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Asymptotic Group Theory – Blatt 2

– a friendly suggestion of exercises, problems and questions arising from the lectures –

In the second lecture we discussed Gromov's characterisation of finitely generated groups of polynomial word growth and we established a classical theorem of Jordan on finite subgroups of $\mathrm{GL}_n(\mathbb{C})$. On the current problem sheet we revisit selected topics from the lecture, e.g. polycyclic groups, and we take a small look at subgroup growth.

Recall that the next lecture is only in mid November. Practical information regarding the course and supplementary material can be found at

`http://reh.math.uni-duesseldorf.de/~internet/GeomMethAsympGrTh_WS2122/`

Aufgabe 2.1

Elementary, but instructive examples of polycyclic groups arise as (subgroups of) semi-direct products $\mathcal{O}^\times \ltimes \mathcal{O}^+$, where $\mathcal{O}$ is the ring of integers in an algebraic number field. Consider $\mathcal{O} = \mathbb{Z}[\sqrt{2}]$, arising from the real quadratic field $\mathbb{Q}(\sqrt{2})$, and the group

$$G = \langle 1 + \sqrt{2} \rangle \ltimes \mathbb{Z}[\sqrt{2}] \cong \langle x, a_1, a_2 \mid [a_1, a_2] = 1,\ a_1^x = a_1 a_2,\ a_2^x = a_1^2 a_2 \rangle.$$

Prove that G has exponential word growth. What bounds do you get for the exponential growth rate $\omega^S(G)$ with respect to the symmetrised generating set $\{1, x^{\pm 1}, a_1^{\pm 1}\}$?

Hint. Work with the indicated generating set and consider group elements of the form $g = \prod_{k=0}^{m} (a_1^{\varepsilon_k})^{x^k}$ for $m \in \mathbb{N}_0$ and exponents $\varepsilon_0, \dots, \varepsilon_m \in \{0, 1\}$.

Aufgabe 2.2

Recall that a group G is said to be *poly-$\mathcal{P}$*, for a group-theoretic property $\mathcal{P}$, if it admits a subnormal series of finite length $G = G_1 \trianglerighteq G_2 \trianglerighteq \dots \trianglerighteq G_{l+1} = 1$, say, such that each factor G_i/G_{i+1}, for $i \in \{1, \dots, l\}$, has $\mathcal{P}$. For instance, poly-abelian groups are more commonly known as soluble groups, and poly-C_∞ groups are built from 1 via finitely many successive split extensions, by adding in each step an infinite cyclic group at the top.
The (Prüfer) *rank* of a group G is the invariant

$$\mathrm{rk}(G) = \sup\{d(H) \mid H \le G \text{ such that } d(H) < \infty\} \in \mathbb{N}_0 \cup \{\infty\},$$

where $d(H)$ denotes the minimal number of generators of a group H. A group G is called *noetherian* if $d(H) < \infty$ for every subgroup $H \le G$.

(a) Prove that every poly-(cyclic or finite) group G is (poly-C_∞)-by-finite: there exists $N \trianglelefteq G$ such that N is poly-C_∞ and G/N is a finite group.

Hint. Observe that subgroups of poly-C_∞ groups are again poly-C_∞ and work by induction on the length of a subnormal series with cyclic factors to reduce to the case, where G is finite-by-C_∞.

(b) Show that, for a poly-(cyclic or finite) group G, there is a number $h(G) \in \mathbb{N}_0$ such the number of C_∞-factors in any subnormal series of finite length with cyclic or finite factors for G is equal to $h(G)$. The invariant $h(G)$ is called the *Hirsch length* of G.

Hint. Given two subnormal series of finite length with cyclic or finite factors $G = G_1 \trianglerighteq \dots \trianglerighteq G_{l+1} = 1$ and $G = H_1 \trianglerighteq \dots \trianglerighteq H_{m+1} = 1$ with $l \le m$, say, consider the induced sections

$H_jG_2/H_{j+1}G_2$ and $(H_j \cap G_2)/(H_{j+1} \cap G_2)$ for $1 \le j \le m$. Use induction on l to infer that both series yield the same Hirsch number.

