Ultraproducts

and

Transferprinciples II

being the last talk in this semerster's seminar on

C_i - fields

Last time: Theorem of Łoś

Let $\mathcal{U}$ be an ultrafilter on I, $(\mathcal{X}_i)_{i\in I}$ a family of $\mathcal{L}$-structures and φ an $\mathcal{L}$-sentence of first order logic. Then

$$ {}_{U}\prod_{I,\mathcal{U}} \mathcal{X}_i \models \varphi \iff \{ i\in I \mid \mathcal{X}_i \models \varphi \} \in \mathcal{U}. $$

"ultraproduct"

Jerzy Łoś
1920 – 98

Application: $\mathcal{U}$ any ultrafilter on an infinite set of primes P.

$$ {}_{U}\prod_{P,\mathcal{U}} \overline{\mathbb{F}_p}^{a} \cong \mathbb{C} $$

Ernst Steinitz
1871 - 1928

Why? Characteristic 0, algebraically closed.

<u>Steinitz' thm:</u> K, L alg. closed fields,
$\operatorname{char} K = \operatorname{char} L$
$|K| = |L| > \aleph_0$
Then $K \cong L$.

+ some cardinality calculations.

A reformulation: The Lefschetz principle

Solomon Lefschetz
1884 – 1972

Let φ be a first order sentence in the language of fields $\mathcal{L}_F$. Then

$$\mathbb{C} \models \varphi \iff \overline{\mathbb{F}_p}^a \models \varphi$$

for all but finitely many primes.

$*$: Let $\mathbb{C} \models \varphi$. Then ${}^{u}\prod_{\mathbb{P},\vartheta} \overline{\mathbb{F}_p}^a \models \varphi$ for all ϑ.

Łoś: $\{ p \in \mathbb{P} \mid \overline{\mathbb{F}_p}^a \not\models \varphi \} \notin \bigcup_{\vartheta \text{ uf}} \vartheta = \mathcal{P}(\mathbb{P}) \setminus \{\text{finite subsets}\}$

ultrafilter lemma

$\square$

Even better: the theory of algebraically closed fields of fixed characteristic is complete.

$\mathbb{C} \models \varphi \iff K \models \varphi$ for all alg. closed fields K with $\operatorname{char} K = 0$

$\iff (K \models \varphi$ for all alg. closed fields K with $\operatorname{char} K = p)$ for all but finitely many primes p

$\iff \overline{\mathbb{F}_p}^a \models \varphi$ for all but finitely many primes p.

The analogue for valued fields:

Thm (Ax-Kochen-Ershov principle) $\qquad \mathcal{L}_F \cup \{v, \leq\}$

Let φ be a first order sentence in the language $\mathcal{L}_{VF}$ of valued fields. Then

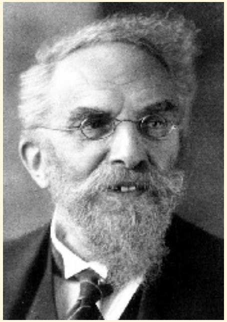
$\mathbb{Q}_p \models \varphi$ for all but finitely many primes p

$\Longleftrightarrow$ $\mathbb{F}_p((t)) \models \varphi$ for all but finitely many primes p.

(This implies the Ax-Kochen thm.)

We want to proceed similarly as we did for the Lefschetz principle.

What have we used?
- Being algebraically closed is a first order property.
- Being of characteristic p ——— " ———.
- Steinitz' theorem (+ some cardinality stuff)
- Łoś' theorem



Kurt Hensel
1861 – 1941

Both $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ are complete valued fields, thus in particular Henselian valued fields.
(Replacement for alg. closedness)

Def: A valued field (F, v) is Henselian if one of the following equivalent conditions hold

i) The valuation can be extended to all algebraic extensions E of F.

ii) If $f \in \mathcal{O}_F[X]$ such that its reduction $\bar{f} \in \bar{F}[X]$ has a root $a \in \bar{F}$ such that $\bar{f}'(a) \neq 0$, then there is a unique $b \in \mathcal{O}_F$ such that $f(b) = 0$ and $a \equiv b \ (\pi \mathcal{O}_F)$.

iii) Let $f, g_0, h_0 \in \mathcal{O}_F[X]$ be monic. If $\bar{g}_0$ and $\bar{h}_0$ are relatively prime and $\bar{g}_0 \bar{h}_0 = \bar{f}$, there exist $g, h \in \mathcal{O}_F[X]$ such that $\bar{g}_0 = \bar{g}$, $\bar{h}_0 = \bar{h}$ and $gh = f$.

Benjamin's Remark: Maybe not so hard: Model finite ext. by polynomials ...

