Tutorium zur Linearen Algebra I

zu Kapitel L8/L9, am 11.11.2019

$$\mathbb{C} := \mathbb{R}^2 = \{(x,u) : x,u \in \mathbb{R}\}, \qquad + : (x_1,u_1)+(x_2,u_2) := (x_1+x_2, u_1+u_2)$$
$$\cdot : (x,u)\cdot(y,v) := (xy-uv, xv+uy)$$

$$\underbrace{(1,0)}_{} = 1_{\mathbb{C}} : \underbrace{(1,0)}_{=1_{\mathbb{C}}=1}\cdot(y,v) = (1\cdot y - 0\cdot v, 1\cdot v + 0\cdot y) = (y,v)$$

$$i = \underbrace{(0,1)}_{} : (0,1)^2 = (0,1)\cdot(0,1) = (0\cdot 0 - 1\cdot 1, 0\cdot 1 + 1\cdot 0)$$
$$= (-1, 0)$$
$$= \underbrace{-(1,0)}_{\text{add. Inv. von } (1,0),} = -1_{\mathbb{C}}$$

$$\boxed{i^2 = -1}$$

d.h. $\big(-(1,0)\big) + (1,0) = (0,0)$ ✓

und $(-1,0) + (1,0) = (0,0)$ ✓

$\Rightarrow -(1,0) = (-1,0) = -1_{\mathbb{C}}$

$$x + iu \overset{!}{=} \underline{\underline{(x,u)}} = x\cdot\underbrace{(1,0)}_{1_{\mathbb{C}}} + u\cdot\underbrace{(0,1)}_{i} = x + ui$$

$$\mathbb{C} = \{x + iu;\ x,u \in \mathbb{R}\}$$

x: Realteil, u: Imag-teil

$i^1 = i$, $i^2 = -1$

$i^3 = -i$, $i^4 = 1$

$$(x+iu)^2 = (x+iu)\cdot(x+iu)$$
$$= x^2 + 2uxi + i^2u^2 = \underline{x^2 - u^2} + \underline{2uxi}$$ ✓

$$i^k = \begin{cases} i, & k \equiv 1 \pmod 4 \\ -1, & k \equiv 2 \pmod 4 \\ -i, & k \equiv 3 \pmod 4 \\ 1, & k \equiv 0 \pmod 4 \end{cases}$$

$$(x+iu)^n = \sum_{k=0}^{n} \binom{n}{k} \underbrace{(iu)^k}_{i^k u^k} x^{n-k} = \sum_{\substack{k=0 \\ k\equiv 1(4)}}^{n} \binom{n}{k} i u^k x^{n-k}$$

$$- \sum_{k\equiv 2(4)} \binom{n}{k} u^k x^{n-k}$$

$$- \sum_{k\equiv 3(4)} \binom{n}{k} i u^k x^{n-k}$$

$$+ \sum_{k\equiv 0(4)} \binom{n}{k} u^k x^{n-k}$$

$$= \left(- \sum_{k\equiv 2(4)} \binom{n}{k} u^k x^{n-k} + \sum_{k\equiv 0(4)} \binom{n}{k} u^k x^{n-k} \right)$$

$$+ \left(\sum_{\substack{k=0 \\ k\equiv 1(4)}}^{n} \binom{n}{k} u^k x^{n-k} - \sum_{k\equiv 3(4)} \binom{n}{k} u^k x^{n-k} \right) i$$

$z = x + iu$

$\bar{z} = \overline{x+iu} = x - iu$

$|z|^2 = z \cdot \bar{z} = (x+iu)\cdot(x-iu) = x^2 + u^2 + \underbrace{(-xu+ux)\, i}_{=0} = \underset{\in \mathbb{R}}{x^2+u^2}$

$|z| = \sqrt{x^2+u^2}$ Betrag von z

$z \neq 0 \Rightarrow \frac{1}{z} = \frac{\bar{z}}{z\cdot\bar{z}} = \frac{1}{\underbrace{z\cdot\bar{z}}_{\in\mathbb{R}}} \cdot \bar{z} = \frac{1}{|z|^2}\cdot \bar{z}$

$z = x + iu \Rightarrow \frac{1}{z} = \frac{1}{x^2+u^2}\cdot(x-iu)$ $\left\lceil = \underset{\in\mathbb{R}^2}{\left(\frac{x}{x^2+u^2}, \frac{-u}{x^2+u^2}\right)} \right\rfloor$

$\frac{1}{i} = -i \;\Big|\; \overset{\cdot i}{\Leftrightarrow} \; \underbrace{\frac{1}{i}\cdot i}_{=1} = (-i)\cdot i = -i^2 = -(-1) = 1$ ✓ $\rfloor$

