For this little preliminary set of reminders, we will be shamelessly copying the definitions given in:

- *Modern Differential Geometry for Physicists*, Chris J. Isham.
- *Geometry, Topology and Physics*, Nakahara Mikio.

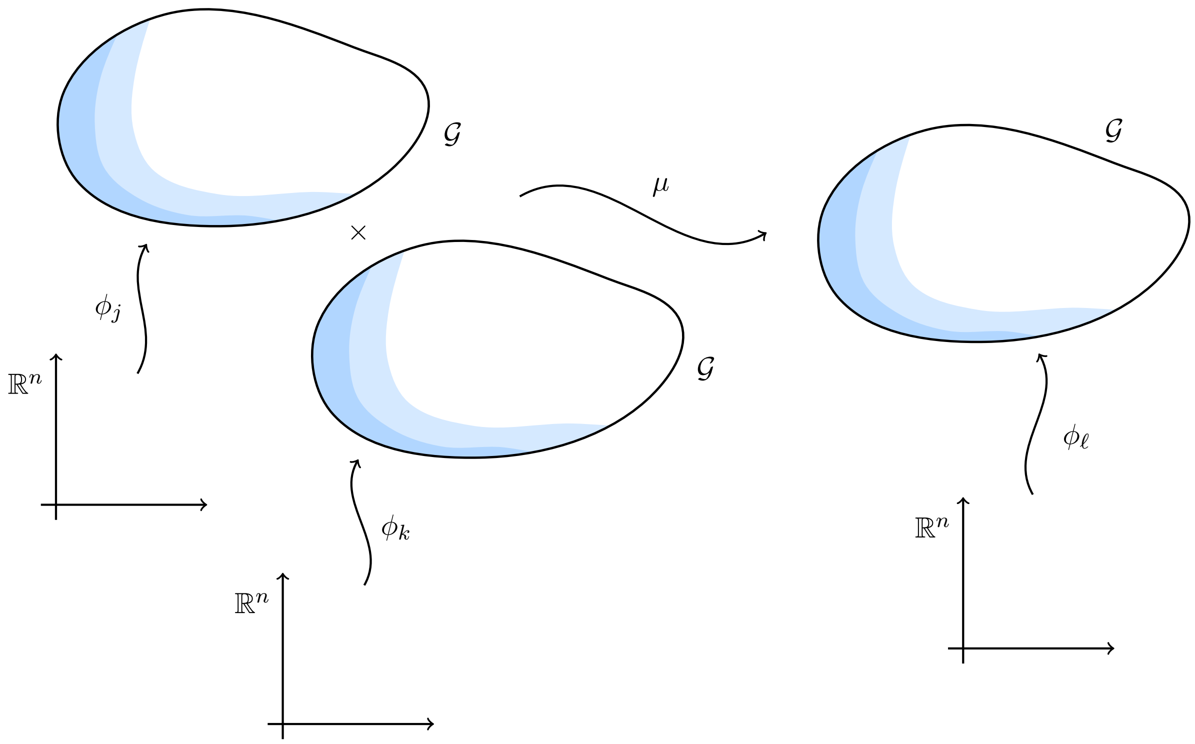
(I would like to add as an aside that Prof. Isham is a fellow that I've long had a great degree of admiration for, since despite him being trained as a physicist, Prof. Isham is a notable proponent of the use of category theory in physics, and has been imploring young physicists for many years to study the topic, as well as algebraic geometry.)

Definition. (Lie group.) A *Lie group* is a smooth manifold which is endowed with a group structure such that the group operations

(1) $\cdot : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, $(g_1, g_2) \mapsto g_1 \cdot g_2$; and,
(2) ${}^{-1} : \mathcal{G} \to \mathcal{G}$, $g \mapsto g^{-1}$

are smooth.

And, of course, it goes unspoken here that what we really mean is that these operations define maps between manifolds, which when composed with the charts produce smooth functions between Euclidean spaces. With the risk of belabouring the point, in the case of the multiplicative operation, if we instead denote the map by μ, and let $\{\phi_i\}_{i \in I}$ be an atlas of $\mathcal{G}$, what we mean to say is that if $g_j \in \operatorname{im} \phi_j$, $g_k \in \operatorname{im} \phi_k$, $g_\ell \in \operatorname{im} \phi_\ell$, $\mu(g_j, g_k) = g_j \cdot g_k = g_\ell$, then the composition $\phi_\ell^{-1} \circ \mu \circ (\phi_j, \phi_k)$ is a smooth function:

