$$A = \begin{pmatrix} 3 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 1 \end{pmatrix} \qquad A : \mathbb{R}^4 \to \mathbb{R}^3$$

$\operatorname{im} A \subset \mathbb{R}^3$

$$\left\langle \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{R}} = \operatorname{im} A$$

$$\parallel$$

$$\left\{ \begin{pmatrix} a_1 \\ 0 \\ a_3 \end{pmatrix} \;\middle|\; a_1, a_3 \in \mathbb{R} \right\} \leftarrow \text{Basis davon: } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= a_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + a_3 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\dim \operatorname{im} A = 2 = \operatorname{rk} A$

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 4 & 2 & 0 \\ 4 & 7 & 3 & 1 \\ 1 & 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & & & \\ 1 & & & \\ \cdots & & & 1 \end{pmatrix}$$

$$A' = \begin{pmatrix} 3 & 4 & 2 & 0 \\ 4 & 7 & 3 & 1 \\ 1 & 3 & 1 & 1 \end{pmatrix} \cdot(-1) \qquad \operatorname{rk} A' = ?$$

$A'' = E_1 \cdot A'$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Da E_1 inv'bar: $\ker A'' = \ker A'$. Also: $\operatorname{rk} A'' = \operatorname{rk} A'$

$$A'' = \begin{pmatrix} 3 & 4 & 2 & 0 \\ 1 & 3 & 1 & 1 \\ 1 & 3 & 1 & 1 \end{pmatrix} \cdot(-1)$$

$A = E_2 \cdot A''$

Da E_2 inv'bar: $\operatorname{rk} A'' = \operatorname{rk} A$

$$A = \begin{pmatrix} 3 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & 4 & 2 & 0 \end{pmatrix} \cdot(-3)$$

$$\begin{pmatrix} 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & -1 & -3 \end{pmatrix}$$

$$\operatorname{rk} A' = \operatorname{rk} A = 2$$
$$\parallel$$
$$\dim(\mathbb{R}^4 / \ker A')$$
$$\parallel$$
$$4 - \dim \ker A'$$

Also: $\dim \ker A' = 2$

$$\begin{pmatrix} 1 & 3 & 1 & 1 \\ 0 & -5 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot\left(-\tfrac{1}{5}\right)$$

Für Normalform bräuchte hier 0

Pivot

$$\begin{pmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & \frac{3}{5} \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot(-3)$$

Zeilenstufenform

zwei Zeilen $\neq 0 \Rightarrow rk = 2$

$$\begin{pmatrix} 1 & 0 & \frac{2}{5} & -\frac{4}{5} \\ 0 & 1 & \frac{1}{5} & \frac{3}{5} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Normalform

$f : V \longrightarrow W$ linear

$\cup$ $\quad$ $\cup$ $\leftarrow$ senkrecht geschriebenes Teilmengen-Zeichen

$\ker f$ $\quad$ $\operatorname{im} f$

$$\underset{\overset{\wedge\!\shortparallel}{3}}{\dim(\operatorname{im} f)} = rk\, f = \dim\left(V / \ker f\right) = \underset{\overset{\shortparallel}{4}}{\dim V} - \underbrace{\dim(\ker f)}_{\geq 1}$$

Bsp: $\dim V = 4$, $\dim W = 3$ $\qquad$ $\operatorname{im} f \subset W$ $\qquad$ Also $\dim \operatorname{im} f \leq 3$

---

ker A' bestimmen:

$= \{ \underline{x} \in \mathbb{R}^4 \mid A' \underline{x} = 0 \}$, also die Lösungen vom LGS mit Koeff-Matrix

$$\left(\begin{array}{cccc|c} 3 & 4 & 2 & 0 & 0 \\ 4 & 7 & 3 & 1 & 0 \\ 1 & 3 & 1 & 1 & 0 \end{array}\right)$$

$\Big\downarrow$ Gauß

$$\left(\begin{array}{cccc|c} 1 & 0 & \frac{2}{5} & -\frac{4}{5} & 0 \\ 0 & 1 & \frac{1}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(*)