$\frac{1}{i} = \frac{1}{0+1\cdot i} = \frac{1}{0^2+1^2}\cdot(-i) = -i$ ✓

$$\mathbb{R}[T] = \left\{ \sum_{i=0}^{n} \alpha_i T^i \; ; \; n \in \mathbb{N}_0 \, , \; \alpha_i \in \mathbb{R} \right\} \qquad \text{Polynome}$$

Polynomdiv.: $P, Q \in \mathbb{R}[T] \Rightarrow \exists!\, S, R \in \mathbb{R}[T] : P = S \cdot Q + R$,
$\deg R < \deg Q$.

$$(T^4 - 2T^3 + 6T - 1) : (T-1) = T^3 - T^2 - T + 5 + \frac{4}{T-1}$$

$$\begin{array}{l}
-(T^4 - T^3) \\
\hline
\quad -T^3 + 6T \\
\quad -(-T^3 + T^2) \\
\hline
\qquad -T^2 + 6T \\
\qquad -(-T^2 + T) \\
\hline
\qquad\quad 5T - 1 \\
\qquad\quad -(5T - 5) \\
\hline
\qquad\qquad 4
\end{array}$$

$$P : Q = S + \frac{R}{Q}$$

$$\Leftrightarrow \quad \underline{P = S \cdot Q + R}$$

$$(T^5 - 5T^3 + 4T^2 - 6T + 4) : (T^2 + 1) = \underline{T^3 - 6T + 4}$$

$$\begin{array}{l}
-(T^5 + T^3) \\
\hline
\quad -6T^3 + 4T^2 \\
\quad -(-6T^3 - 6T) \\
\hline
\qquad 4T^2 + \underbrace{6T - 6T} + 4 \\
\qquad -(4T^2 + 4) \\
\hline
\qquad\qquad 0
\end{array}$$

$$T^2 + pT + q$$

$$T_{1/2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

$$T^2 + 1 = \underbrace{(T - i)}_{\text{hat Nst. } i} \cdot \underbrace{(T + i)}_{\text{hat Nst. } -i}$$

$$T^2 - 6T + 2 = (T - 3 - \sqrt{7}) \cdot (T - 3 + \sqrt{7}), \quad T_{1/2} = 3 \pm \sqrt{9 - 2} = 3 \pm \sqrt{7} \in \mathbb{R}$$

$$T^2 - 2T + 10 = (T - 1 - 3i) \cdot (T - 1 + 3i)$$

$$\ulcorner T_{1/2} = 1 \pm \sqrt{1-10} = 1 \pm \sqrt{-9} = 1 \pm \sqrt{-1} \cdot \sqrt{9} = 1 \pm 3i \lrcorner$$

$T^3 - 3T^2 + 2T - 6$ hat Nst. $T = 3$

$\ulcorner 27 - 3 \cdot 9 + 2 \cdot 3 - 6 = 0 \checkmark \lrcorner$

$$(T^3 - 3T^2 + 2T - 6) : (T - 3) = T^2 + 2$$
$$\underline{-(T^3 - 3T^2)}$$
$$2T - 6$$
$$\underline{-(2T - 6)}$$
$$0$$

$$1 \cdot T^3 - 3T^2 + 2T - 6 = (T-3) \cdot (T^2 + 2)$$
$$= (T-3) \cdot (T - i\sqrt{2}) \cdot (T + i\sqrt{2})$$

$\sum_{i=0}^{3} \alpha_i T^i = \alpha_3 T^3 + \alpha_2 T^2 + \alpha_1 T + \alpha_0$ ← allg. Pol. vom Grad 3, falls $\alpha_3 \neq 0$

$0 \cdot T^3 + \underline{2T^2} - 1$ hat Grad 2

$$\left(\sum_i \alpha_i T^i\right) \cdot \left(\sum_j \beta_j T^j\right) = \sum_k c_k T^k$$

wo $c_k = \sum_{i=0}^{k} \alpha_i \beta_{k-i}$

$k = i + j \Leftrightarrow j = k - i$

$$\ulcorner \text{l.S.} = \sum_i \sum_j \alpha_i T^i \beta_j T^j = \sum_{i,j} \alpha_i \beta_j T^{i+j} = \sum_k \underbrace{\left(\sum_i \alpha_i \beta_{k-i}\right)}_{c_k} T^k \lrcorner$$

Zwei Körper K_1, K_2 heißen <u>isomorph</u>, falls es einen <u>Körperisom.</u> $f: K_1 \to K_2$ gibt, | $K_1 \cong K_2$

d.h. falls $f(x_1 + x_1') = f(x_1) + f(x_1')$

und $f(x_1 \cdot x_1') = f(x_1) \cdot f(x_1')$ für alle $x_1, x_1' \in K_1$

und f bijektiv.