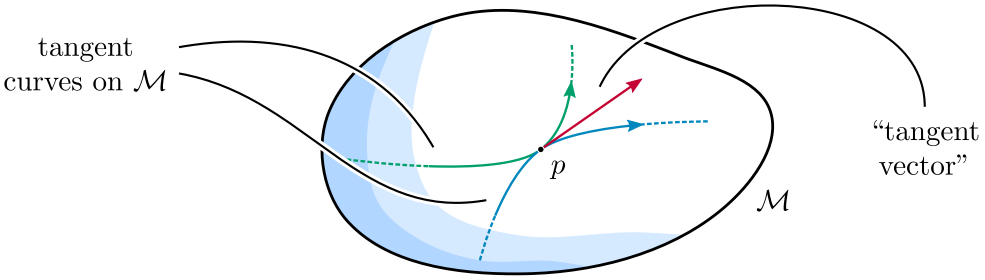


(The above illustration is what I like to call a "*blubber diagram,*" and Prof. Isham makes great use of them in *Modern Differential Geometry for Physicists*. Since this was the way I first learned differential geometry, you're going to be seeing a lot of them.)

Let $\mathcal{M}$ be a manifold and p be a point thereon.

DEFINITION. (VECTOR AT A POINT ON A MANIFOLD.) A vector at p is an equivalence class of curves that are tangent at the point p.

This is traditionally illustrated by the following blubber diagram:



Since I got some questions about this during the presentation, allow me to go into some further details on this. As Prof. Isham writes in his book, a *curve* on a manifold $\mathcal{M}$ is a smooth map σ from some interval $(-\varepsilon, \varepsilon)$ of the real line into $\mathcal{M}$. Two curves σ_1, σ_2 are tangent at a point $p \in \mathcal{M}$ if:

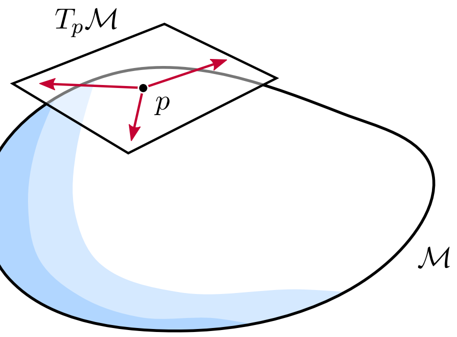
(a) $\sigma_1(0) = \sigma_2(0) = p$;
(b) in some local coordinate system $(x^1, x^2, \ldots, x^n)$ around the point, the two curves are 'tangent' in the usual sense as curves in $\mathbb{R}^n$:

$$\frac{dx^i}{dt}(\sigma_1(t))\bigg|_{t=0} = \frac{dx^i}{dt}(\sigma_2(t))\bigg|_{t=0}$$

for $i = 1, 2, \ldots, n$.

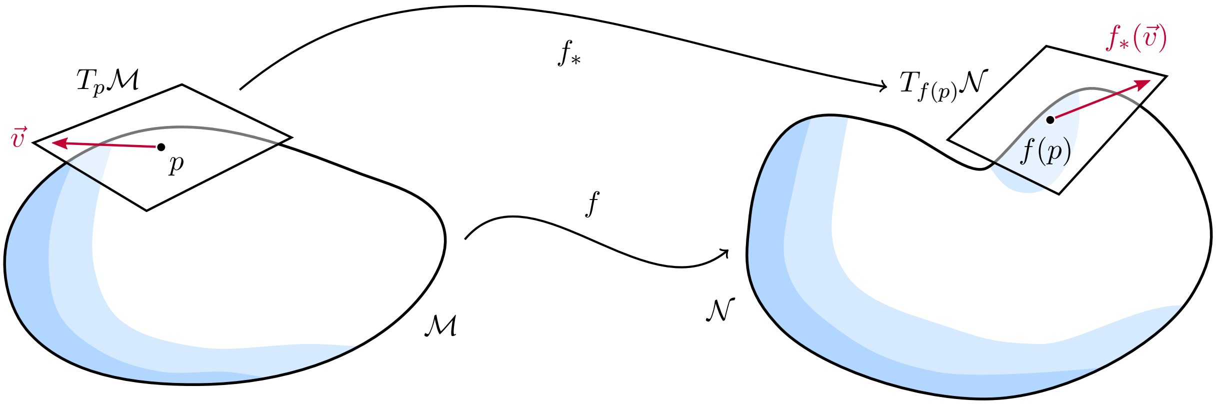
DEFINITION. (TANGENT SPACE AT A POINT ON A MANIFOLD.) The tangent space $T_p\mathcal{M}$ is the set of all tangent vectors at the point p. It is an easy matter to prove that $T_p\mathcal{M}$ indeed has the structure of a vector space.

On a blubber diagram, with manifolds are illustrated as two-dimensional surfaces, the tangent space at a point is illustrated as a two-dimensional plane intersecting the manifold at that point and tangent to that point in three dimensions. The vectors are then illustrated as arrows on that plane.