Remark. The Hirsch length behaves like and can be thought of as a dimension function.

(c) Show that every poly-(cyclic or finite) group is finitely presented, soluble-by-finite, residually-finite, noetherian and of finite rank. – Give an example of a finitely presented, metabelian group of finite rank that is *not* poly-(cyclic or finite).

Hint. Use (a) to simplify matters.

(d) Show that every soluble noetherian group is polycyclic.

Remark. In view of Tarski monster groups and similar examples, it seems hopeless to characterise noetherian groups in general. It is known that the linear noetherian groups are precisely the polycyclic-by-finite groups, and it is an open problem, whether – more generally – residually finite, noetherian groups are polycyclic-by-finite. Due to results of Mal'cev (1951) and Auslander (1967) the soluble-by-finite groups that are linear over $\mathbb{Z}$ are precisely the polycyclic-by-finite groups.

Aufgabe 2.3

The aim of this exercise is to prove a technical lemma that we used in the lecture to establish Jordan's theorem.

Let $n \in \mathbb{N}$ and let $x, z \in \mathrm{Mat}_n(\mathbb{C})$ and $y \in \mathrm{GL}_n(\mathbb{C})$ such that

$$x^y = y^{-1}xy = xz \quad \text{and} \quad z^y = y^{-1}zy = z.$$

Let $\langle \cdot, \cdot \rangle$ denote the standard hermitian form on $\mathbb{C}^n$ and let $\|\cdot\|$ denote the norm on $\mathrm{Mat}_n(\mathbb{C})$ given by $\|t\|^2 = \sup\{\langle v.t, v.t\rangle \mid v \in \mathbb{C}^n \text{ such that } \langle v, v\rangle = 1\}$ for $t \in \mathrm{Mat}_n(\mathbb{C})$.

(a) Show that, if $\mathrm{Tr}(x) \neq 0$, then z has eigenvalue 1.

Hint. Use induction to see that $\mathrm{Tr}(xz^k) = \mathrm{Tr}(x)$ for all $k \in \mathbb{N}_0$ and 'compute' $\mathrm{Tr}(xf(z))$ for the characteristic polynomial f of z in two ways.

(b) Show that, if x, y, z are unitary and $\|1 - x\| < 1$, then $z = 1$.

Hint. Observe that, after conjugation by a suitable unitary matrix u, we may assume that, for suitable $r \in \{0, \ldots, n\}$,

$$z = \begin{pmatrix} 1 & 0 \\ 0 & z_2 \end{pmatrix}, \quad y = \begin{pmatrix} * & 0 \\ 0 & y_2 \end{pmatrix}, \quad x = \begin{pmatrix} * & * \\ * & x_2 \end{pmatrix},$$

where the entry 1 denotes the $r \times r$ identity matrix, $z_2, y_2, x_2 \in \mathrm{Mat}_{n-r}(\mathbb{C})$ are unitary and z_2 does not have eigenvalue 1. Assume for a contradiction that $r < n$. Deduce from (a) that $\mathrm{Tr}(x_2) = 0$. Observe that $\|1 - x_2\| < 1$ and use $|\mathrm{Tr}(x_2)| \ge |\mathrm{Tr}(1)| - |\mathrm{Tr}(1 - x_2)|$ to deduce that $|\mathrm{Tr}(x_2)| > 0$, a contradiction.

Aufgabe 2.4

The description of finite subgroups of $\mathrm{SO}(3, \mathbb{R})$, up to conjugacy, is a classical and beautiful result. It yields, essentially, also a description of finite subgroups of $\mathrm{SL}(2, \mathbb{C})$. In this exercise we re-visit some of the ideas involved.[1]

[1] Thanks to Djurre, for suggesting that I illustrate the meaning of Jordan's theorem in the special case of degree 2, which is rather well understood. The version of the theorem that we proved yields that every finite subgroup $G \le \mathrm{GL}(2, \mathbb{C})$ contains an abelian normal subgroup A of index $|G : A| \le 10^8$ ...

