07.2015	12	

Introduction

In this seminar we want to study some fundamental problems in geometry. Loosely speaking, the problems are as follows: given a set (a variety), does there exist a nice enough (regular) finite-to-one parametrisation of this set by a k-vector space (unirationality)? Can one find a parametrisation which is injective (rationality)? We also want to compare the two notions. We shall also see that these notions strongly depend on the field of definition k.

The above problems for plane curve are already of importance in the theory of integrals: functions which can be parametrised in terms of rational functions are easily integrated while other may cause (lots of) trouble (like elliptic integrals).

Here are some very classical results in this active field of research: Whether an unirational variety is rational was answered positively long ago for curves (Lüroth's Theorem 1876). The corresponding result for surfaces (Castelnuovo's Theorem) is only true for k of characteristic zero and algebrically closed (Zariski gave counterexamples in 1958). It was a longstanding open problem to give (correct) counterexamples in higher dimension. This was done by Several people (Clemens-Griffiths [2], Iskovkih-Manin [4] and Artin-Mumford [1]) in the early seventies.

In this seminar we want to deal with some more down-to-earth problems along the above lines. Especially we want to understand simple examples of rational and almost rational varieties (mostly in dimension 1 or 2) with a special emphasis on non algebraically closed fields. This should be the occasion to learn some basics on algebraic geometry via classical examples. The last two talks will give a (partial) proof of the results of Iskovkih-Manin [4].

Note that, even though we will not really see this, there is a non-trivial intersection with the Düsseldorf-Wuppertal seminar: For example the Artin-Mumford technique [1] for proving non rationality uses the Brauer group and very recent important results in this direction use this circle of ideas.

TALKS

This seminar is mostly concerned with algebraic geometry. However we will deal most of the time with very simple concepts and objects from algebraic geometry and we should try as much as possible to recall the basic definitions and results that are used in the book [5]. Our basic reference for algebraic geometry will be [7]. I will try to give precise references for each talk.

1. Rational and unirational varieties: The case of curves

Since this is the first talk, there will be several reminders on algebraic geometry to be done. Here is the list that should be mixed with the results appearingin the book [5]: we want to explain the two Sections 1.1 and 1.2 of the book. Especially some details should be given on the classification of rational curves on algebraically closed fields especially the equivalence of the statements given at the beginning of Section 1.2.

Here are some usefull references concerning algebraic geometry (in some cases one could restrict to the case of curves but for most of the definitions this is not necessary and will serve as reference for further talks).

Algebraic Varieties

- Variety over a field (Shafarevich, Chapter 1. Section 4.1 – Definition on page 46 and pages before).
- Projective variety (closed subset in the projective space).
- Regular varieties (Shafarevich, Chapter 2. Section 1.4 – Definition on page 92 and pages before for the definition of the tangent space).

Morhisms and rational maps

- Morphism (over a field) and isomorphisms (Shafarevich, Chapter 1. Section 4.2 – Definition on page 47 and pages after).
- Rational maps (over a field) and birational maps (Shafarevich, Chapter 1. Section 4.3).
- Extension of rational maps: explain Theorem 2.12 (page 109 in Shafarevich) and Corollaries 2.3 and 2.4 on the same page.

Divisors

- Definition of the group of divisors and of divisors associated to a rational function (Shafarevich, Chapter 3 Section 1.1). Locally principal divisors and the Picard group or group of Cartier divisors (Shafarevich, Chapter 3 Section 1.2 – Definition on page 153).
- Definition of the degree of a divisor on a curve (Shafarevich, Chapter 3 Section 2.1).
- The case of $\mathbb{P}^1$ and that the Picard group is $\mathbb{Z}$ (generated by the class of a point) could be of use.
- Linear system $\mathcal{L}(D)$ of a divisor D (Shafarevich, Chapter 3 Section 1.5 – first line on page 157).
- Shafarevich, Chapter 3 Section 1.5 Theorem 3.3 is Proposition 1.6 of the book [5].
- Construction of the canonical divisor (This is Proposition 1.5 of the book [5]). Here to simplify one could chose to only present the case of curves since this is then simpler. One should emphasis the case of $\mathbb{P}^1$ especially that the anticanonical divisor $-K_{\mathbb{P}^1}$ is equivalent to 2pt and has a vector space of global sections of dimension 3.
- Riemann-Roch Theorem (Theorem 3.23 page 210 in Shafarevich).

2. Quadric hypersurfaces

Explain Sections 1.3 and 1.4 of the book [5]. For Theorem 1.11 one could restrict to the case where k is infinite. Exercice 1.13 should be proven since it gives the structure of singular quadrics according to their rank. Exercise 1.14 is also important while Theorem 1.15 should only be stated without proof. In this section we shall need much less results of algebraic geometry. One could however expand the discussion of the previous talk on divisors to get more intuition on intersection of divisors in the proof of Theorem 1.23. This is explained in Shafarevich, Chapter 4 Section 1.

3. Cubic hypersurfaces

Explain Section 1.5 of the book [5]. This talk again uses less algebraic geometry.

The beginning of Section 1.5 mainly gives examples of cubic hypersurfaces that are or are not rational. Especially the rationality statements are proved when the hypersurface has some non regular points.

The second part of the section proves that cubic surfaces having a k-point are unirational.

4. Other examples and some numerical criteria

Explain Sections 1.6 and 1.7 of the book [5].