Arguably, this is a deeply misleading way of illustrating things, since the very point of defining vectors in the way we do (as equivalence classes of curves) is to avoid having to resort to embedding our manifolds into higher-dimensional Euclidean space. Nevertheless, it is a very useful schematic for conveying ideas.

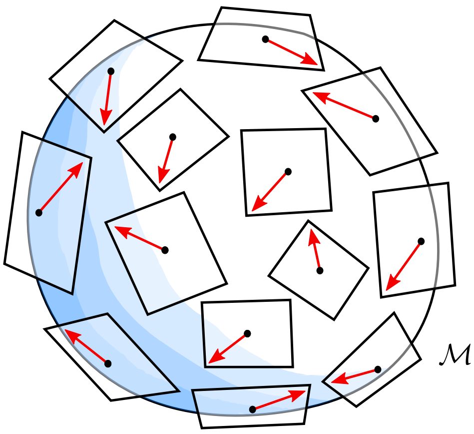
DEFINITION. (PUSHFORWARD.) A smooth map $f : \mathcal{M} \to \mathcal{N}$ naturally induces a map $f_* : T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$ called the pushforward.



The precise form of how this works out in the nitty-gritty algebra comes from how the vectors are defined as tangents, that is, as directional derivatives, but I won't bore you with that, it's a standard thing, for now, all you need to know is that there's an unambiguous way to define this thing naturally. See either Isham's or Nakahara's book for the details.

Definition. (Vector field on a manifold.) A vector field X on a manifold is a smooth assignment of a tangent vector $X_p \in T_p\mathcal{M}$ at each point $p \in \mathcal{M}$.

In terms of blubber diagrams, a vector field can be drawn as follows:



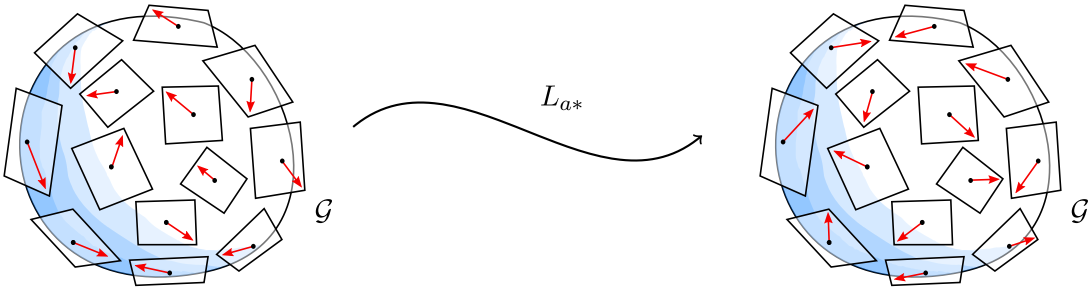
Definition. (Left-translation.) Let a, g be elements of a Lie group G. The left-translation $L_a : \mathcal{G} \to \mathcal{G}$ of g by a is defined by

$$L_a(g) = ag.$$

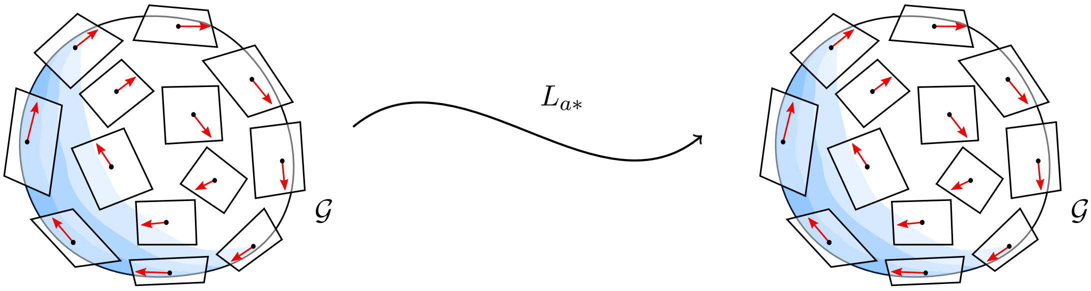
The fact that this merits a definition of its own makes it seem far more profound than it really is. Essentially, all we're doing is being very careful about pointing out that, yes, $\mathcal{G}$ is a group, but it's also a manifold.

Definition. (Left-invariant vector field.) Let X be a vector field on a Lie group $\mathcal{G}$. X is said to be a left invariant vector field if $L_{a*}X|_g = X|_{ag}$ for all $a, g \in \mathcal{G}$.