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zu LG: Sei K Körper.

Ein K-VR V hat $+ : V \times V \to V$ innere Verkn.

$\cdot : K \times V \to V$ äußere Verkn.

so, dass
(1) $(V, +)$ ab. Gr., neutr. El. 0
(2) $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta x$, $x \in V$, $\alpha, \beta \in K$
(3) $\alpha \cdot (x + y) = \alpha x + \alpha y$, $x, y \in V$, $\alpha \in K$
(4) $\alpha \cdot (\beta \cdot x) = (\alpha \cdot \beta) \cdot x$, $x \in V$, $\alpha, \beta \in K$
(5) $1 \cdot x = x$, $x \in V$.

z.B. $\mathbb{R}[t]$ ist $\mathbb{R}$-VR

- $\mathbb{R}^n$, $\mathbb{C}^n$ sind $\mathbb{R}$-VRe,

$\mathbb{R}^n \subseteq \mathbb{C}^n$: $\mathbb{R}^n$ ist UVR des $\mathbb{R}$-VRes $\mathbb{C}^n$

$\to$ $\ulcorner$ $0 \in \mathbb{R}^n$ ✓, $\forall x, y \in \mathbb{R}^n \Rightarrow x + y \in \mathbb{R}^n$ ✓, $\forall \alpha \in \mathbb{R}\ \forall x \in \mathbb{R}^n: \alpha x \in \mathbb{R}^n$ $\lrcorner$

- $\mathbb{C}^n$ ist $\mathbb{C}$-VR ✓. Ist $\mathbb{R}^n$ ein UVR des $\mathbb{C}$-VRes $\mathbb{C}^n$? Nein

$\ulcorner$ $0 \in \mathbb{R}^n$ ✓, $\forall x, y \in \mathbb{R}^n \Rightarrow x + y \in \mathbb{R}^n$ ✓, $\forall \alpha \in \mathbb{C}\ \forall x \in \mathbb{R}^n: \alpha x \in \mathbb{R}^n$? f

<u>$\mathbb{R}^n$ ist kein $\mathbb{C}$-VR:</u> Bsp.: $\alpha = i$, $x = \binom{1}{0} \in \mathbb{R}^2$: $\alpha x = i \binom{1}{0} \notin \mathbb{R}^2$ $\lrcorner$

$V = \mathbb{R}[T]$ ist $\mathbb{R}$-VR

$U = \{ P \in \mathbb{R}[T];\ \alpha_0 = 0 \text{ in } P = \sum_{i=0}^{m} \alpha_i T^i,\ m \in \mathbb{N}_0 \}$

$U = \{ P \in \mathbb{R}[T];\ P = \sum_{i=1}^{m} \alpha_i T^i,\ m \in \mathbb{N}_0 \}$

Frage: ist U ein UVR von V?

Beh.: Ja.

Bew.: • $0 \in U$ ✓ ⌜0 ist das Nullpolynom $\sum_{i=0}^{m} 0 \cdot T^i$⌟

• Seien $P, Q \in U$, etwa $P = \sum_{i=1}^{n} \alpha_i T^i$, $Q = \sum_{i=1}^{m} \beta_i T^i$.

Dann $P + Q = \sum_{i=1}^{n} \alpha_i T^i + \sum_{i=1}^{m} \beta_i T^i = \sum_{i=1}^{\max\{n,m\}} (\alpha_i + \beta_i) T^i \in U$. ✓

• Sei $P \in U$, $\alpha \in \mathbb{R}$, etwa $P = \sum_{i=1}^{n} \alpha_i T^i$.

Dann $\alpha \cdot P = \alpha \cdot \left(\sum_{i=1}^{n} \alpha_i T^i \right) = \sum_{i=1}^{n} (\alpha \alpha_i) T^i \in U$. ✓ $\square$

Das griechische Alphabet

A	α	alpha	E	ε	epsilon	I	ι	iota
B	β	beta	Z	ζ	zeta	K	$\kappa, \varkappa$	kappa
Γ	γ	gamma	H	η	eta	Λ	λ	lambda
Δ	δ	delta	Θ	ϑ, θ	theta	M	μ	mü
N	ν	nü	P	ϱ	rho	Φ	φ	phi
Ξ	ξ	xi	Σ	σ	sigma	X	χ	chi
O	o	omikron	T	τ	tau	Ψ	ψ	psi
Π	π	pi	Y	υ	ypsilon	Ω	ω	omega