In Section 1.6, one could shrtly recall how to obtain the 27 lines on a cubic from its description as the blow-up of 9 points in the plane and also explain roughly what the blow-up of a point in the plane is (Shafarevich, Chapter 4. Section 3.1). This will appear again in latter talks.

Section 1.7 uses more algebraic geometry. One should here explain the definition of the sheaf of differentials Ω_X and its m-th tensor product $\Omega_X^{\otimes m}$ (Shafarevich, Chapter 3. Section 5.) and its behaviour under morphisms and restrictions. One could prove for example Exercise 1.53 as an example of the computation of these sheaves in an easy case. We shall also need the computation of the canonical divisor of a product (this is explained for curves in Shafaravich page 252). The statements in Exercise 1.59 are nice numerical statements.

5. Segre and Manin Theorems: statements

Explain Sections 2.1 and 2.2 of the book [5].

In the first section we shall explain the statement of Segre and Manin theorems on cubis surfaces. Reminders on the Picard group and the geometry of cubic surfaces over algebraically closed fields could help here (Picard group: Shafarevich, Chapter 3. Section 1.2, cubic surface: Shafarevich, Chapter 4. Section 2.5).

In the second section we consider linear systems and their connection to morphisms (Shafarevich, Chapter 3. Section 1.3-1.5). In particular we will consider intersection of divisors (=curves) on surfaces (Shafarevich, Chapter 4. Section 3). Blow-ups are of importance in this section.

6. Segre and Manin Theorems: proofs

Explain Sections 2.3 and 2.4 of the book [5].

In both section we will play a lot with intersection numbers on surfaces. Useful results are contains in Shafarevich, Chapter 4. The definition and basic properties of intersection numbers is contained in Shafarevich, Chapter 4. Section 1. Shafarevich, Chapter 4. Section 3 for surfaces will also be useful. In the second part we prove the existence of surfaces with Picard number one concerned by the Theorems of Segre and Manin.

7. Surfaces: Rationality criterion of Castelnuovo

Explain Sections 3.1 and 3.2 of the book [5].

Section 3.1 is mainly expository. Some results should be presented like Adjunction Formula (Shafarevich 2 [8], Chapter 6. Section 1.4.).

Section 3.2 explains parts of the philosophy of the minimal model program (MMP), one of the major research theme in modern algebraic geometry, in the special case of surfaces. One needed result is the Castelnuovo's contractibility criterion. This statement is not proved in Shafarevich by explained Chapter 4 Section

3.5 on page 267. A complete proof is given in Hartshorne's book ([3] Chapter 5 Section 5). We shall use this freely. The author first recalls the notion of a conic bundle (sketch of proof for the results of Exercise 3.13 could be given) and then proves castelnuovo's criterion for rational surfaces modulo the MMP (Theorem 3.14). Some basic Hodge-Theoretic results are used here.

8. Surfaces: minimal model

Explain Section 3.3 of the book [5].

Section 3.3 prove the MMP part of the previous talk: Theorem 3.14. Several general techniques of birational geometry in the case of surfaces will be used. In particular we will use Riemann-Roch theorem for surfaces. This can be found in Harshorne's book (Chapter 5 Section 5) or in Reid's book [6] page 77. We will also use Adjunction Formula (Shafarevich 2 [8], Chapter 6. Section 1.4.). We will also use Bertini's Theorem (Shafarevich, Chapter 2. Section 6.1. Theorem 2.26).

9. Noether-Fano method

Explain Sections 5.1 and 5.2 of the book [5].

The last two talks deal with algebraic geometry in higher dimension. We shall assume several results and a (important) technical part of the proof will be left aside (Theorem 5.20).

Section 5.1 reviews the maximal centers techniques while Section 5.2 draws some of its consequences.

10. Proof of Iskovskih-Manin result on quartics

Explain Sections 5.3 and 5.4 of the book [5].

Section 5.3 defines birationally rigid Fano varieties (examples of those are cubic surfaces with Picard number 1) and proves results on their (non)rationality. Section 5.4 applies these results to quartic threefolds.

References

[1] Artin, M.; Mumford, D. (1972), *Some elementary examples of unirational varieties which are not rational*, Proceedings of the London Mathematical Society. Third Series 25: 75–95.

[2] Clemens, H.C.; Griffiths, P.A. (1972), *The intermediate Jacobian of the cubic threefold*, Annals of Mathematics. Second Series (The Annals of Mathematics, Vol. 95, No. 2) 95 (2): 281–356.

[3] Hartshorne, R. *Algebraic Geometry*, Graduate Texts in Mathematics, No. 52, New York, Springer-Verlag 1977.

[4] Iskovskih, V. A.; Manin, Y.I. (1971), *Three-dimensional quartics and counterexamples to the Lüroth problem*, Matematicheskii Sbornik, Novaya Seriya 86: 140–166.

[5] Kollár J.; Smith K.E.; Corti A. *Rational and Nearly Rational Varieties*. Cambridge Studies in Advanced Mathematics, 92. Cambridge University Press, Cambridge, 2004.

[6] Reid, M. *Chapters on algebraic surfaces, Complex algebraic geometry*. Providence, RI, American Mathematical Society 1997.

[7] Shafarevich, I.R. *Basic algebraic geometry. 1*. Varieties in projective space. Third edition. Translated from the 2007 third Russian edition. Springer, Heidelberg, 2013.

[8] Shafarevich, I.R. *Basic algebraic geometry. 2*. Varieties in projective space. Third edition. Translated from the 2007 third Russian edition. Springer, Heidelberg, 2013.