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Asymptotic Group Theory – Blatt 1

– a friendly suggestion of exercises, problems and questions arising from the lectures –

The first lecture introduced some ideas from the subject of word growth in groups. We looked at specific groups and discussed, by means of examples, various features, highlighting growth type on the one hand and arithmetic aspects on the other hand.
Practical information regarding the course and supplementary material can be found at

`http://reh.math.uni-duesseldorf.de/~internet/GeomMethAsympGrTh_WS2122/`

Aufgabe 1.1

Let G be a finitely generated group G, equipped with a symmetric finite generating set S. Recall that the word length function $\ell^S\colon G \to \mathbb{N}_0$ provides for each $g \in G$ the minimal length $l = \ell^S(g)$ of a product representation $g = y_1 \cdots y_l$ in terms of generators y_i from S. The *word growth series* of G with respect to S is the generating series

$$\sum_{n=0}^{\infty} \dot{s}_n(G)\, z^n, \quad \text{where } \dot{s}_n(G) = \dot{s}_n^S(G) = |\{g \in G \mid \ell^S(g) \le n\}|.$$

Verify that the growth series of the free abelian groups $G_1 = \mathbb{Z}$ with respect to $S_1 = \{0, \pm 1\}$ and of $G_2 = \mathbb{Z} \times \mathbb{Z}$ with respect to $S_2 = \{(0,0), \pm(1,0), \pm(0,1)\}$ are

$$\frac{1+z}{(1-z)^2} \quad \text{and} \quad \frac{1+2z+z^2}{(1-z)^3}.$$

Can you work out a rational function that gives the growth series of the free abelian group $G_d = \mathbb{Z}^d$ with respect to the symmetrised standard generating set S_d?

Hint. Guess what it could be and justify by induction on d; it would be counterproductive to compute the coefficients $\dot{s}_n(G_d)$ in the first place.

Remark. What you compute is also known as the Ehrhart series of the d-dimensional cross-polytope. It has a nice geometric interpretation.

Aufgabe 1.2

Two non-decreasing sequences $a, b\colon \mathbb{N}_0 \to \mathbb{N}$ are said to have the *same growth type* if there exists a scaling constant $k \in \mathbb{N}$ such that, for all $n \in \mathbb{N}_0$,

$$a_n \le k\, b_{kn} \quad \text{and} \quad b_n \le k\, a_{kn}.$$

Two groups $G, \widetilde{G}$ are said to be *quasi-commensurable* if they admit isomorphic sections

$$H/N \cong \widetilde{H}/\widetilde{N}, \quad \text{where}$$
$$N \trianglelefteq H \le G \text{ with } |N|, |G:H| < \infty \quad \text{and} \quad \widetilde{N} \trianglelefteq \widetilde{H} \le \widetilde{G} \text{ with } |\widetilde{N}|, |\widetilde{G}:\widetilde{H}| < \infty.$$

Show that, if $G, \widetilde{G}$ are quasi-commensurable finitely generated groups equipped with symmetric finite generating sets $S, \widetilde{S}$, then the associated word growth sequences $\dot{s}_n^S(G)$ and $\dot{s}_n^{\widetilde{S}}(\widetilde{G})$ display the same growth type.

Remark. This shows that, if we focus on growth type rather than on finer arithmetic properties of the growth sequence, the particular choice of a generating set does not matter; furthermore, the growth type is stable under quasi-commensurability.

Hint. Start with the case $G = \widetilde{G}$ to see that the choice of generating sets does not matter. Then convince yourself that working modulo a finite normal subgroup does not change the growth type. Observe that passing to a finite index subgroup does not 'increase' the growth type. The trickiest bit is to see that, conversely, passing to a finite index subgroup does not 'decrease' the growth type. For this you may wish to recall (or look up) a standard procedure for securing a generating set of size at most $d|G:H|$ for a finite index subgroup H of a d-generated group G.

Aufgabe 1.3

The point of this exercise is to revisit Peter Neumann's construction[1] presented in the lecture and to go through some of the details which we skipped.

For simplicity, consider $S = \mathrm{Alt}(6)$ with its natural action on $\Sigma = \{1, \ldots, 6\}$. We wish to construct a finitely generated, residually finite, perfect group G with the additional property $G \cong G \wr_\Sigma S$.

Remark. For applications of related wreath product constructions in asymptotic group theory see, for instance:

- L. Bartholdi and P. de la Harpe, Representation zeta functions of wreath products with finite groups, Groups Geom. Dyn. **4** (2010), 209–249.
- D. Segal, The finite images of finitely generated groups, Proc. London Math. Soc. **82** (2001), 597–613.
- J. S. Wilson, On exponential growth and uniformly exponential growth for groups, Invent. Math. **155** (2004), 287–303.

