CHAIN COMPLEXES:

22/04/2021.

$\mathcal{A} := \mathrm{Ab}$

Definition

A chain complex is a sequence of homomorphisms of abelian groups

$$C_\bullet : \quad \cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

n: Degree of C_n

∂_n: Differencials.

such that $\partial_n \circ \partial_{n+1} = 0$, $\forall\, n \in \mathbb{Z}$.

$$\Updownarrow$$

$$\operatorname{Im} \partial_{n+1} \subseteq \operatorname{Ker} \partial_n.$$

Note.

1. Given a complex $C_\bullet$, by defining $C^n := C_{-n}$ and $\partial^n := \partial_{-n}$ we get:

$$C^\bullet : \quad \cdots \longrightarrow C^{n-1} \xrightarrow{\partial^{n-1}} C^n \xrightarrow{\partial^n} C^{n+1} \longrightarrow \cdots$$

where $\partial^n \circ \partial^{n-1} = 0$, $\forall n \in \mathbb{Z}$.

This is known as cochain complex.

2. Given a group $G \in \mathcal{A}$, we have:

$$\cdots \longrightarrow 0 \to G \to 0 \to \cdots$$

If $G = 0$, then 0. (zero complex).

Definition.

A complex $C_\bullet$ is bounded above (below) if $C_n = 0$, $\forall\ n \geq N$ ($n \leq N$).

A complex $C_\bullet$ is bounded is it is bounded both below and above.

Morphisms of chain complexes:

A morphism $f: C_\bullet \to D_\bullet$ is a family of morphisms $\{f_n : C_n \to D_n\}_{n\in\mathbb{Z}}$ making the following diagram commutative:

$$\begin{tikzcd}
C_\bullet : \arrow[d, "f"'] & \cdots \arrow[r] & C_{n+1} \arrow[r, "\partial^C_{n+1}"] \arrow[d, "f_{n+1}"] \arrow[dr, phantom, "\circlearrowright"] & C_n \arrow[r, "\partial^C_n"] \arrow[d, "f_n"] \arrow[dr, phantom, "\circlearrowright"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
D_\bullet : & \cdots \arrow[r] & D_{n+1} \arrow[r, "\partial^D_{n+1}"'] & D_n \arrow[r, "\partial^D_n"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

This is $\partial^D_n \circ f_n = f_{n-1} \circ \partial^C_n$, $\forall n \in \mathbb{Z}$.

(Bounded)
Chain complexes together with chain morphisms define a category $Ch^b(A)$ which is called the category of (bounded) chain complexes.

Remark:

$Ch(A)$ is abelian since A is abelian.

$$\begin{tikzcd}
\operatorname{Ker} f \arrow[d, hook] & \Leftrightarrow & & \operatorname{Ker} f_n \arrow[d, hook] & & \\
C_\bullet : \arrow[d, "f"'] & \cdots \arrow[r] & C_{n+1} \arrow[r, "\partial^C_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial^C_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
D_\bullet : \arrow[d, two heads] & \cdots \arrow[r] & D_{n+1} \arrow[r, "\partial^D_{n+1}"'] & D_n \arrow[r, "\partial^D_n"'] \arrow[d, two heads] & D_{n-1} \arrow[r] & \cdots \\
\operatorname{Coker} f & & & \operatorname{Coker} f_n & &
\end{tikzcd}$$

Tensor product of chain complexes.

Given two complexes $C_\bullet$ and $D_\bullet$ we have that the tensor product of $C_\bullet \otimes D_\bullet$ is given by the following data:

$$(C_\bullet \otimes D_\bullet)_n := \bigoplus_{i+j=n} C_i \otimes_{\mathbb{Z}} D_j$$

Let $c \otimes d \in C_i \otimes D_j$, then we have that:

$$\partial_n(c\otimes d) := \partial_i^C(c) \otimes d + (-1)^i c \otimes \partial_j^D(d)$$

Lets check that $C_\bullet \otimes D_\bullet$ is in fact a chain complex:

$$\partial_{n-1}\circ\partial_n(c\otimes d) = \overset{\to 0}{\cancel{\partial_{i-1}^C\partial_i^C(c)}} \otimes d + \cancel{(-1)^{\boxed{i}}\partial_i^C(c)\otimes\partial_j^D(d)} + \cancel{(-1)^{\boxed{i-1}}\partial_i^C(c)}\otimes\partial_j^D(d)$$
$$+ c \otimes \cancel{\partial_{j-1}^D\partial_j^D(d)} \to 0$$
$$= 0.$$

