§3 Riemann-Roch Thm

Fix k = field

Cohomology:

X scheme, $\mathcal{F}$ sh of ab gps on X

$(X \underset{\text{open}}{\supset} \mathcal{U} \mapsto \mathcal{F}(\mathcal{U}))$

$\rightsquigarrow$ can define

$H^i(X, \mathcal{F})$ = i-th cohomology gp of $\mathcal{F}$

- functorial in $\mathcal{F}$
- s.e.s $\to$ l.e.s:

$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \Rightarrow$

$\ldots \to H^i(X, \mathcal{F}') \to H^i(X, \mathcal{F}) \to H^i(X, \mathcal{F}'') \to H^{i+1}(X, \mathcal{F}') \to \ldots$

- Assume X proj$/k$, $\mathcal{F}$ coh $\mathcal{O}_X$-Mod

$\Rightarrow H^i(X, \mathcal{F})$ fin dim'l k-vsp.

$= 0$ for $i > \dim X$.

In this case define

$$\chi(\mathcal{F}) = \sum_{i \geq 0} (-1)^i \dim_k H^i(X, \mathcal{F})$$

Have

$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ s.e.s of coh

$$\Rightarrow \quad \chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$$

[by l.e.s. + dim formulae]

<u>Def.</u> C sm, proj, geom conn $/k$

$$g := g(C) = \text{genus of } C$$
$$= \dim_k H^1(C, \mathcal{O}_C)$$

<u>Ex:</u>

i) $g(\mathbb{P}^1) = 0$

ii) $C = \{F = 0\}$, $F \in k[X, Y, Z]$ hom of deg d

$(\partial_X F, \partial_Y F, \partial_Z F)$ no non-triv common zero in $\bar{k}$

$$\Rightarrow g(C) = \frac{(d-1)(d-2)}{2}$$

Weil-Divisors

$X/_k$ sm/k conn. $\dim X = d$

Weil Divisor is a formal sum

$$D = \sum_{i=1}^{r} n_i D_i$$

$n_i \in \mathbb{Z}$, $D_i \subset X$ irred, closed

$\dim D_i = d-1 \iff \operatorname{codim}(D_i, X) = 1$

Consider:

$$X \supset \mathcal{U} \longmapsto \mathcal{O}_X(D)(\mathcal{U}) = \{ f \in k(X)^* \mid \operatorname{div}(f)_{|\mathcal{U}} \geq -D_{|\mathcal{U}} \}$$

$$\sum_{\substack{Z \subset \mathcal{U} \\ 1\text{-codim}}} v_Z(f) \cdot Z$$

$\rightsquigarrow$ locally free sheaf on X of rk 1

Note:

$$\mathcal{O}_X \xrightarrow{\cong} \mathcal{O}_X(\operatorname{div}(g))$$
$$f \longmapsto f g^{-1}$$

Degree: C sm curve $/_k$ geom conn.

$D = \sum n_i x_i$ Weil div on C

$$\deg(D) = \sum n_i [k(x_i) : k] \in \mathbb{Z}$$

Riemann-Roch Thm (1. version)

C sm proj curve $/k$, geom con

$g = g(C)$, D Weil div

$$\Rightarrow \quad \chi(\mathcal{O}_C(D)) = \dim_k H^0(C, \mathcal{O}_C(D)) - \dim H^1(C, \mathcal{O}_C(D))$$

$$= \underbrace{\deg(D) - g(C) + 1}_{R(D)}$$

Pf: wlog $k = \bar{k}$

$D = 0$: ✓

$x \in C$ ($\Rightarrow \deg x = 1$) $\Rightarrow$ $R(D + x) = R(D) + 1$

suff to show: $\chi(D + x) = \chi(D) + 1$

s.e.s. $\quad 0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D+x) \to k_x \to 0$

supp on x $\to$ has no H^1

$$\Rightarrow \quad \chi(D+x) = \chi(D) + \underbrace{\dim H^0(X, k_x)}_{=1}$$

□

Kor: $f \in k(C)^\times \Rightarrow \deg(\mathrm{div}(f)) = 0$

$$\deg(\mathrm{div}(f)) = \sum_{x \in C} v_x(f)\,[k(x):k]$$

Bew: Take $D = \mathrm{div}(f)$ in RR □

Trace

C/k as above, $K = k(C)$

$\omega_C := \Omega^1_{C/k}$

- If $t \in \mathcal{O}_{C,x}$ is a loc parameter

$\Rightarrow \omega_{C,x} = \mathcal{O}_{C,x} \cdot dt$

$\Rightarrow \underset{\substack{\uparrow \\ \text{gen pt}}}{\omega_{C,\xi}} = K \cdot dt$

For $\alpha \in \omega_{C,\xi}$ can def $\mathrm{Res}_x(\alpha) \in k$

satisfying $\sum_{x \in C} \mathrm{Res}_x(\alpha) = 0$ (Res-Thm)

[if: $x \in C(k) \Rightarrow$ $\omega_{C,\eta} \overset{\text{choice of } t \text{ at } x}{\hookrightarrow} k((t))\, dt$

$\alpha \longmapsto \sum_{i \gg -\infty} a_i t^i dt$

then $\mathrm{Res}_x(\alpha) := a_{-1}$]