(a) Recall the meaning of the permutational wreath product:

$$G \wr_\Sigma S = G^\Sigma \rtimes S,$$

where the action of the top group S on the base group G^Σ by conjugation is given by

$$s^{-1} \cdot (g_\sigma)_{\sigma \in \Sigma} \cdot s = (g_{\sigma s^{-1}})_{\sigma \in \Sigma} \qquad \text{for } (g_\sigma)_{\sigma \in \Sigma} \in G^\Sigma \text{ and } s \in S$$

(b) For $n \in \mathbb{N}_0$, picture the n-fold iterated wreath power (built from below)

$$W_0 = 1 \quad \text{and} \quad W_n = W_{n-1} \wr_\Sigma S = ((\cdots(S \wr_\Sigma S) \wr_\Sigma S) \wr_\Sigma \cdots \cdots S) \wr_\Sigma S) \wr_\Sigma S, \quad \text{for } n \geq 1,$$

as a permutation group on the set of maximal paths $\Delta_n = \Sigma^n$ of a finite $|\Sigma|$-regular rooted directed tree: for $n \geq 1$ the image of $(\tau_1, \ldots, \tau_n) \in \Delta_n$ under the element $(h_\sigma)_{\sigma \in \Sigma} \cdot s \in W_n$, with $(h_\sigma)_{\sigma \in \Sigma} \in W_{n-1}^\Sigma$ and $s \in S$, is

$$(\tau_1, \tau_2, \ldots, \tau_n).\big((h_\sigma)_{\sigma \in \Sigma} \cdot s\big) = (\theta_1, \theta_2, \ldots, \theta_n),$$
$$\text{where } \theta_1 = \tau_1.s \text{ and } (\theta_2, \ldots, \theta_n) = (\tau_2, \ldots, \tau_n).h_{\tau_1}.$$

[1]Illinois J. Math. 30 (1986), §5

Observe that these actions reveal natural homomorphisms from W_n onto W_{n-1} which reflect that $W_n \cong S \wr_{\Delta_{n-1}} W_{n-1} \cong S \wr_{\Sigma^{n-1}} (S \wr_{\Sigma^{n-2}} (S \cdots \cdots \wr_{\Sigma^3} (S \wr_{\Sigma^2} (S \wr_\Sigma S))\cdots))$.

Aside. Describe the Sylow p-subgroup of the symmetric group $\mathrm{Sym}(p^k)$ of prime power degree p^k in terms of an iterated wreath product. Can you pin down the Sylow p-subgroup of a finite symmetric group $\mathrm{Sym}(n)$, based on the p-ary expansion of its degree n?

(c) Equip each set Δ_n with the discrete topology and consider the inverse limit

$$\Delta = \varprojlim_n \Delta_n \quad \text{with respect to the maps } \Delta_n \to \Delta_{n-1},\ (\tau_1, \ldots, \tau_n) \mapsto (\tau_1, \ldots, \tau_{n-1}).$$

Picture Δ, whose elements we represent as infinite sequences in Σ, as the boundary $\partial\mathcal{T}$ of the infinite $|\Sigma|$-regular rooted directed tree $\mathcal{T}$ with vertex set $\Delta_0 \cup \Delta_1 \cup \ldots$ filtered in layers; as a topological space, $\partial\mathcal{T}$ is a Cantor space. Convince yourself that the automorphism group $\mathrm{Aut}(\mathcal{T})$ acts faithfully by homeomorphisms on $\partial\mathcal{T}$; it is a huge profinite group (an infinitely iterated wreath product of symmetric groups).

The actions of the finite groups W_n on the finite sets Δ_n described in (b) induce naturally compatible faithful actions of the groups W_n on Δ as follows:

$$(\tau_1, \ldots, \tau_n, \omega_{n+1}, \omega_{n+2}, \ldots).g = (\theta_1, \ldots, \theta_n, \omega_{n+1}, \omega_{n+2}, \ldots),$$
$$\text{where } (\theta_1, \ldots, \theta_n) = (\tau_1, \ldots, \tau_n).g \text{ for } g \in W_n.$$

In this way we may form the direct limit $W = \varinjlim_n W_n$ as the union of an ascending chain of subgroups $\dot{W}_1, \dot{W}_2, \ldots$ isomorphic to $W_1, W_2, \ldots$ inside $\mathrm{Aut}(\mathcal{T})$. On the other hand, we can also consider the inverse limit $\overline{W} = \varprojlim_n W_n$ with respect to the natural surjections described in (b). Convince yourself that $\overline{W}$ is (isomorphic to) the topological closure of W in the profinite group $\mathrm{Aut}(\mathcal{T})$.

