Algebraic K-Theory

Oberseminar Algebra, Geometrie und Zahlentheorie WS 2017/18

The seminar will cover a small range of mostly historical/classical topics in algebraic K-theory. The main reference used in the abstracts is [Ros94], but further text books are mentioned at the end [Sri96, Wei13]. Half of the talks, marked with a $\star$, are more optional than the others.

Vector bundles & projective modules (K_0)

- Talk 1: Projective modules and vector bundles
- Talk $\star$: Cancellation theorems
- Talk 2: Class groups of Dedekind domains
- Talk $\star$: Projective modules over group rings
- Talk $\star$: Wall's finiteness obstruction

Automorphisms (K_1)

- Talk 3: K_1 of division rings
- Talk $\star$: K_1 of Dedekind domains
- Talk 4: Whitehead torsion *or* The s-Cobordism Theorem
- Talk 5: The short exact sequence
- Talk $\star$: The Fundamental Theorem

Universal abelian extensions (K_2)

- Talk 6: Universal abelian extensions
- Talk 7: The Steinberg group
- Talk $\star$: Matsumoto's Theorem

Higher algebraic K-theory (K_n)

- Talk 8: Classifying spaces
- Talk 9: Quillen's constructions of higher algebraic K-theory
- Talk 10: K-theory of finite fields
- Talk $\star$: K-theory of the integers
- Talk $\star$: K-theory of schemes

Requested topics

- Talk $\star$: The Farrel-Jones Conjectures
- Talk $\star$: Vandiver's Conjecture

Vector bundles & projective modules (K_0)

Talk 1: Projective modules and vector bundles

Introduce vector bundles over topological spaces and projective modules over rings. Explain that these concepts are closely related *(a)* via Swan's theorem [Ros94, § 1.6; Cla15, §§ 6.1–6.4] and *(b)* via the correspondence between modules over a ring and quasicoherent sheaves over its spectrum [Wei13, § I.5]. Optionally, give a rudimentary introduction to *topological* K-theory.

Talk ⋆: Cancellation theorems

Discuss the cancellation theorems for continuous vector bundles and for projective modules described in [BS62]. The analogy illustrates the philosphy of the previous talk and moreover motivates the definition of K_0. For the topological result, see the references provided in [BS62]. The algebraic result, which is now known as the "Serre-Bass Cancellation Theorem", is merely announced there; details may be found in [Bas64] or [Bas68, § IV.3].

Talk 2: Class groups of Dedekind domains

Explain the arithmetic roots of algebraic K-theory, following for example [Ros94, § 1.4]. It's probably wise to also include some simpler computations from [Ros94, § 1.3].

Talk ⋆: Projective modules over group rings

For applications of algebraic K-theory in geometric topology, the most relevant rings are group rings of fundamental groups. Present Swan's result on the structure of projective modules over group rings of *finite* groups [Swa59; Swa60; Bas68, XI]. Highlight the "inspirational" rôle of Brauer's induction theorem. Brauer's induction theorem will appear again much later in the seminar: it is the basis of Green's construction of the Brauer lift, which Quillen uses to compute the higher K-theory of fields.

Talk ⋆: Wall's finiteness obstruction

Discuss Wall's obstruction for a topological space to be homotopy equivalent to a *finite* CW complex. See [Ped17] for a recent self-contained exposition. Other secondary sources include [Wei13, Thm 2.4.1; Ros05, § 1; Ros94, § 1.7; FR01].

Automorphisms (K_1)

Talk 3: K_1 of division rings

Cover [Ros94, §§ 2.1–2.2]. It may be helpful to have a look at [BS62] or [Bas64] for motivation.

Talk ⋆: K_1 of Dedekind domains

Cover [Ros94, § 2.3].

Talk 4: Whitehead torsion *or* The s-Cobordism Theorem

Define the Whitehead torsion of a homotopy equivalence and discuss its significance for detecting simple homotopy equivalences [Ros05, § 3][Ros94, § 2.4]. Discuss some examples/computations of Whitehead groups.