Have $0 \to \omega_C \to \underbrace{i_{\xi *}\omega_{C,\xi}}_{\text{cst sheaf}} \to \bigoplus_{x \in C \text{ cl.}} \underbrace{i_{x*}\left(\frac{\omega_{C,\xi}}{\omega_{C,x}}\right)}_{\text{skyscraper sheaf at } x} \to 0$ inj res

$\rightsquigarrow \mathrm{Tr}: H^1(C, \omega_C) = \left(\bigoplus_x \frac{\omega_{C,\xi}}{\omega_{C,x}}\right) \Big/ \omega_{C,\xi} \xrightarrow{\sum \mathrm{Res}_x} k$

Serre Duality:

We have perfect pairing

$$H^0(C, \omega_C \otimes \mathcal{O}(-D)) \otimes H^1(C, \mathcal{O}(D)) \longrightarrow H^1(C, \omega_C) \xrightarrow{\mathrm{Tr}} k$$

$$H^1(C, \mathcal{O}(D)) = \left(\bigoplus_x \frac{\mathcal{O}(D)_\eta}{\mathcal{O}(D)_x} \right) \Big/ \mathcal{O}(D)_\eta$$

$$\alpha \otimes (\lambda_x) \longmapsto (\alpha \lambda_x)$$

Notation We can write $\omega_C \cong \mathcal{O}(K_C)$

K_C: Weil div called "canonical div" (only defined up to + div(f))

set $h^i(D) := \dim_k H^i(X, \mathcal{O}_X(D))$

Kor $h^0(K_C - D) = h^1(D)$

Riemann-Roch (2 version)

$$h^0(D) - h^0(K_C - D) = 1 - g + \deg D$$

Kor: $h^0(K_C) = g$ and $\deg K_C = 2g - 2$

Kor: $\deg D \geq 2g - 1 \implies h^0(D) = 1 - g + \deg D$ (use $h^0(E) = 0$ if $E < 0$)

§4 Rationality of Zeta Fctn

- $k = \mathbb{F}_q$, $q = p^r$
- C sm proj geom cur $/k$, $g = g(C)$
- $Z_0(C)$ = Weil divisors on C

Recall

$$Z(C/k, t) = 1 + \sum_{d \geq 1} b_d t^d$$

with $b_d = \left| \{ D \in Z_0(C)^{\geq 0} \mid \deg D = d \} \right|$

<u>Notation</u>: $m = \gcd \{ [k(x) : k] \mid x \in C \text{ closed pt} \}$

(we will see next time $m = 1$)

$$\Rightarrow \boxed{m\mathbb{Z} = \deg(Z_0(C))}$$

<u>Prop</u>: $\exists\ f \in \mathbb{Q}[t]$ with $f(0) = 1$ and $\deg f \leq 2g - 2 + 2m$

s.t. $$Z(C/k, t) = \frac{f(t)}{(1 - t^m)(1 - q^m t^m)}$$

first:

Lem:
$D \in Z_0(C) \qquad \deg D = n$

$$\left| \left\{ E \in Z_0(C)^{\geq 0} \mid \underset{E = D + \operatorname{div}(f)}{E \sim D} \right\} \right| = \frac{q^{h^0(D)} - 1}{q - 1}$$

Bew.

$0 \leq E = D + \operatorname{div}(f) \iff f \in H^0(C, \mathcal{O}(D)) \setminus \{0\}$

and $\operatorname{div}(f) = \operatorname{div}(g) \iff f/g \in k^\times$

pf Prop.: $Z(C, t) = 1 + \sum_{d \geq 1} b_d \, t^d$

$b_d = |Z_0(C)^{\geq 0}_{\deg = d}|$

Have $m\mathbb{Z} = \deg(CH_0(C))$

$\Rightarrow \quad Z(C, t) = 1 + \sum_{d \geq 1} b_{md} \, t^{md}$

Fix D_d with $\deg D_d = m d$

$$\Rightarrow \quad Z_0(C)^{\mathrm{eff}}_{\deg md} = \coprod_{\alpha \in \mathrm{Pic}_0(C)^0} \{\beta \in Z_0(C)^{\mathrm{eff}} \mid \beta \sim D_d + \alpha\}$$

$$\underset{(1)}{\Rightarrow} \quad b_{md} = \sum_{\alpha \in \mathrm{Pic}_0(C)^0} \frac{q^{h^0(D_d + \alpha)} - 1}{q - 1}$$

<u>if $md \geq 2g-1$</u> $\Rightarrow$ $h^0(\alpha + D_d) = 1 - g + md$

$$\Rightarrow \quad b_{md} := \underbrace{|\mathrm{Pic}_0(C)^0|}_{= A} \frac{q^{1-g+md} - 1}{q - 1}$$

$(\Rightarrow A < \infty)$

$\Rightarrow$

$$Z(C,t) = \ell(t) + h(t)$$

with $\ell(t) \in \mathbb{Z}[t]$, $\deg \ell(t) \leq 2g-2$, $\ell(0) = 1$

and $$h(t) = A \cdot \sum_{d > \frac{2g-2}{m}} \frac{q^{1-g+md} - 1}{q - 1} t^{md}$$

<u>Check</u>: $(1-t^m)(1-q^m t^m) Z(C,t) = f(t)$ as in statement.

$\square$

Similarly RR implies

Thm (Functional equation)

$$Z\left(C, \frac{1}{qt}\right) = q^{1-g}\, t^{2-2g}\, Z(C, t)$$