§ 5 Intersection theory on surfaces

$k = \bar{k}$

X sm proj surface $/k$ ($\dim X = 2$, connected), $K = k(X)$

- $Z^1(X) = \bigoplus_{\substack{C \subset X \\ \text{integral curve}}} \mathbb{Z} \cdot [C]$

- $CH^1(X) = \operatorname{Coker}\big(\operatorname{div} : k(X)^\times \longrightarrow Z^1(X)\big)$
  $f \longmapsto \sum_C v_C(f) \cdot [C]$

- $\operatorname{Pic}(X) =$ Isom classes of $\underbrace{\text{invertible } \mathcal{O}_X\text{-mod of rk } 1}_{\text{line bundles}}$.

- $c_1 : \operatorname{Pic} X \xrightarrow{\cong} CH^1(X)$ (also for curves)
  $\mathcal{O}_X(D) \longleftarrow D$

  $\mathcal{O}_X(D)(U) = \{ f \in K^\times \mid \operatorname{div}(f) \geq -D \}$

- For $C_1, C_2 \subset X$ integral curves

  define $C_1 \cdot C_2 = \deg \big(\underbrace{c_1(\mathcal{O}_X(C_1)|_{\tilde{C}_2})}_{\in CH_0(\tilde{C}_2)} \big) \in \mathbb{Z}$

<u>Intersection number</u>

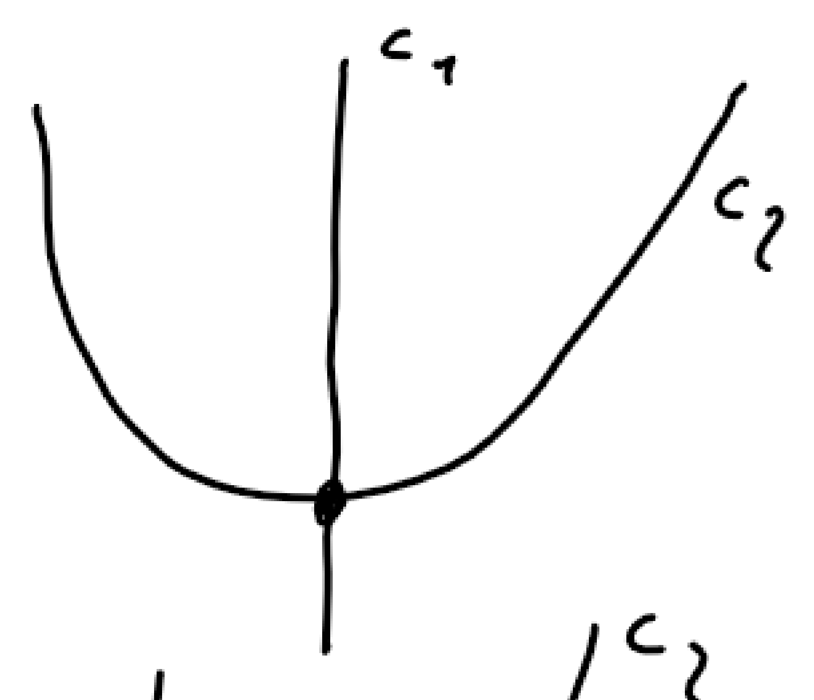
Properties:

- extends to
$$CH^1(X) \times CH^1(X) \longrightarrow \mathbb{Z}$$
$$(D, E) \longmapsto D.E \quad \text{bilinear.}$$