Remark. As explained in the lecture it is not difficult to show that $\overline{W}$ is actually the profinite completion of the locally finite group W, because the non-trivial normal subgroups of W are precisely the level stabilisers and the finite quotients of W are correspondingly the groups W_n.

(d) Next we want to write elements of the profinite group $\overline{W}$ in a convenient and unique way as (converging) infinite products. This requires some notation which one should again picture geometrically. For each $s \in S$, each $n \in \mathbb{N}_0$ and $\delta \in \Delta_n$ we consider the element $s^{[\mathrm{at}\,\delta]} \in W$ which "acts like s in position δ": the image of $(\tau_1, \ldots, \tau_n, \alpha, \omega_{n+2}, \ldots) \in \Delta$ under the action of $s^{[\mathrm{at}\,\delta]}$ is

$$(\tau_1, \ldots, \tau_n, \alpha, \omega_{n+2}, \ldots).s^{[\mathrm{at}\,\delta]} = \begin{cases} (\tau_1, \ldots, \tau_n,\ \alpha.s\,, \omega_{n+2}, \ldots) & \text{if } (\tau_1, \ldots, \tau_n) = \delta, \\ (\tau_1, \ldots, \tau_n,\ \alpha\quad, \omega_{n+2}, \ldots) & \text{if } (\tau_1, \ldots, \tau_n) \neq \delta. \end{cases}$$

Verify that each element of $\dot{W}_n$ can be written in the form

$$\prod_{\delta \in \Delta_{n-1}} s_\delta^{[\mathrm{at}\,\delta]} \cdots \prod_{\delta \in \Delta_1} s_\delta^{[\mathrm{at}\,\delta]} \prod_{\delta \in \Delta_0} s_\delta^{[\mathrm{at}\,\delta]}$$

for a uniquely determined parameter family $(s_\delta)_{\delta \in \bigcup_{k=0}^{n-1} \Delta_k}$. In the limit, every element $g = \overline{W}$ can be written as a converging product

$$\cdots \prod_{\delta \in \Delta_k} s_\delta^{[\mathrm{at}\,\delta]} \cdots \prod_{\delta \in \Delta_1} s_\delta^{[\mathrm{at}\,\delta]} \prod_{\delta \in \Delta_0} s_\delta^{[\mathrm{at}\,\delta]} \qquad (\dagger)$$

for a uniquely determined parameter family $(s_\delta)_{\delta \in \bigcup_{k=0}^{\infty} \Delta_k}$, called the 'portrait' of g.
The groups W and $\overline{W}$ are highly self-similar; one manifestation of this feature is the following:

$$\overline{V_1} = \{g \in \overline{W} \mid g \text{ has the form } (\dagger) \text{ with } s_\delta \neq 1 \text{ only for } \delta = (\delta_1, \ldots, \delta_k) \text{ with } k \geq 1 \text{ and } \delta_1 = 1\}$$

is a closed normal subgroup of $\overline{W}$ such that $\overline{V_1} \cong \overline{W}$ and $\overline{W} = \overline{V_1} \wr_\Sigma \dot{W}_1$ (as an internal wreath product); an explicit isomorphism from $\overline{V_1}$ onto $\overline{W}$ is induced by

$$s^{[\text{at}\,(1,\delta_2,\ldots,\delta_k)]} \mapsto s^{[\text{at}\,(\delta_2,\ldots,\delta_k)]} \qquad \text{for } s \in S \text{ and } (1, \delta_2, \ldots, \delta_k) \in \Delta_k. \tag{$\ddagger$}$$

Intersection with W yields an isomorphism from $V_1 = W \cap \overline{V_1}$ onto W and a decomposition $W = V_1 \wr_\Sigma \dot{W}_1$, in accordance with (b).
(e) Finally, we arrive at our candidate group. For $t \in S$ set $\text{fix}(t) = \{\sigma \in \Sigma \mid \sigma.t = \sigma\}$ and define

$$G = \langle w(t,\tau) \mid t \in S, \tau \in \text{fix}(t) \rangle \leq \overline{W},$$

where $w(t,\tau)$ denotes the element whose product decomposition $(\dagger)$ has the particular shape

$$w(t,\tau) = \cdots t_{(\tau,\tau,\tau)}\, t_{(\tau,\tau)}\, t_{(\tau)}\, t_{()}.$$