Example:

Shift of a chain complex:

Given $n\in\mathbb{Z}$ and the chain complexes:

$$\begin{tikzcd}
 & & -n+1 & -n & -n-1 & \\
\mathbb{Z}[-n]: & \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[r, bend right] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
C_\bullet: & \cdots \arrow[r] & C_{n+1} \arrow[r] & C_n \arrow[r] & C_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

we have that $\mathbb{Z}[-n]\otimes C_\bullet$ is given by:

$$\begin{tikzcd}
 & & 1 & 0 & -1 & \\
\mathbb{Z}[-n]\otimes C_\bullet: & \cdots \arrow[r] & \mathbb{Z}\otimes_{\mathbb{Z}} C_{n+1} \arrow[r, "\partial_1"] \arrow[d, phantom, "\cong"] & \mathbb{Z}\otimes_{\mathbb{Z}} C_n \arrow[r, "\partial_0"] \arrow[d, phantom, "\cong"] & \mathbb{Z}\otimes_{\mathbb{Z}} C_{n-1} \arrow[r] \arrow[d, phantom, "\cong"] & \cdots \\
C_\bullet[n]: & \cdots \arrow[r] & C_{n+1} \arrow[r, "(-1)^n\partial_{n+1}"] & C_{\boxed{n}} \arrow[r, "(-1)^n\partial_n"] & C_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

$$\partial_0(\mathbb{Z}\otimes C_n) = \cancelto{0}{\partial_{-n}(\mathbb{Z})\otimes C_n} + (-1)^n \mathbb{Z}\otimes\partial_n(C_n)$$
$$\cong (-1)^n \partial_n(C_n)$$

In fact, in general in arbitrary abelian categories the n-shift chain complex $C_\cdot[n]$ is defined by:

$$(C_\cdot[n])_k := C_{n+k}$$

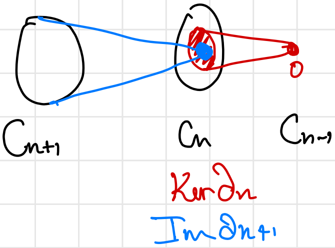
$$\partial_k^{C_\cdot[n]} := (-1)^n \partial_{n+k}.$$

Homology of chain complexes:

Given a complex $C_\cdot$:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

we that $\operatorname{Im}\partial_{n+1} \subseteq \operatorname{Ker}\partial_n$



$$\frac{\operatorname{Ker}\partial_n}{\operatorname{Im}\partial_{n+1}} = 0 \quad \text{if}$$

$$\operatorname{Im}\partial_{n+1} = \operatorname{Ker}\partial_n.$$

The n-th homology group of $C_\cdot$ is defined by:

$$H_n(C_\cdot) := \frac{\operatorname{Ker}\partial_n}{\operatorname{Im}\partial_{n+1}}$$

Given a map $f: C_\cdot \longrightarrow D_\cdot$ in $Ch(\mathcal{A})$, it induces a natural morphism $H_n(f): H_n(C_\cdot) \longrightarrow H_n(D_\cdot)$

$$\begin{tikzcd}
 & & & x \in \operatorname{Ker}\partial_n^C \arrow[r, mapsto] \arrow[d, hook] \arrow[ddd, mapsto, bend right] & 0 \arrow[ddd, mapsto, bend left] & \\
C_\bullet : \arrow[d, "f"'] & \cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}^C"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n^C"] \arrow[d, "f_n"] \arrow[dr, phantom, "\circlearrowright"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
D_\bullet : & \cdots \arrow[r] & D_{n+1} \arrow[r, "\partial_{n+1}^D"'] & D_n \arrow[r, "\partial_n^D"'] & D_{n-1} \arrow[r] & \cdots \\
 & & & f_n(x) \in \operatorname{Ker}\partial_n^D \arrow[u, hook] \arrow[r, mapsto] & 0 & \\
 & & & & \partial_n^D \circ f_n(x) \arrow[u, phantom, "\rotatebox{90}{$=$}"] &
\end{tikzcd}$$

$\Rightarrow f_n(\operatorname{Ker}\partial_n^C) \subseteq \operatorname{Ker}\partial_n^D.$

Similarly, $f_n(\operatorname{Im}\partial_{n+1}^C) \subseteq \operatorname{Im}\partial_{n+1}^D.$

Hence, we get a well-defined map, for each $n \in \mathbb{Z}$

$$H_n(f) : H_n(C_\bullet) \longrightarrow H_n(D_\bullet)$$
$$x + \operatorname{Im}\partial_{n+1}^C \longmapsto f_n(x) + \operatorname{Im}\partial_{n+1}^D$$

In fact, $H_n : Ch(\mathcal{A}) \longrightarrow \mathcal{A}$ is a functor.