- $D.E = E.D$

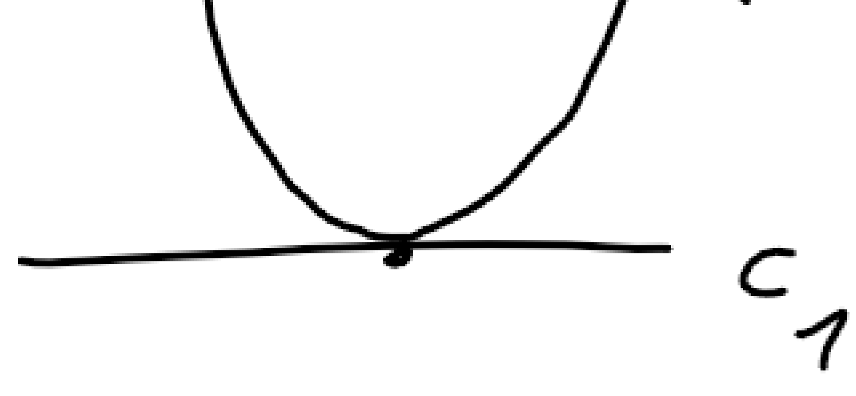
- $C_1, C_2 \subset X$ integral curves, $C_1 \neq C_2$
$$\Rightarrow C_1 . C_2 = \sum_{x \in |C_1 \cap C_2|} \ell_{\mathcal{O}_{X,x}} \left(\frac{\mathcal{O}_{X,x}}{(t_1, t_2)} \right)$$
local eq of C_1, C_2 around x

Ex: . $X = \mathbb{P}^2 \supset \mathbb{A}^2 = \operatorname{Spec} k[t, s]$



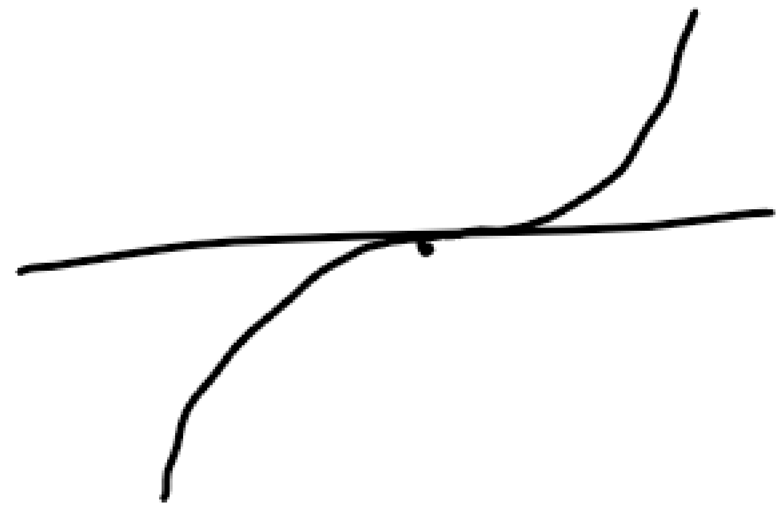
1) $C_1 = \{t = 0\}$, $C_2 = \{s - t^2 = 0\}$