So, for instance, this would *not* be an invariant vector field on $\mathcal{G}$:



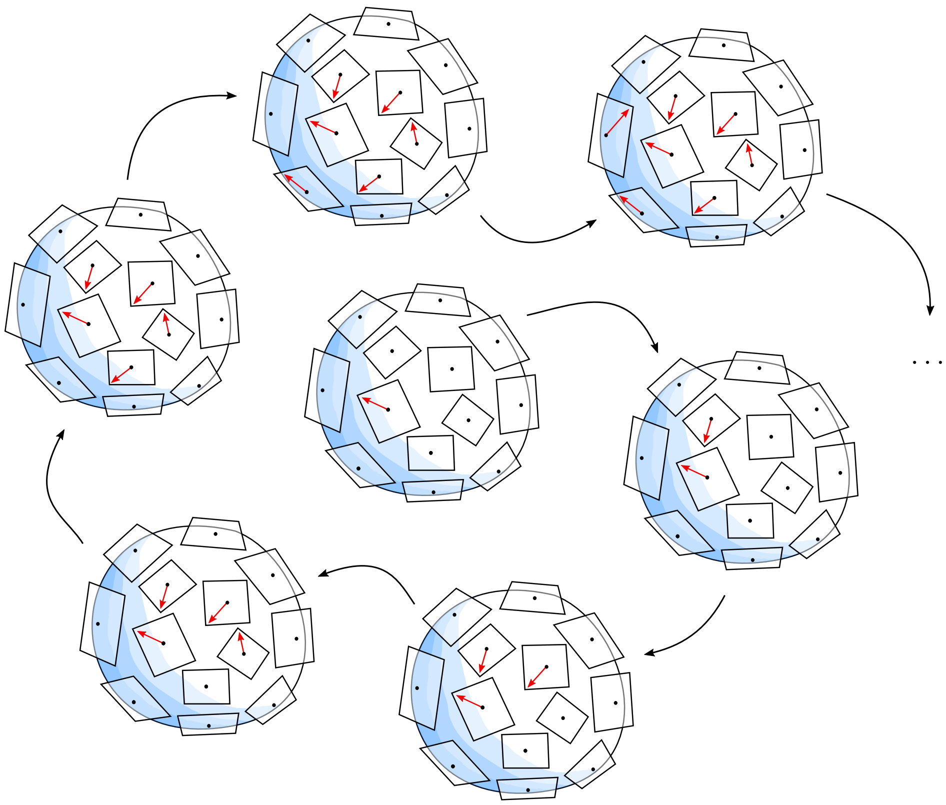
But if this holds true for all $a \in \mathcal{G}$:



then we are indeed looking at an invariant vector field.

Lemma. A single vector at a given point of a Lie group defines a left-invariant vector field on said Lie group and vice versa.

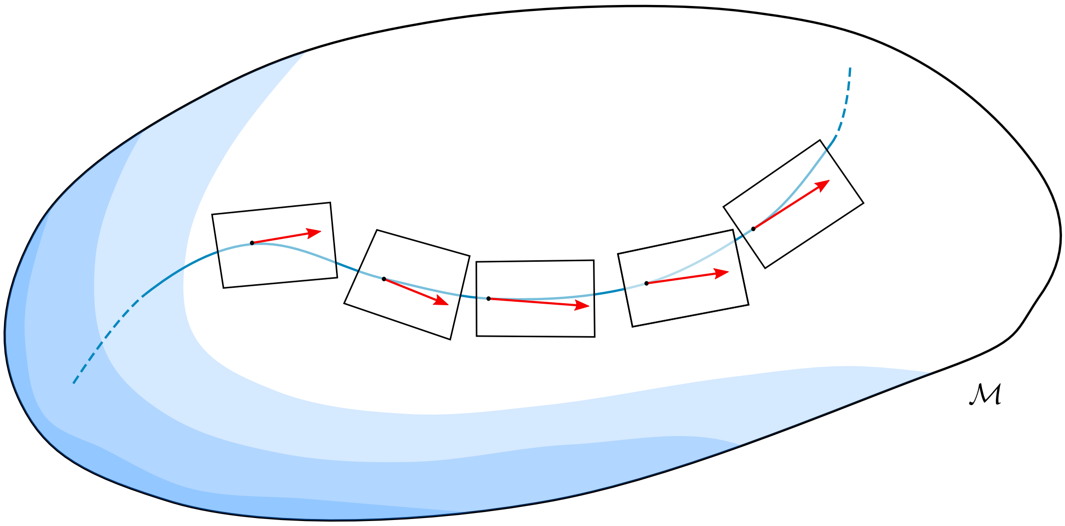
Having picked some arbitrary point, the vice versa thing is obvious. For the first part, just take a vector at that point on the manifold and left-translate it to every other point on said manifold, and presto, there you have it! The smoothness of the vector field being guaranteed by the smoothness of the multiplication operation.



Definition. (Lie algebra, preliminary.) The set of left-invariant vector fields on a Lie group $\mathcal{G}$ is called its Lie algebra.

As per previous spoken remarks, this is of course a vector space. But why then call it an algebra? A vector space on it's own isn't an algebra. Sure, we've got the additive structure, but for something to be an algebra, it has to have a bilinear form to it as well! Fortunately, there exists a very natural bilinear form to be found here!