Example:

Let $C_\bullet$ be the complex given by:

$$\cdots \longrightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \rightarrow \cdots$$

where $f(x) = 2x$ and $g(x) = \bar{x}$.

Then, $\operatorname{Im}(g) = \mathbb{Z}/2\mathbb{Z}$, $\operatorname{Im}(f) = 2\mathbb{Z} = \operatorname{Ker}(g)$ and $\operatorname{Ker}(f) = 0$.

Thus we get $H_n(C_\bullet) = 0 \;\; \forall\, n \in \mathbb{Z}.$

Definition:

A chain morphism $f: C_\bullet \to D_\bullet$ in $Ch(\mathcal{A})$ is a quasi-isomorphism if $H_n(f)$ is an isomorphism $\forall n \in \mathbb{Z}$.

Remark:

$0_\bullet \longrightarrow C_\bullet$ is a quasi-isomorphism if and only if $H_n(C_\bullet) = 0$.

Chain homotopies:

Two maps $f, g: C_\bullet \to D_\bullet$ in $Ch(\mathcal{A})$ are homotopic if there is a morphism $h: C_\bullet \to D_\bullet[1]$ s.t.

$$\begin{tikzcd}
C_\bullet : \arrow[d, "f-g"'] & \cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}^{C}"] \arrow[d] & C_n \arrow[r, "\partial_n^{C}"] \arrow[d, "f_n - g_n"] \arrow[dl, dashed, "h_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] \arrow[dl, dashed, "h_{n-1}"'] & \cdots \\
D_\bullet : & \cdots \arrow[r] & D_{n+1} \arrow[r, "\partial_{n+1}^{D}"'] & D_n \arrow[r, "\partial_n^{D}"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

$$h_{n-1} \circ \partial_n^C + \partial_{n+1}^D \circ h_n = f_n - g_n \quad \forall\, n \in \mathbb{Z}.$$

h is called homotopy. We write $f \sim g$.

"Topological way":

Define an interval object $I_\bullet$ in $Ch(\mathcal{A})$ as the chain complex concentrated in degrees $[0,1]$ given by:

$$I_\bullet : \quad \cdots \to 0 \to \overset{1}{\mathbb{Z}} \xrightarrow{f} \overset{0}{\mathbb{Z} \oplus \mathbb{Z}} \longrightarrow 0 \to \cdots$$

with $f(x) = (x, -x)$.

Given a chain complex $C_\bullet$ in $Ch(\mathcal{A})$ we have $I_\bullet \otimes C_\bullet$ will be defined as follows:

$$(I.\otimes C.)_n := C_n \oplus C_n \oplus C_{n-1}$$

$\partial_{n+1} : (I.\otimes C.)_{n+1} \longrightarrow (I.\otimes C.)_n$ is :

$$\partial_{n+1}(x,y,z) := (\partial^C_{n+1}(x) + z, \partial^C_{n+1}(x) - z, -\partial^C_n(z))$$

Proposition:

A chain homotopy between two maps $f, g : C. \to D.$ corresponds to a commutative diagram:

$$\begin{tikzcd}
& (0,x,0) & x \arrow[l, mapsto] \\
x \arrow[r, mapsto] & (x,0,0) & \\
C. \arrow[r] \arrow[dr, "f"'] \arrow[r, phantom, shift right=6ex, "\circlearrowright"] & I.\otimes C. \arrow[d, "{(f,g,h)}"] \arrow[dr, phantom, "\circlearrowright"] & C. \arrow[l] \arrow[dl, "g"] \\
& D. &
\end{tikzcd}$$

where $h : C. \longrightarrow D.[1]$ defines a homotopy.

Proof.

The chain maps (f,g,h) are :

$$\begin{tikzcd}
I.\otimes C. : \arrow[d, "{(f,g,h)}"'] & \cdots \arrow[r] & C_{n+1}\oplus C_{n+1}\oplus C_n \arrow[r, "\partial_{n+1}"] \arrow[d, "{(f_{n+1},g_{n+1},h_n)}"'] \arrow[dr, phantom, "\circlearrowright"] & C_n\oplus C_n\oplus C_{n-1} \arrow[r] \arrow[d, "{(f_n,g_n,h_{n-1})}"] & {} \\
D. : & \cdots \arrow[r] & D_{n+1} \arrow[r, "\partial^D_{n+1}"'] & D_n \arrow[r] & {}
\end{tikzcd}$$

commutative if

$$\partial^D_{n+1} \circ (f_{n+1}, g_{n+1}, h_n) = (f_n, g_n, h_{n-1}) \circ \partial_{n+1}$$

which happens if:

For elements of the form $(x,0,0)$ it commutes. ✓ Since f is a chain map.