$$C_1 . C_2 = 1$$



2) $C_1 = \{s = 0\}$, $C_2 = \{s - t^2 = 0\}$

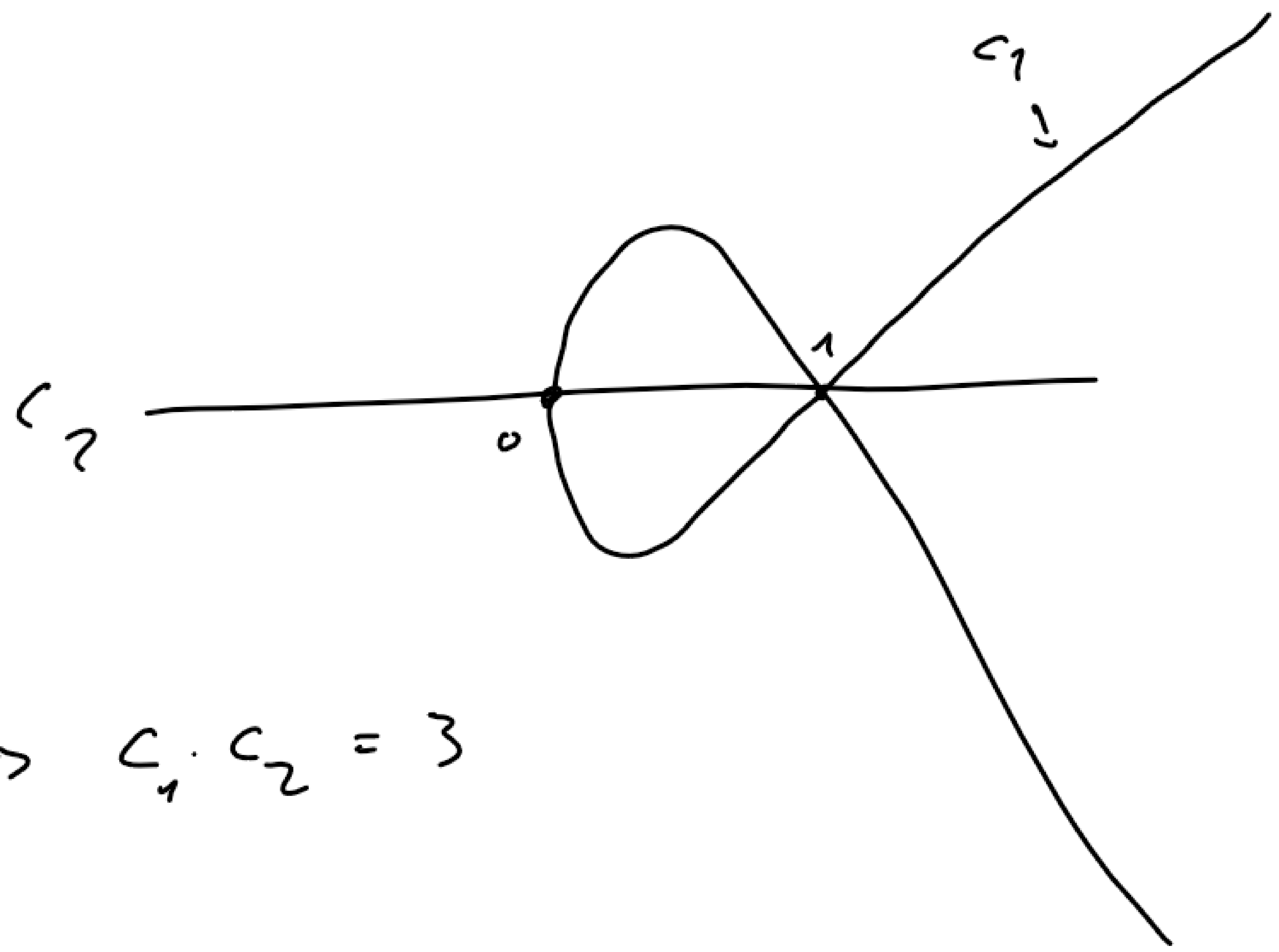
$$C_1 . C_2 = 2$$



3) $C_1 = \{s = 0\}$, $C_2 = \{s - t^5 = 0\}$

$$C_1 . C_2 = 5$$

4) $C_1 = \{ s^2 - t(t-1)^2 = 0 \}$ $\qquad C_2 = \{ s = 0 \}$



$\Rightarrow C_1 \cdot C_2 = 3$

5)
$$\begin{tikzcd} \mathrm{Bl}_x \arrow[r] & \mathbb{P}^2 \\ E \arrow[u, phantom, "\cup" rotate=90] \arrow[r] & x \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

ex div

Haben $\mathcal{O}(-E)_{|E} \underset{E \cong \mathbb{P}^1}{\cong} \mathcal{O}_{\mathbb{P}^1}(1) \Rightarrow E \cdot E = -1$

Cor (Bezout's Thm)

$\mathbb{P}^2$, $\quad C_i = \{ F_i = 0 \}$, $\quad F_i \in k[X_0, X_1, X_2]$
$\deg F_i = d_i$

$\Rightarrow C_1 \cdot C_2 = d_1 \cdot d_2$

Pf.: $H_i = \{ X_i = 0 \}$
$C_i = d_i H_i + \operatorname{div}\left(\frac{F_i}{X_i^{d_i}} \right) = d_i H_i$ in $CH^1(X)$

$\Rightarrow C_1 \cdot C_2 = d_1 d_2 \underbrace{H_1 \cdot H_2}_{=1} = d_1 d_2$ $\square$

§ 6 Hodge Index

X sm proj $/ k = \bar{k}$

- a divisor A on X is <u>ample</u>

$\Leftrightarrow \exists \; X \overset{i}{\hookrightarrow} \mathbb{P}^N$ s.d.