Definition. (Integral curves.) Let X be a vector field on a manifold $\mathcal{M}$. An integral curve $x(t)$ of X is a curve on $\mathcal{M}$ whose tangent vector at $x(t)$ is $X|_{x(t)}$. Note: Given a point p on a manifold $\mathcal{M}$ and a vector field X, there exists a unique integral curve passing through p.



The uniqueness follows from the smoothness criteria for the definition of a vector field, which renders the existence and uniqueness problem equivalent to the existence and uniqueness problem of solutions for ODEs, which is a problem that has been solved long ago.

The next two definitions I give are somewhat qualitative and informal. I would once again advice you to have a look at either Nakahara or Isham for the definition (or, just Google it, there are good stuff on the web too...)

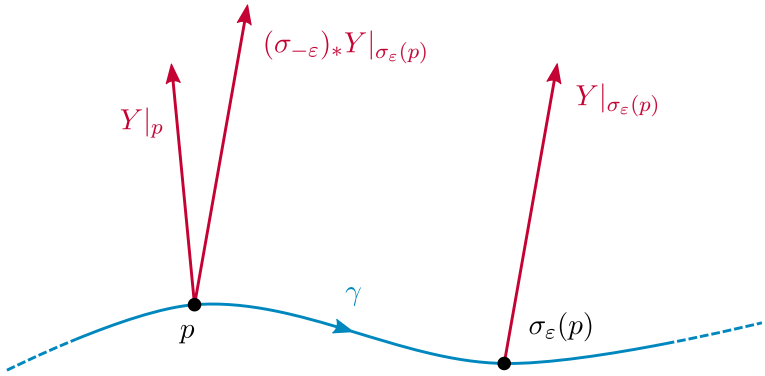
Definition. (Lie derivative.) Vectors $\vec{v}_1$ and $\vec{v}_2$ at points p_1 and p_2 along an integral curve on a manifold $\mathcal{M}$ cannot be *directly* compared with one another, as they live in different vector spaces, $T_{p_1}\mathcal{M}$ and $T_{p_2}\mathcal{M}$. However, given said integral curve $\gamma : \mathbb{R} \to \mathcal{M}$, we can define a map $\sigma_\varepsilon : \mathcal{M} \to \mathcal{M}$, $\varepsilon \in \mathbb{R}$, which is such that if $p = \gamma(t)$ is a point on the curve, then σ_ε maps it to $\gamma(t + \varepsilon)$. We then define the *Lie derivative of the vector field* Y *with respect to the vector field* X *at the point* p to be

$$\mathcal{L}_X Y|_p = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[(\sigma_{-\varepsilon})_* Y|_{\sigma_\varepsilon(p)} - Y|_p],$$

where σ_ε is defined by the integral curve defined by X that passes through p.

To give a better understanding of this formula, allow me to explain it in words and pictures. What we are doing is taking the vector $Y|_{\sigma_\varepsilon(p)} \in T_{\sigma_\varepsilon(p)}\mathcal{M}$, pushforwarding it to $T_p\mathcal{M}$ with $(\sigma_{-\varepsilon})_*$, subtracting the vector $Y|_p$ and dividing by ε. In a sense, this gives a measure of how much the vector field Y "changes along the vector field X." The illustration below has been

~~shamelessly copied~~ *adapted* from Nakahara:



Definition. (Lie bracket.) The Lie bracket is then given the rather straightforward definition

$$[X, Y]|_p = \mathcal{L}_X Y|_p.$$

Picking a coordinate system and expressing everything therein, it becomes readily apparent that the Lie bracket obeys

- $[x, x] = 0$,
- $[x, y] = -[y, x]$

and it is then barely an inconvenience to verify that it is bilinear, and with just a little more tediousness, one can show that it satisfies the Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Specifically, see Nakahara, 2nd edition, p. 191-3.

Definition. (Lie algebra, midway.) The set of left-invariant vector fields on a Lie group $\mathcal{G}$, together with its Lie bracket $[\cdot, \cdot]$ is called its Lie algebra, and is denoted $\mathfrak{g}$ (in Gothic Fraktur).

Why the Gothic Fraktur for denoting Lie algebras? Well, despite the fact that old Sophus Lie was a good Norwegian who wrote all his works in French, much early work on Lie algebras were done by the Germans Killing and Weyl, and back in those days, Germans liked to print everything in Gothic Fraktur.

So there you have it, that's where Lie algebras originally came from, and I personally always like to include this little historical interlude, because I feel that they help to make Lie algebras appear more tangible to some extent or another. They become something you can visualize in your head. There is of course also the fact that Lie algebras are tremendously important in quantum and particle physics, and when they show up, it because there is some wider Lie group acting in the background, the Lie group reflecting some continuous symmetry exercising its influence over the form of the equations and their solutions.