For elements of the form $(0,x,0)$ it commutes ✓ Since g is a chain map

For elements of the form $(0,0,x)$ it commutes if:

$$\underline{\partial^D_{n+1}} \circ h_n(x) = (f_n, g_n, h_{n-1}) \circ \partial_{n+1}(x)$$

$$= (f_n, g_n, h_{n-1})(x, -x, -\partial^C_n(x))$$

$$\underline{=}\ f_n(x) - g_n(x) - h_{n-1} \circ \partial^C_n(x)$$

$\Leftrightarrow$ h is a homotopy

(Bounded)

Chain complexes together with homotopic classes define a category $K^b(A)$ which is called the ↓ homotopic category.
bounded

Comparison between homotopy equivalences and quasi-isomorphisms.

Definition:

A map $f: C_\bullet \to D_\bullet$ in $Ch(A)$ is homotopy equivalence if $\exists$ $g: D_\bullet \to C_\bullet$ s.t. $g \circ f \sim id_{C_\bullet}$ and $f \circ g \sim id_{D_\bullet}$.

* If $f \sim g \Rightarrow H_n(f) = H_n(g)$ $\forall n \in \mathbb{Z}$.

Then if $g \circ f \sim id_{C.}$ and $f \circ g \sim id_{D.}$ we get for each $n \in \mathbb{Z}$:

$$\underline{H}_n(g \circ f) = H_n(id_{C.}) \text{ and } H_n(f \circ g) = H_n(id_{D.})$$

$$\Rightarrow H_n(g) \circ H_n(f) = id_{H_n(C.)} \text{ and } H_n(f) \circ H_n(g) = id_{H_n(D.)}$$

$\Rightarrow$ $H_n(g)$ and $H_n(f)$ mutually inverse isom.

$\Rightarrow$ g and f are quasi-isom.

Lema:

If $f: C_\cdot \to D_\cdot$ in $Ch(\mathcal{A})$ is a homotopy equivalence, the f is a quasi-isom.

In general

Quasi-isomorphisms $\not\Rightarrow$ Homotopy equivalences.

Counter-example:

Let $C_\cdot$ be the chain complex

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "f"] & \mathbb{Z} \arrow[r, "g"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

where $f(x) = 2x$ and $g(x) = \overline{x}$.

Since $H_n(C_\cdot) = 0$ $\forall n \in \mathbb{Z}$, then $0_\cdot \to C_\cdot$ is a quasi-isomorphism.

Claim: $0_\cdot \to C_\cdot$ is not a homotopy equivalence.

$0_\cdot \to C_\cdot \to 0_\cdot$

Proof:

Assume $0_\cdot \to C_\cdot$ to be a homotopy equivalence, then we have $0 \sim id_{C}$.
On the other hand, consider a chain map $\tilde{f}: C_\cdot \to C_\cdot$ such that it is homotopic to $0: C_\cdot \to C_\cdot$ in $Ch(\mathcal{A})$. Thus, there exists a chain homotopy $h: C_\cdot \to C_\cdot[1]$.

$$\begin{tikzcd} C_\cdot : \arrow[d, "\tilde{f}"'] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d, "0"'] & \mathbb{Z} \arrow[r, "f"] \arrow[d, "\tilde{f}_1"'] & \mathbb{Z} \arrow[r, "g"] \arrow[d, "\tilde{f}_0"] \arrow[dl, dashed, "h_0"'] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "\tilde{f}_{-1}"] \arrow[dl, dashed, "h_{-1}"'] & 0 \arrow[r] \arrow[d, "0"] \arrow[dl, dashed, "h_{-2}"'] & \cdots \\ C_\cdot : & \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "f"'] & \mathbb{Z} \arrow[r, "g"'] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

with $h_{n-1} = 0$, because $\mathrm{Hom}_A(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$.

Thus, $\tilde{f}_{-1} = h_{-2} \circ 0 + g \circ h_{n-1} = 0$. Implying that $\tilde{f} \not\sim \mathrm{id}_c$. and by assumption $0 \sim \tilde{f}$.
This yields to a contradiction to $0 \sim \mathrm{id}c$..