Also: $\ker A' = \left\{ \begin{pmatrix} \overbrace{\frac{4}{5}x_4 - \frac{2}{5}x_3}^{(*)} \\ -\frac{1}{5}x_3 - \frac{3}{5}x_4 \\ x_3 \\ x_4 \end{pmatrix} \;\middle|\; x_3, x_4 \in \mathbb{R} \right\}$

zwei Var. frei wählbar $\Rightarrow$ dim = 2

Erklärung:

Bsp. Lsg: $x_3 = 1$, $x_4 = 0$ ergibt: $\begin{pmatrix} -\frac{2}{5} \\ -\frac{1}{5} \\ 1 \\ 0 \end{pmatrix} = v_1$

Bsp. Lsg: $x_3 = 0$, $x_4 = 1$ ergibt $\begin{pmatrix} \frac{4}{5} \\ -\frac{3}{5} \\ 0 \\ 1 \end{pmatrix} = v_2$

v_1, v_2 sind l. u.

$$\ker A' = \{ \overbrace{x_3 v_1 + x_4 \cdot v_2}^{(*)} \mid x_3, x_4 \in \mathbb{R} \} = \langle v_1, v_2 \rangle_{\mathbb{R}}$$

Also: v_1, v_2 ist Basis von $\ker A'$

Lösung eines LGS mit Koeff-Matrix $\left(A \,\middle|\, \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix} \right)$ ist eindeutig gdw dim LSG Menge = dim Ker A = 0

$$A''' = \left.\begin{pmatrix} 0 & 1 & 5 & 0 & 0 & 3 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right\} 5 \in \mathbb{R}^{7 \times 9}$$

$x_1 \; x_2 \; x_3 \; x_4 \; x_5 \; x_6 \; x_7 \; x_8 \; x_9$

frei wählbar (4 Var.)

Anzahl berechneter Variablen = Anzahl Pivot-Elem = rk A'''

$V = \mathbb{R}^9$ $\quad W = \mathbb{R}^7$

$A : V \longrightarrow W$

rk $A''' = 5$

$$\dim(\text{Lösungsmenge von } A''' \underline{x} = 0) = \dim \ker A'''$$

$$\dim(\ker A''') + \text{rk } A''' = V = 9$$

Matrix invertieren:

$$B = \begin{pmatrix} 2 & -1 & -2 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix} \qquad B^{-1} = ?$$

$$\left(\begin{array}{ccc|ccc} \multicolumn{3}{c|}{\overbrace{\hphantom{2 \quad -1 \quad -2}}^{B}} & \multicolumn{3}{c}{\overbrace{\hphantom{1 \quad 0 \quad 0}}^{I_3}} \\ 2 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 2 & -1 & -2 & 1 & 0 & 0 \end{array}\right) \cdot(-2)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & -1 & -4 & 1 & 0 & -2 \end{array}\right) \cdot 1$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & -2 \end{array}\right) \cdot(-1)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right) \cdot(-3) \quad \cdot(-1)$$

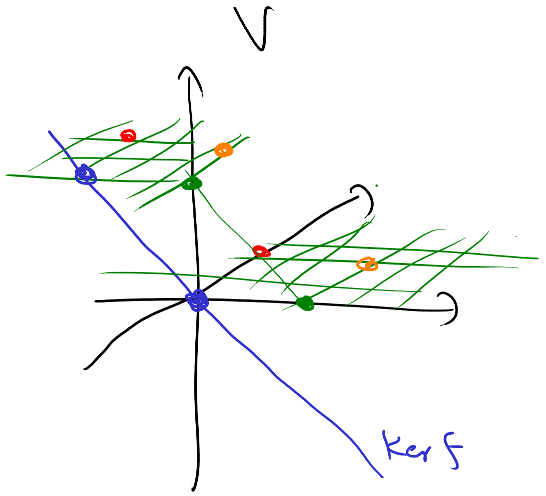
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 3 & 4 & -6 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right)$$