But mathematicians will not accept such silly straight jackets! Mathematicians like to abstract and to generalize! And just as we went from having logarithms defined first only for integers to natural powers, to having them defined for, well, eventually the whole complex plane, so nowadays, Lie algebras are generally defined in this way, as M. Benoist defines them.

Definition. (Lie algebra, final.) Letting $\mathbb{K}$ be a field, a Lie algebra is a $\mathbb{K}$-vector space $\mathfrak{g}$ together with a bilinear antisymmetric form, denoted $[\cdot, \cdot]$ obeying the Jacobi identity.

Example. Given a field $\mathbb{K}$, we may give the vector space $\mathrm{End}(\mathbb{K}^d)$ the structure of a vector space by means of the commutator $[A, B] = AB - BA$ for all $A, B \in \mathrm{End}(\mathbb{K}^d)$.

Theorem. (Ado.) Every finite dimensional Lie algebra over a field $\mathbb{K}$ can be viewed as a subalgebra of $\mathrm{End}(V)$ for some vector space V over $\mathbb{K}$ under the commutator bracket.

(*We'll get back to why I underlined that.*)

Definition. (Abelian, ideal, nilpotence.) It is easy to extend the notion of an abelian group to a Lie algebra. For an abelian group G, we have $a \cdot b = b \cdot a$ for all $a, b \in G$, so for a Lie algebra $\mathfrak{g}$, we should have $[x, y] = [y, x]$ for all $x, y \in \mathfrak{g}$. Since antisymmetry already gives us $[x, y] = -[y, x]$, this is equivalent to saying that $[\mathfrak{g}, \mathfrak{g}] = 0$.

(*I'd like to make a comment here that whereas every abelian Lie group has an abelian Lie algebra, there are non-abelian Lie groups that nevertheless have abelian Lie algebras.*)

The notion of an ideal from ring theory can similarly be extended to Lie algebras. Given a Lie algebra $\mathfrak{g}$, a subspace $\mathfrak{h} \subset \mathfrak{g}$ constitutes a Lie algebra ideal if $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.

While the notion of a nilpotent element cannot be extended to a Lie algebra ...

(*Well, eventually we can, and we can already now, but it would be really trivial and stupid, since the bracket of the same two elements is always zero.*)

... we can extend the notion of a nilpotent (resp. solvable) group to Lie algebras. We say that a Lie algebra is nilpotent (resp. solvable) if there exists a flag of ideals

$$0 = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_i \subset \cdots \subset \mathfrak{g}_p = \mathfrak{g}$$

such that $[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i-1}$ (resp. $\mathfrak{g}_i/\mathfrak{g}_{i-1}$ is abelian) for all i.

Let V be a vector space over a field $\mathbb{K}$ of dimension d and let $\mathfrak{g} \subset \mathrm{End}(V)$ be a Lie subalgebra.

Theorem. (Engel.) If every element of $\mathfrak{g}$ is nilpotent when regarded as a matrix, i.e., $M^n = 0$ for some n, then there exists a basis of V such that every element of $\mathfrak{g}$ has the form of a strictly upper triangular matrix.

Theorem. (Lie.) If $\mathfrak{g}$ is solvable and $\mathbb{K}$ be algebraically closed, then there exists a basis of V such that every element of $\mathfrak{g}$ has the form of an upper triangular matrix.

M. Benoist gives the proofs of both theorems in his text, but a quick qualitative description should be enough to convince you that these "ought" to be correct.

What is a strictly upper triangular matrix? Well, it's a matrix which maps every vector to a lower and lower subspace. In linear algebra, it's a standard homework question to prove that for any nilpotent matrix, you can find a basis such that the matrix is strictly upper diagonal.

What is an upper triangular matrix? Well, it's a matrix with a set of invariant subspaces of varying dimensions, each one contained within the next. From that insight, the rest almost becomes a linear algebra problem.

Okay, kids! Now comes the tricky part!

A misunderstanding that has been the bane of many a young mathematician, and even more physicists, as they first venture into Lie algebras, is keeping track of what the different matrices mean! In fact, it's a bane for many people when they start working with matrices in the first place! Allow me to explain.

As noted earlier from Ado's theorem, any finite-dimensional Lie algebra can be encoded in terms of endomorphisms of some vector space under the commutator bracket.

Example. Consider the Lie algebra

$$\mathfrak{su}(2) := \left\{ \begin{pmatrix} ia & -\overline{z} \\ z & -ia \end{pmatrix} \;\middle|\; a \in \mathbb{R}, z \in \mathbb{C} \right\},$$

which has basis

$$u_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad u_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

and we may note

$$[u_1, u_1] = 0, \qquad [u_1, u_2] = 2u_3, \qquad [u_1, u_3] = -2u_2.$$

The thing is of course, that these endomorphisms over a vector space themselves define a vector space, and so we have endomorphisms of endomorphisms, and because we are absolutely **Satanic**, we encode these in terms of matrices.