Alternatively, give an introduction to the h-cobordism theorem and its generalization to non-simply-connected manifolds, the s-cobordism theorem. The statement can be found in [Ros94, § 2.4] or [Ros94, § 3]. For details, see [Mil65] or [Ran02, § 8].

In either case, do not forget to discuss some examples/computations of Whitehead groups. (See also the talk on the "Fundamental Theorem" below.)

Talk 5: The short exact sequence

Explain how K_0 and K_1 "fit together" in certain exact sequences [Ros94, § 2.5]. Point out the analogy with topological K-theory. Exude an air of mystery about what happens "further to the left".

Talk ⋆: The Fundamental Theorem

Motivate the passage from "K-theory of rings" to "K-theory of categories" [Ros94, § 3.1] by discussing the K-theory of $R[t]$ or $R[t, t^{-1}]$ in terms of the K-theory of a base ring R [Ros94, § 3.2]. The idea is to pass from the K-theory of a ring R to a variant known as G-theory, to show that we have an isomorphism $G_i(R) \cong G_i(R[t])$ in high generality (Thms 3.3.12 & 3.3.16), and that, for regular rings, we can translate this back into K-theory (Thm 3.1.16).

A nice application is discussed in [Ros94, Ex. 3.2.27]: the Whitehead group of any free abelian group vanishes.

Universal abelian extensions (K_2)

Talk 6: Universal abelian extensions

Cover [Ros94, § 4.1].

Talk 7: The Steinberg group

Cover [Ros94, § 4.2]. In particular, define K_2 of a ring. Mention Matsumoto's Theorem (but say nothing about its proof).

Talk ⋆: Matsumoto's Theorem

Discuss Matsumoto's Theorem [Ros94, Thm 4.3.15] and its proof.

Higher algebraic K-theory (K_n)

Talk 8: Classifying spaces

Give an introduction to the theory of classifying spaces of groups and categories. Classifying spaces of groups are covered in [Ros94, § 5.1]; see [McC01, § 6.3] for a much briefer discussion and [May99, §§ 16.2, 16.5 & 23.8] for an account that emphasizes the simplicial point of view. This will be useful in the generalization to categories, see [Qui73, § 1] or [Sch11, App. A.1] and the references given there.

Talk 9: Quillen's constructions of higher algebraic K-theory

Discuss Quillen's two constructions of higher algebraic K-theory: the +-construction [Ros94, § 5.2] and the Q-construction [Qui73; Ros94, § 5.3; Sch11, § 2.2]. Go into the details of at most one of these, but try to emphasize what their respective advantages/disadvantages are. (The +-construction works on the "ring level" and is used in the computation of the K-theory of finite fields; for the definition of the higher K-theory of schemes, the more flexible Q-construction is essential.)

Talk 10: K-theory of finite fields

Present Quillen's computation of the K-theory of finite fields [Qui72]. You will find lots of expository notes on these computations on the web; for example, the short and neat [Hai17] or the detailed [Mit].

Talk $\star$: K-theory of the integers

Give a survey of what is known and what is not known about the K-theory of the integers [Wei05]. Possibly also say why we care.

Talk $\star$: K-theory of schemes

Discuss K-theory as a cohomology theory on algebraic varieties/schemes. A brief introduction can be found in [Gei05, § 1.4]. For details, see [Qui73] or [Sch11].

Requested topics

If there is a topic in K-theory that you would like to talk/hear about and which is not listed here, please tell me!

Talk $\star$: The Farrel-Jones Conjectures

According to the Farrel-Jones conjectures, it should be possible to compute the K-theory of a group ring from the K-theory of the group rings of certain subgroups. Remind your audience why this would be wonderful, especially with respect to applications in geometric topology. Then state and explain the conjectures in more detail, and indicate what is known. One possible starting point is [BLR08].

Talk ⋆: Vandiver's Conjecture

Discuss Soulé's recent advances towards the Vandiver Conjecture in number theory [Sou99]. The notes by Eknath Ghate [Gha] look comparatively readable (and honest).

References

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