(a) Recall (explain!) that every finite subgroup of $\mathrm{SL}(2,\mathbb{C})$ is conjugate to a subgroup of $\mathrm{SU}(2)$, the special unitary subgroup. Writing $\mathbb{H} = \mathbb{R}.1 + \mathbb{R}.i + \mathbb{R}.j + \mathbb{R}.k = \mathbb{C}.1 + \mathbb{C}.j$ for the Hamiltonian quaternions, verify that the map $\mathrm{SL}(1,\mathbb{H}) \to \mathrm{SU}(2)$, $z + w.j \mapsto \left(\begin{smallmatrix} z & w \\ -\overline{z} & \overline{w} \end{smallmatrix}\right)$ yields an isomorphism between the norm-1 group of $\mathbb{H}$ and the special unitary group.

(b) Check that the isomorphism from (a), coupled with the natural action of $\mathrm{SL}(1,\mathbb{H})$ on the space of pure quaternions $V = \mathbb{R}.i + \mathbb{R}.j + \mathbb{R}.k \cong \mathbb{R}^3$ by conjugation, induces a short exact sequence of groups

$$1 \to \{1,-1\} \to \mathrm{SU}(2) \to \mathrm{SO}(3,\mathbb{R}) \to 1.$$

Thus finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ are, modulo a central subgroup of order at most 2, finite subgroups of $\mathrm{SO}(3,\mathbb{R})$.

(c) Let $G \le \mathrm{SO}(3,\mathbb{R})$ be finite. Explain that each $g \in G \smallsetminus \{1\}$ is a rotation with two antipodal fixed points of euclidean length 1. Consider $X = \{x \in \mathbb{R}^3 \mid \|x\| = 1 \text{ and } \exists g \in G \smallsetminus \{1\} : x.g = x\}$. Justify that G acts on X and use the elementary fixed-point count

$$2(|G|-1) = \textstyle\sum_{x \in X} \big(|\mathrm{Stab}_G(x)| - 1\big)$$

to deduce that

$$2 - 2|G|^{-1} = \sum_{i=1}^{k} (1 - a_i^{-1}),$$

where $a_1, \ldots, a_k \in \mathbb{N}_{\ge 2}$ are the sizes of stabilisers of representatives of distinct G-orbits. Observe that we may arrange $a_1 \ge \ldots \ge a_k$ and, in any case, a_i divides $|G|$ for each i.

(d) Deduce from (c) that a finite subgroup of $\mathrm{SO}(3,\mathbb{R})$ is the image of a so-called *spherical von Dyck group*

$$H_{l,m,n} = \langle x, y, z \mid x^l = y^m = z^n = xyz = 1 \rangle,$$

where $(l,m,n) \in \mathbb{N}^3$ satisfies $l \ge m \ge n$ and $l^{-1} + m^{-1} + n^{-1} > 1$. Determine all possible parameter triples and as many of the related finite subgroups of $\mathrm{SO}(3,\mathbb{R})$ as you can.

Remark. Naturally, there are also euclidean and hyperbolic von Dyck groups; explore …

Aufgabe 2.5

The aim of this exercise is to justify a remark that appears, for instance, in a paper of Platonov and Sury on adelic profinite groups.[2] Call a profinite group G *adelic* if there exists n such that G embeds as a closed subgroup into the profinite group $\prod_p \mathrm{SL}_n(\mathbb{Z}_p)$, where p runs through all primes and $\mathbb{Z}_p$ denotes the ring of p-adic integers.

Show that a non-abelian free profinite group is not adelic.

Hint. Fix a prime q and look at a Sylow pro-q group Q of $\prod_p \mathrm{SL}_n(\mathbb{Z}_p)$ in order to show that this cannot contain a non-abelian free pro-q group. Use Jordan's theorem to conclude that Q is an extension of the pro-q group $\mathrm{SL}_n^1(\mathbb{Z}_q) = \{g \in \mathrm{SL}_n(\mathbb{Z}_q) \mid g \equiv_q 1\}$ by an abelian-by-nilpotent group. Use also the fact that $\mathrm{SL}_n^1(\mathbb{Z}_q)$ is known to have rank $n^2 - 1$; compare Aufgabe 2.2.