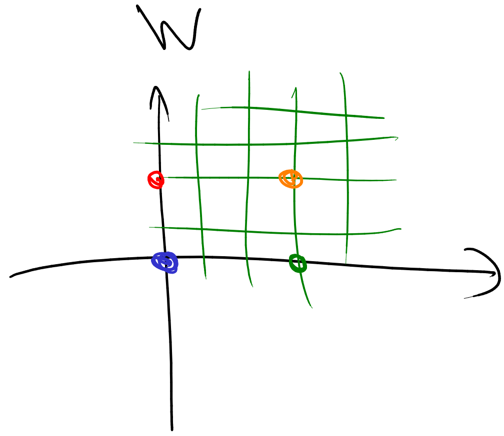
$$\underbrace{\begin{pmatrix} 1 & 1 & -1 \\ 3 & 4 & -6 \\ -1 & -1 & 2 \end{pmatrix}}_{B^{-1}}$$

Wenn wir

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{array}\right)$$

erhalten hätten, würde das bedeuten, dass die Matrix nicht inv'bar ist, da rk = 2





$\dim V = 3$

$\dim \ker f = 1$

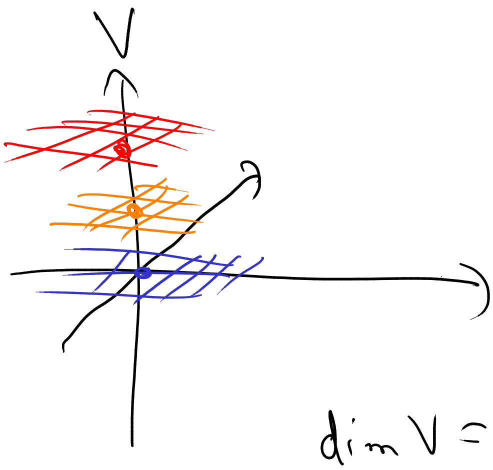
$\dim V - \dim \ker f = \operatorname{rk} f$

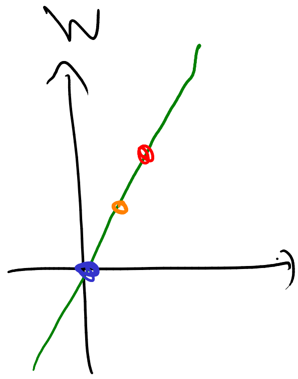
$\dim W = 2$

$\parallel$

$\dim \operatorname{im} f$

$/\!/$





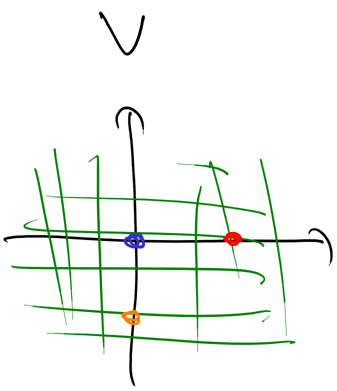
$\dim V = 3$

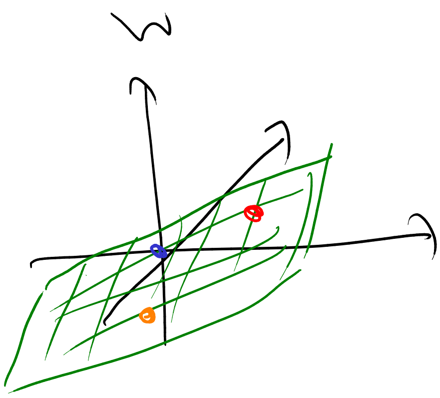
$\dim \ker f = 2$

$\dim W = 2$

$\dim \operatorname{im} f = 1$

$\dim V - \dim \ker f = \operatorname{rk} f$ $/\!/$





$\dim V = 2$

$\dim W = 3$

$\dim \operatorname{im} f = 2 = \operatorname{rk} f$

$\dim \ker f = 0$