$nA \sim H \cap X$ for some $n \geq 1$

$\uparrow$ Hyperplane in $\mathbb{P}^N$

- D, E divisors on X

$D \equiv E \Leftrightarrow (D-E).C = 0$ for any integral curve $C \subset X$

numerical equiv.

$$Num(X) = \frac{Z^1(X)}{\equiv} \equiv \frac{CH^1(X)}{\equiv}$$

$\rightsquigarrow$ non-degenerate bil pairing

$$Num(X) \times Num(X) \longrightarrow \mathbb{Z}$$
$$(D, E) \longmapsto D.E$$

can show: $Num(X)$ free $\mathbb{Z}$-Mod of fin rk.

Hodge Index Thm

A ample on X , $0 \neq D \in Num(X)$

If $D.A = 0 \implies D.D < 0$

(pf uses R.R. for surfaces)

Explanation of name:

$\langle , \rangle : Num(X)_{\mathbb{R}} \times Num(X)_{\mathbb{R}} \longrightarrow \mathbb{R}$

non deg quadratic form.

A ample $\Rightarrow A.A > 0$

$\Rightarrow$ extend A to orthogonal basis $(A, D_1, \dots D_{g-1})$

$\underset{HIT}{\Rightarrow}$ Matrix of $\langle , \rangle$ in $\frac{A}{A.A}, \frac{D_1}{|D_1 . D_1|}, \dots, \frac{D_{g-1}}{|D_{g-1} . D_{g-1}|}$

is $\begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & \ddots & -1 \end{pmatrix}$

i.e. Index $(\langle \cdot , \cdot \rangle)$ = signature

$= (1, g-1)$

§7 Proof of Riemann Hyp for curves over fin fields

- $C / \mathbb{F}_q$, $q = p^r$, $\Omega = \overline{\mathbb{F}}_q$, $\overline{C} = C \otimes_{\mathbb{F}_q} \Omega$

sm proj, geom conn

- $\varphi : C \longrightarrow C$ q-power Frob

i.e. ind by $\mathcal{O}_C \longrightarrow \mathcal{O}_C$ $\mathbb{F}_q$-linear.

$a \longmapsto a^q$

$\rightsquigarrow \overline{\varphi} = \varphi \otimes \mathrm{id}_\Omega : \overline{C} \longrightarrow \overline{C}$

Assume $C \subset \mathbb{P}^n_{\mathbb{F}_q} \Rightarrow \overline{C} \subset \mathbb{P}^n_\Omega$

$\overline{C} \ni x = (a_0 : a_1 : \ldots : a_n)$

$\overline{\varphi}(x) = (a_0^q : a_1^q : \ldots : a_n^q)$

Thus $x \in C(\mathbb{F}_{q^n}) \iff \overline{\varphi}^n(x) = x$

$\rightsquigarrow$ Key-lemma $\Gamma = \text{graph } \overline{\varphi}^n \subset \overline{C} \times \overline{C}$

$\Delta = \text{diagonal} \subset \overline{C} \times \overline{C}$

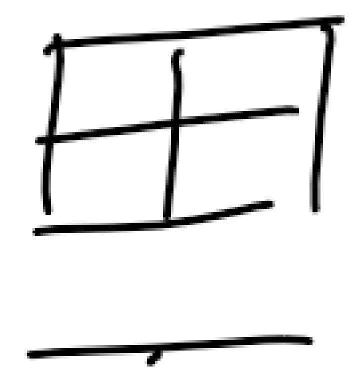
$\Rightarrow \Gamma \cdot \Delta = | C(\mathbb{F}_{q^n}) |$

Set

$$X = \overline{C} \times \overline{C} \quad \text{sm proj surf}$$

Pick $z \in \overline{C}$, $F_1 := \overline{C} \times \{z\} \subset X$

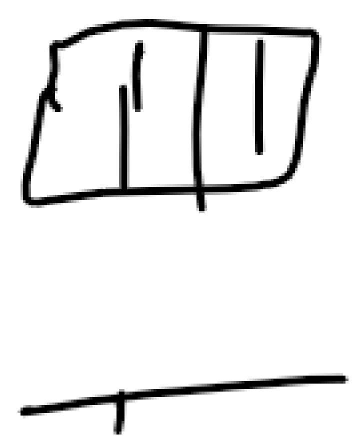
$F_2 := \{z\} \times \overline{C}$



Have:

- $F_1 . F_2 = 1$
- $(F_1 . F_1) = (F_2 . F_2) = 0$

[$F_1 \sim$ bunch of other fibers



]

- $F_1 + F_2$ ample

Thm (Castelnuovo - Severi - Inequality)

$D \in CH^1(X)$, $(D . F_i) := d_i$

$$\Rightarrow \quad D.D \leq 2 d_1 d_2$$

Pf. $A := F_1 + F_2 \Rightarrow$

$$(\underbrace{D - d_2 F_1 - d_1 F_2}_{= D'}) . A = 0 \underset{HIT}{\Rightarrow} D' . D' \leq 0$$

□

Define

$$\operatorname{def}(D) = 2 d_1 d_2 - (D.D) \qquad \geq 0$$

defect of D

Cor: $D, E \in CH^1(X)$ $\qquad e_i = E.F_i$, $\quad d_i = D.F_i$

$$\Rightarrow \quad | D.E - d_1 e_2 - d_2 e_1 | \leq \sqrt{\operatorname{def}(E)\operatorname{def}(D)}$$

Pf: Consider $0 \leq \operatorname{def}(mD + nE)$ $\quad m, n \in \mathbb{Z}$

and compute $\square$

Prop: $f : C \to C$ dom of degree n

$$\Rightarrow \quad \operatorname{def}(\operatorname{graph}(f)) = 2n\, g(C)$$

Recall

<u>Thm (*)</u>

$$\left| 1 + q^n - |C(\mathbb{F}_{q^n})| \right| \leq 2g\sqrt{q^n}$$

<u>Pf</u>:

$\Delta = \mathrm{graph}(\mathrm{id}_C)$ has degree $1 \underset{\text{Prop}}{\Rightarrow} \mathrm{def}(\Delta) = 2g(C)$

$\Gamma = \mathrm{graph}(\varphi^n)$ has degree $q^n \Rightarrow \mathrm{def}(\Gamma) = 2g\, q^n$

$$\Delta . F_1 = \Delta . F_2 = 1$$

$$\Gamma . F_1 = q^n \,,\quad \Gamma . F_2 = 1$$

$$\underset{\text{Cor}}{\Rightarrow} \quad |\Delta.\Gamma - q^n - 1| \leq \sqrt{\mathrm{def}(\Delta)\,\mathrm{def}(\Gamma)} = 2g\sqrt{q^n}$$

// key lem

$$|C(\mathbb{F}_{q^n}) - q^n - 1|$$

$\square$

End of pf of Weil conj.

last time

(*) $\left\{ \begin{array}{l} Z(C/\mathbb{F}_q, t) = \dfrac{f(t)}{(1-t^m)(1-q^m t^m)} \\ \text{where } f(0)=1, \ \deg(f) \le 2g-2+2m \\ m\mathbb{Z} = \deg(Z_0(C)) \end{array} \right.$

set $H(t) := (1-t)(1-qt)\, Z(C/\mathbb{F}_q, t) \quad \in \mathbb{Q}(t)$

$a_n := 1 + q^n - |C(\mathbb{F}_{q^n})|$

$\Rightarrow \dfrac{H'(t)}{H(t)} = -\sum_{n=1}^{\infty} a_n t^{n-1}$

(*) $\Rightarrow \dfrac{H'(t)}{H(t)}$ converges for $|t| < q^{-1/2}$

$\Rightarrow$ $H(t)$ has no pole or zero in $|t| \neq q^{-1/2}$

+ Fct eqn

$\Rightarrow Z(C/\mathbb{F}_q, t) = \dfrac{H(t)}{(1-t)(1-qt)}$ has a simple poles at $1, q^{-1}$

all other poles and zeros are in $|t| = \frac{1}{\sqrt{q}}$

$\Rightarrow$ Inspection of f (last time) + Fct eqn $\quad H = f, \ m = 1, \ \deg H = 2g$

$\square$