Clearly, G is finitely generated and it inherits the property of being residually finite, i.e. $\bigcap\{N \trianglelefteq G \mid |G:N| < \infty\} = 1$, as a subgroup from $\overline{W}$.
Check that, for any $\sigma \in \Sigma$, the subgroup

$$\langle w(t,\sigma) \mid t \in S \text{ such that } \sigma \in \text{fix}(t) \rangle$$

of G is isomorphic to $\{t \in S \mid \sigma \in \text{fix}(t)\} \cong \text{Alt}(5)$. Conclude that the group G is perfect, i.e. that $[G,G] = G$.
(f) It remains to see that $G \cong G \wr_\Sigma S$. For $t \in S$ and $\tau \in \text{fix}(t)$ write

$$w(t,\tau) = w^*(t,\tau)\, t_{()}, \qquad \text{where} \quad w^*(t,\tau) = \cdots t_{(\tau,\tau,\tau)}\, t_{(\tau,\tau)}\, t_{(\tau)}.$$

Show that, for $s, t \in S$ and $\tau_1, \tau_2 \in \text{fix}(s) \cap \text{fix}(t)$, the group commutator of $w^*(s,\tau_1)$ and $w^*(t,\tau_2)$ satisfies

$$[w^*(s,\tau_1), w^*(t,\tau_2)] = [s_{()}, t_{()}] = [s,t]_{()}.$$

By considering suitable 3-cycles $s, t \in S$, conclude that G contains $u_{()}$ for every double-transposition $u \in S$, hence $\dot{W}_1 \leq G$. Observe that this yields $w^*(t,\tau) \in G$ for all $t \in S$ and $\tau \in \text{fix}(t)$ and consequently

$$G = \langle \{t_{()} \mid t \in S\} \cup \{w^*(t,\tau) \mid t \in S, \tau \in \text{fix}(t)\} \rangle.$$

Write $\overline{L_1}$ for the kernel of the natural homomorphism

$$\overline{W} \to \dot{W}_1, \qquad g = \cdots \prod_{\delta \in \Delta_k} s_\delta^{[\text{at}\,\delta]} \cdots \prod_{\delta \in \Delta_1} s_\delta^{[\text{at}\,\delta]} \prod_{\delta \in \Delta_0} s_\delta^{[\text{at}\,\delta]} \mapsto s_{()}$$

and deduce that $G = (\overline{L_1} \cap G) \rtimes \dot{W}_1$.
For each $\tau \in \Sigma$ fix an element $s(\tau) \in S$ such that $\tau.s(\tau) = 1$ and check that, for $t \in S$ with $\tau \in \text{fix}(t)$,

$$s(\tau)_{()}^{-1} \cdot w^*(t,\tau) \cdot s(\tau)_{()} = \cdots t_{(\tau,\tau,1)}\, t_{(\tau,1)}\, t_{(1)} = w_1(t,\tau).$$

Deduce that

$$G = \langle \{w_1(t,\tau) \mid t \in S, \tau \in \mathrm{fix}(t)\} \cup \{t_{()} \mid t \in S\}\rangle$$

and observe that the subgroup

$$G_1 = \langle w_1(t,\tau) \mid t \in S, \tau \in \mathrm{fix}(t)\rangle \leq \overline{V_1} \cap G \leq \overline{L_1} \cap G$$

is isomorphic to G, by restriction of the isomorphism $\overline{V_1} \to \overline{W}$ induced by (‡).
Finally conclude that G decomposes as an internal wreath product $G = G_1 \wr_\Sigma \dot{W}_1$ so that $G \cong G \wr_\Sigma S$.

(g) As mentioned above, the normal subgroup structure of W and the (closed) normal subgroup structure of $\overline{W}$ are very transparent. The non-trivial (closed) normal subgroups of $\overline{W}$ are the groups $\overline{L_n}$, $n \in \mathbb{N}_0$, which are the kernels of the natural homomorphisms

$$\overline{W} \to \dot{W}_n, \qquad g = \cdots \prod_{\delta \in \Delta_k} s_\delta^{[\mathrm{at}\,\delta]} \cdots \prod_{\delta \in \Delta_1} s_\delta^{[\mathrm{at}\,\delta]} \prod_{\delta \in \Delta_0} s_\delta^{[\mathrm{at}\,\delta]} \mapsto \prod_{k=1}^{n} \prod_{\delta \in \Delta_{n-k}} s_\delta^{[\mathrm{at}\,\delta]}.$$

Does this rigid normal subgroup structure transfer to our group G?
P. M. Neumann proves a more general result which specialises to the assertion: if $1 \neq N \trianglelefteq G$ then there exists $n \in \mathbb{N}_0$ such that $N = \overline{L_n} \cap G \cong G^{\Delta_n}$. Try to establish this result.

Hint. Use the fact that $G \cong G_n \wr_{\Delta_n} \dot{W}_n$ with $G_n \cong G$ for any $n \in \mathbb{N}_0$.

Remark. This shows that the inverse limit wreath product $\overline{W}$ is the profinite completion of the finitely generated group G. In general it is a difficult task to decide whether a given (topologically) finitely generated profinite group is the profinite completion of a finitely generated discrete group.