Can we express being Henselian in first order logic?

i) seems hard: $\forall E \in \{\text{algebraic extension of } F\}$

ii) problematic: $\exists a \in \bar{F}$

So let's try iii): $\boxed{\text{Let } f, g_0, h_0 \in \mathcal{O}_F[X] \text{ be monic. If } \overline{g_0} \text{ and } \overline{h_0} \text{ are relatively prime and } \overline{g_0}\,\overline{h_0} = \overline{f}, \text{ there exist } g, h \in \mathcal{O}_F[X] \text{ such that } \overline{g_0} = \overline{g},\ \overline{h_0} = \overline{h} \text{ and } gh = f.}$

$\underline{1.\text{Step}}$: iii) $\Longleftrightarrow$ for all $n \in \mathbb{N}$ iii)$_n$ holds:

iii)$_n$ Let $f, g, h \in \mathcal{O}_F[X]$ be monic of degree at most n. If $\overline{g}$ and $\overline{h}$ are relatively prime and $\overline{g}\,\overline{h} = \overline{f}$, there exist $\hat{g}, \hat{h} \in \mathcal{O}_F[X]$ of degree at most n such that $\overline{\hat{g}} = \overline{g}$, $\overline{\hat{h}} = \overline{h}$ and $\hat{g}\hat{h} = f$.

$\underline{2.\text{Step}}$: We can model polynominals of degree up to n by their coëfficients:

$\forall_n f \in F[X]$ is an abbr. for $\forall f_0 \forall f_1 \forall f_2 \ldots \forall f_n$
$\exists_n f \in F[X]$ is an abbr. for $\exists f_0 \exists f_1 \exists f_2 \ldots \exists f_n$

$\forall_n f \in \mathcal{O}_F[X]$ abbr. $\forall_n f \in F[X]\ v(f_0) \geq 0 \wedge \ldots v(f_n) \geq 0$

&c.

$f \doteq gh$ abbr. $f_0 \doteq g_0 \cdot h_0 \wedge f_1 \doteq g_0 h_1 + h_0 g_1 \wedge \ldots \wedge f_n \doteq \ldots$
$\wedge\ 0 \doteq g_1 h_n + \ldots \wedge\ 0 \doteq g_n h_n$

3. Step: We can measure equality in the residue field!

$$f \doteq gh \quad \text{abbr.} \quad v(f_0 - g_0h_0) > 1 \wedge v(f_1 - g_0h_1 - g_1h_0) > 1 \wedge \ldots \wedge v(f_n - \ldots) > 1 \wedge 0 \doteq \ldots \wedge \ldots \wedge 0 \doteq g_nh_n$$

4. Step: Expressing being relatively prime in the residue field:

$$\mathrm{RelPriRes}(f,g) \quad \text{abbr.} \quad \neg(\exists_n p \in \mathcal{O}_F[x]\ \exists_n q \in \mathcal{O}_F[x]\ \exists_n r \in \mathcal{O}_F[x]\ f \doteq pq \wedge g \doteq pr)$$

5. Step: Being a monic polynomial can be expressed:

$$\mathrm{Mon}(f) \quad \text{abbr.} \quad f_n \doteq 1 \vee (f_n \doteq 0 \wedge f_{n-1} \doteq 1) \vee \ldots \vee (f_n \doteq 0 \wedge f_{n-1} \doteq 0 \wedge \ldots \wedge f_1 \doteq 0 \wedge f_0 \doteq 1)$$

All together:

iii)$_n$ is modelled by $\forall_n f \in \mathcal{O}_F[x]\ \forall g \in \mathcal{O}_F[x]\ \forall h \in \mathcal{O}_F[x]$

$$\mathrm{Mon}(f) \wedge \mathrm{Mon}(g) \wedge \mathrm{Mon}(h) \wedge f \doteq gh \wedge \mathrm{RelPriRes}(g,h)$$

$$\longrightarrow \exists_n \hat{g} \in \mathcal{O}_F[x]\ \exists_n \hat{h} \in \mathcal{O}_F[x]\ g \doteq \hat{g} \wedge h \doteq \hat{h} \wedge f \doteq gh.$$

What have we used?

- Being ~~algebraically closed~~ (a) Henselian (valued field) is a first order property.
- Being of characteristic p —— " —— .
- Steinitz' theorem (+ some cardinality stuff)
- Łoś' theorem

Thm. (Ax-Kochen) Let (F, ν) and (G, μ) be henselian valued fields such that

language of gps

$$\nu(F^*) \equiv \mu(G^*) \text{ as } \mathcal{L}_G\text{-structures, and}$$

$$\overline{F} \equiv \overline{G} \text{ as } \mathcal{L}_F\text{-structures.}$$

$$\operatorname{char} F = \operatorname{char} G = \operatorname{char} \overline{F} = 0$$

Then $F \equiv G$ as valued fields.