And the worst thing of all is that the endomorphisms *themselves* define endomorphisms of endomorphisms!

Definition. (Adjoint representation.) Let X, Y be two elements of a Lie algebra $\mathfrak{g}$. Then we may define an action of X on Y known as the adjoint action by

$$\mathrm{ad}_X(Y) = [X, Y].$$

First viewing u_1, u_2, u_3 above as basis vectors, we can then represent u_1 when viewed as the element of the algebra by the matrix

$$[\mathrm{ad}_{u_1}] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}.$$

This is known as the adjoint representation.

(*I have here of course assumed that I didn't need to remind you of how the action of an algebra corresponds to modules and representations. M. Benoist first gives that definition on p. 19, but I felt the need to start using it already here.*)

I cannot stress how much this managed to confuse me once upon a time! Anyway, this is why it makes sense to make the following definition.

Definition. (Killing form.) Given a Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$, we define the Killing form $B = B_{\mathfrak{g}}$ to be the symmetric bilinear form $\mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ given by

$$B(X, Y) = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}_X \mathrm{ad}_Y).$$

(*The subscript $\mathfrak{g}$ means that we're looking at $\mathfrak{g}$ as a vector space. Since we're taking a trace, we obviously don't have to specify a basis.*)

(*Earlier we noted that we could extend the notion of an ideal to a Lie algebra. Now we note that we can also extend the notion of a radical ideal.*)

Definition. (Radical.) The radical of a Lie algebra $\mathfrak{g}$, denoted $\mathrm{rad}(\mathfrak{g})$, is the maximal solvable ideal of $\mathfrak{g}$.

(*In the case of modules over rings and algebras, we define simple modules to be modules which are non-zero and without proper submodules. We define semisimple modules to be modules which are direct sums of simple modules. Similarly, we have as follows.*)

Definition. (Simple and semisimple Lie algebras.) We say that a Lie algebra is simple if it contains no proper ideals and if its dimension is greater than 1.

(*The dimension demand is analogous to the non-zero demand for modules. Note that one-dimensional Lie algebras are always abelian for obvious reasons, but, well, for some probably well founded reason, akin to why 1 is not considered a prime, they are traditionally not considered simple Lie algebras.*)

We say that a Lie algebra $\mathfrak{g}$ is semisimple if it fulfills any and all of the following equivalent conditions:

(iii) $\mathfrak{g}$ is a direct sum of simple ideals.
(i) Every abelian ideal of $\mathfrak{g}$ is trivial.
(ii) The radical of $\mathfrak{g}$ is trivial.
(iv) The Killing form $B_{\mathfrak{g}}$ is non-degenerate.

(*The connection to the adjoint representation is, as mentioned, very important in Lie algebra. So much so that we may make the following definition.*)

Definition. (Nilpotent and semisimple elements.) Let X be an element of a Lie algebra. We say that it is nilpotent if the endomorphism ad_X is nilpotent. We say that it is semisimple if the endomorphism ad_X is semisimple.

(*This then leads to the following proposition.*)

Proposition. (Jordan decomposition.) Let $\mathfrak{g}$ be a semisimple Lie algebra. Every element X of $\mathfrak{g}$ admits a unique decomposition $X = X_s + X_n$ with X_s semisimple, X_n, and $[X_s, X_n] = 0$.

(*This really is the same as the standard Jordan-Chevalley decomposition in linear algebra, and the proof can be found in most decent books out there on the subject.*)

Representations of $\mathfrak{sl}_2$.

$$\mathfrak{sl}(2, \mathbb{K}) := \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{K} \right\}$$

(*Now, you all know about modules and representations and simple modules and irreducible representations and so forth, so let me begin this section with the very example that M. Benoist decides to lead with.*)

Example. The Lie algebra $\mathfrak{sl}(2, \mathbb{K})$ permits, for all $d \geq 0$, a representation of dimension $d + 1$ over the $\mathbb{K}$-vector space V_d which is made up of homogeneous polynomials of degree d over $\mathbb{K}^2$ as follows.

A base of the algebra is given by the matrices

$$X := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and they obey the relations

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H.$$

Now, a basis for the vector space V_d is given by the monomials $\{x^d, x^{d-1}y, \ldots, xy^{d-1}, y^d\}$ and so a representation of the algebra is given by the assignment

$$X \mapsto x\frac{\partial}{\partial y}, \qquad H \mapsto x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \qquad Y \mapsto y\frac{\partial}{\partial x}.$$

In this representation, X, Y only permits trivial eigenvectors, but for H, we have for a given monomial $x^{d-k}y^k$ that

$$Hx^{d-k}y^k = (d-k)x^{d-k}y^k - kx^{d-k}y^k = (d-2k)x^{d-k}y^k.$$

We say that V_d is the representation of weight d because d is the highest possible eigenvalue of H. The representations V_d are in fact the irreducible representations of $\mathfrak{sl}(2, \mathbb{K})$.