[2] Thanks to Moritz, for bringing the subject up in his recent Oberseminar talk; the precise reference for the paper is: J. Algebra **193** (1997), 757–763.

Aufgabe 2.6

The aim of this exercise is to take a first look at the subgroup growth and the subgroup zeta functions of finitely generated groups.

Let G be a finitely generated group. Convince yourself that, for each $n \in \mathbb{N}$, the numbers

$$a_n(G) = \#\{H \mid H \le G \text{ with } |G:H| = n\} \quad \text{and} \quad s_n(G) = \sum_{k=1}^{n} a_k(G)$$

are finite. The *subgroup zeta function* of G is the formal Dirichlet series

$$\zeta_G^{\le}(s) = \sum_{n=1}^{\infty} a_n(G)\, n^{-s}.$$

It is known that $\zeta_G^{\le}(s)$ converges absolutely for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \alpha(G)$, where

$$\alpha(G) = \overline{\lim_{n\to\infty}} \frac{\log s_n(G)}{\log n} \in \mathbb{R}_{\ge 0} \cup \{\infty\};$$

indeed, unless $s_n(G)$ is bounded, $\alpha(G)$ is the so-called abscissa of convergence of $\zeta_G(s)$. In any case, it is true that $\alpha(G)$ is finite if and only if $s_n(G)$ is polynomially bounded (we say, "G has polynomial subgroup growth (PSG)") and, in this case,

$$\alpha(G) = \min\left\{\beta \in \mathbb{R}_{\ge 0} \mid s_n(G) = O(n^{\beta+\varepsilon}) \text{ for every } \varepsilon > 0\right\}$$

so that we may regard $\alpha(G)$ as the *degree* of polynomial growth of finite index subgroups in G.

(a) What are $\zeta_G(s)$ and $\alpha(G)$ for the infinite cyclic group $G = C_\infty$?

(b) Compute $\zeta_G(s)$ and determine $\alpha(G)$ for the infinite dihedral group $G = D_\infty$. Compare with (a), based on the observation that $D_\infty \cong C_2 \ltimes C_\infty$ and C_∞ are commensurable.

(c) Next consider $G = \langle x_1, \ldots, x_d\rangle \cong C_\infty \times \ldots \times C_\infty$, a free abelian group of rank $d \ge 2$. Put $K = \langle x_d \rangle \trianglelefteq G$ and establish, for $n \in \mathbb{N}$, the recursive formula

$$a_n(G) = \sum_{m \mid n} a_{n/m}(G/K)\, a_m(K)\, m^{d-1}$$

by counting subgroups $H \le G$ of index $|G:H| = n$ sorted by the associated data HK/K and $H \cap K$.

Use the formula to produce an explicit description of $\zeta_G^{\le}(s)$ as a product of familiar Dirichlet series – compare (a), (b) – and read off $\alpha(G)$.

Extra challenge. Can you find a finitely generated, residually finite group that is not isomorphic to $C_\infty \times C_\infty$, but nevertheless has the same subgroup zeta function?

(d) Let $G = \langle x, y\rangle$ be a non-abelian free group of rank 2. According to the Schreier formula, the minimal number of generators of a finite index subgroup $H \le G$ is $d(H) = |G:H| + 1$. Deduce from this that $a_{2n}(G) \ge 2^{n+1}$ for each $n \in \mathbb{N}$ so that G does not have polynomial subgroup growth.

(e) Let $G = C_2 \wr C_\infty$, the so-called lamplighter group. Show that $a_{k2^{k+1}}(G) \ge \binom{2k}{k}_2$ for each $k \in \mathbb{N}$, where

$$\binom{2k}{k}_X = \frac{(X^{2k}-1)(X^{2k}-X)\cdots(X^{2k}-X^{k-1})}{(X^k-1)(X^k-X)\cdots(X^k-X^{k-1})}$$

denotes the X-binomial coefficient. Deduce that $a_{k2^{k+1}}(G) \ge 2^{k^2}$ so that G does not have polynomial subgroup growth.

Hint. The group G maps onto $C_2 \wr C_{2k}$; look for subgroups in the base group, which is elementary abelian.