Let A, B be $\mathcal{L}$-structures. We say A and B are elementary equivalent as $\mathcal{L}$-structures and write $A \equiv B$ if for all first order $\mathcal{L}$-sentences φ

$$A \models \varphi \iff B \models \varphi.$$

What have we used?

(a) Henselian (valued field)

- Being ~~algebraically closed~~ is a first order property.
- Being of characteristic p —— " —— .
- ~~Steinitz' theorem (+ some cardinality stuff)~~
- Łoś' theorem

Theorem A

What is missing? Thm A has stronger assumptions than Steinitz' Thm.

<u>Lemma</u>: For any ultrafilter ϑ on $\mathbb{P}$,

$$v\Big(\big({}_u\prod_{\mathbb{P},\vartheta} \mathbb{Q}_p\big)^*\Big) \cong v\Big(\big({}_u\prod_{\mathbb{P},\vartheta} \mathbb{F}_p((t))\big)^*\Big) \cong {}_u\prod_{\mathbb{P},\vartheta} \mathbb{Z} \quad \text{as groups,}$$

$$\overline{{}_u\prod_{\mathbb{P},\vartheta} \mathbb{Q}_p} \cong \overline{{}_u\prod_{\mathbb{P},\vartheta} \mathbb{F}_p((t))} \cong {}_u\prod_{\mathbb{P},\vartheta} \mathbb{F}_p \quad \text{as fields.}$$

Sketch *: $\mathbb{Q} = {}_u\prod_{\mathbb{P},\vartheta} \mathbb{Q}_p$, $f \in \mathbb{Q}^*$, $f = [(f_p)_{p\in\mathbb{P}}]$ wlog $f_p \neq 0 \ \forall p$

why? $\{p \in \mathbb{P} \mid f_p \neq 0\} \in \vartheta$

$$v_{\mathbb{Q}}([(f_p)_{p\in\mathbb{P}}]) = [(v_{\mathbb{Q}_p}(f_p))_{p\in\mathbb{P}}] = [(p^{n_p})_{p\in\mathbb{P}}]$$

$$v_{\mathbb{Q}}(\mathbb{Q}^*) \longrightarrow {}_u\prod_{\mathbb{P},\vartheta} \mathbb{Z}, \quad f \longmapsto [(n_p)_{p\in\mathbb{P}}]$$

Thm (Ax - Kochen - Ershov principle)

Let φ be a first order sentence in the language $\mathcal{L}_{VF}$ of valued fields. Then

$$\mathbb{Q}_p \models \varphi \text{ for all but finitely many primes } p$$

$$\Longleftrightarrow \quad \mathbb{F}_p((t)) \models \varphi \text{ for all but finitely many primes } p.$$

$*$: Let ϑ be any ultrafilter on $\mathbb{P}$. Then

$$\mathbb{Q}^* := u\prod_{\mathbb{P},\vartheta} \mathbb{Q}_p \quad \text{and} \quad \mathbb{F}^* := u\prod_{\mathbb{P},\vartheta} \mathbb{F}_p((t))$$

are Henselian valued fields of characteristic 0 (Henselian is first order) with

$$\nu(\mathbb{Q}^*) \cong \nu(\mathbb{F}^*) \implies \nu(\mathbb{Q}^*) \equiv \nu(\mathbb{F}^*)$$

$$\overline{\mathbb{Q}} \cong \overline{\mathbb{F}} \implies \overline{\mathbb{Q}} \equiv \overline{\mathbb{F}}$$

(Lemma)

Now Thm A implies $\mathbb{Q} \equiv \mathbb{F}$, i.e. for a $\mathcal{L}_{VF}$-sentence φ

$$\{p \in \mathbb{P} \mid \mathbb{Q}_p \not\models \varphi\} \notin \vartheta \iff \mathbb{Q} \models \varphi$$

$$\iff \mathbb{F} \models \varphi$$

$$\iff \{p \in \mathbb{P} \mid \mathbb{F}_p((t)) \models \varphi\} \notin \vartheta$$

But ϑ is arbitrary, hence

$$\{p \in \mathbb{P} \mid \mathbb{Q}_p \not\models \varphi\} \notin \bigcup \vartheta = \mathcal{P}(\mathbb{P}) \setminus \{\text{finite subsets}\}$$

$$\{p \in \mathbb{P} \mid \mathbb{F}_p((t)) \not\models \varphi\} \notin \bigcup \vartheta = \mathcal{P}(\mathbb{P}) \setminus \{\text{finite subsets}\}.$$

$\square$

~ Fin ~