(*Now, if you've taken a course or two of quantum mechanics, you're probably thinking that all of this looks very familiar. X raises the power of x in a given monomial and lowers the power of y, Y does vice-versa, and H gives the difference between the x power and the y power in the monomial. This looks very similar to the ladder operators and the square of the magnitude operator for orbital angular momentum.*)

(*And if you think that, you're not wrong, because that is in fact exactly what this is! And we'll get there to explaining why that is!*)

(*And if you haven't done quantum mechanics, no worries, but then this all probably looks very arbitrary to you. Like, why come up with such a system, and give them these names and introduce this term called weight or whatever? Where is this leading? Well...*)

DEFINITION. ($\mathfrak{sl}_2$-TRIPLETS.) A triplet of elements (X, H, Y) of a Lie algebra satisfying the above commutation relations is called a $\mathfrak{sl}_2$-triplet.

(*It goes without saying that a Lie algebra containing such a triplet has a Lie subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{K})$.*)

Now then comes the mind-boggling thing!

THEOREM. (JACOBSON, MOROZOV.) Every nilpotent element x in a semisimple Lie algebra $\mathfrak{g}$ can be extended to a $\mathfrak{sl}_2$-triplet (x, h, y).

That is, for every nilpotent element x can we find elements h, y in the Lie algebra such that the three of them obey the commutation relations!

(*This is why we can recognize the behaviour of them from orbital angular momentum in quantum mechanics (where we're actually dealing with the Lie algebra of the group $SO(3)$! That is why this particular Lie algebra and its representations are so important to study!*)

ROOT SYSTEMS.

Let $\mathfrak{g}$ be a semisimple Lie algebra.

So what to do next? Well, in his book "Lie Algebra in Particle Physics," Howard Georgi of Harvard gives a good description: "Our ultimate goal," he says, "is to completely reduce the Hilbert space of the world to block diagonal form." We want to, as closely as possible, diagonalize everything! Why? Well, because it makes the computations so much easier! And the way to start out is to look at commutative properties.

Definition. (Cartan subalgebra.) A Cartan subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a maximal commutative subalgebra of semisimple elements.

The Cartan subalgebra is not necessarily unique! But remarkably that doesn't matter! They all give the same final results.

Next, we really double down on diagonalization!

For $\alpha \in \mathfrak{h}^*$ (*that is the dual of* $\mathfrak{h}$) denote

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | [H, X] = \alpha(H)X \quad \text{for all } H \in \mathfrak{h}\}.$$

Definition. (System of roots.) We define the set

$$\Delta = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0 \text{ and } \alpha \neq 0\}$$

to be the system of roots of $\mathfrak{g}_\alpha$.

Why is this of interest? Well, because if every $X \in \mathfrak{g}$ was in one of the $\mathfrak{g}_\alpha$, then we truly could diagonalize everything! But, alas, obviously, we are not always going to be that fortunate. Some little final module $\mathfrak{g}_0$ will remain, and we will have decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus (\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha).$$

M. Benoist's Theorem 2.20 contains a wide number of statements, all of which are of course very much interesting on their own. Nevertheless, if you were to ask me what I think are the two most important points to take away, it is that:

- $\mathfrak{g}_0 = \mathfrak{h}$; and,
- The roots corresponds to weights in $\mathfrak{sl}_2$-triplets.

Example. (Colour charge, adapted from Howard Georgi.) The Lie group SU(3) is of interest in physics as it is a symmetry group of the quarks, giving rise to what is known as *colour charge*. Therefore, the Lie algebra, $\mathfrak{su}(3)$, turns out to be of interest too. It may be defined as the set of 3×3 hermitian, traceless matrices, and traditionally, it's basis is given in terms of the so-called *Gell-Mann matrices:*

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

We find a Cartan subalgebra $\mathfrak{h}$ for $\mathfrak{su}(3)$ generated by the matrices λ_3 and λ_8. These have three simultaneous eigenvectors, with corresponding eigenvalues

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mapsto (1/2, \sqrt{3}/6), \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mapsto (-1/2, \sqrt{3}/6), \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto (1/2, -\sqrt{3}/3).$$

Since

$$(\lambda_1 + i\lambda_2)^2 = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we should, by Jacobson and Morozov's theorem, be able to extend this to an $\mathfrak{sl}_2$-triplet. And indeed we can. Writing $X = \lambda_1 + i\lambda_2$, $Y = \lambda_1 - i\lambda_2$, and $H = \lambda_3$, we recover the desired commutation relations.