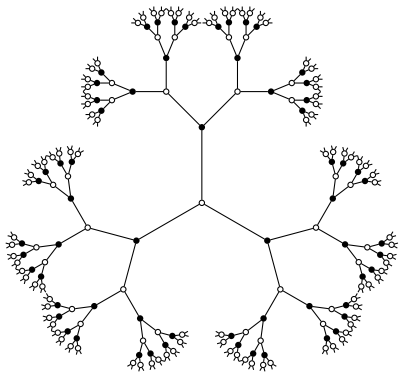
GRK WORKSHOP: BUILDINGS

SUMMER SEMESTER 2022

GEORG LINDEN AND YINGYING WANG[1]



Sketch of the Bruhat-Tits building of $\mathrm{SL}_2(\mathbb{Q}_2)$.

Introduction

Buildings were originally introduced by Jacques Tits beginning in the 1950s to study semi-simple Lie groups and semi-simple algebraic groups via geometric means. He defined them as simplicial complexes which consist of subcomplexes called "apartments", and satisfy certain axioms that prescribe a high degree of symmetry. The aforementioned groups then give rise to such buildings in a canonical manner.

From there the theory of buildings and their uses spread out to other areas of mathematics. For example François Bruhat and Jacques Tits constructed buildings associated to reductive algebraic groups over a field K with non-Archimedean valuation. In contrast to Tits construction these take the topology of K into account, and serve as an analogue of real symmetric spaces over non-Archimedean fields. Other applications include the representation theory of finite and p-adic groups, the classification of finite groups, arithmetic groups, and the compactification of symmetric spaces.

Furthermore, instead of Tits original approach via simplicial complexes, different ones only relying on the "chambers" (the top-dimensional simplices of a building) or in terms of certain metric spaces have been developed.

The main source for the program is the book of Abramenko and Brown on buildings [1] which is based on an earlier book by Brown [2]. A short introduction to the topic of buildings can be found in [3].

We want to begin the program with a recollection on finite reflection groups and more generally Coxeter groups. To those, we want to associate a certain simplicial complex[2], the Coxeter complex, which encapsulates the structure of a Coxeter group. The relevance of Coxeter complexes lies in the fact that these will be the components which constitute a building.

March 13, 2022

[1]Email addresses: linden@uni-wuppertal.de, ywang@uni-wuppertal.de.

[2]We follow the convention of [1] and mean by a simplicial complex the poset that describes the face relations. The geometric realization of this simplicial complex then yields a topological object.

The second talk then introduces buildings via Tits original simplicial approach. They are the union of apartments[3] which are Coxeter complexes and which are arranged in a very regular way. Moreover, we will consider well-behaved group actions on buildings and deduce that groups affording such an action possess a Bruhat decomposition.

In the third talk, we will encounter the notion of a BN-pair of a group G, and reproduce Tits construction of the building $\Delta(G, B)$ associated to this datum. It turns out that this yields a correspondence between the well-behaved group actions of G on buildings from the second talk, and BN-pairs of G. Then we will look at linear algebraic groups and the combinatorial Tits building that arises via the above method. As an application we will obtain a description of the Steinberg representation of $G(k)$, for G a connected reductive groups over a finite field k, as the top-dimensional rational homology group of the combinatorial Tits building of G.

In the last talk, we will turn towards the Bruhat-Tits building for reductive groups over a field with a discrete valuation. In contrast to the examples considered earlier this is an Euclidean building with the apartments being associated to affine Weyl groups. We will focus on the example of SL_n and give a more concrete description.

The first two talks should be well accessible without specific preknowledge. For the third and the last talk, familiarity with linear algebraic groups is beneficial. Also knowledge of valued fields is helpful for the fourth talk.

Do not hesitate to contact the organizers if you have any questions regarding the talks or the references.

Talk 1: Coxeter Groups

Briefly recall reflections at hyperplanes and finite reflection groups [1, Def. 1.3], [4, p. 3]. Explain the example of the reflection group with root system of type A_2 [1, Expl. 1.9] and draw its hyperplanes. It might be nice to include the examples for type B_2, G_2, or a general dihedral group if time permits. Give the definition of a Coxeter system (W, S) resp. a Coxeter group W [4, p. 105] (cf. [1, §2.4]) as a group W with generators S of order 2 only subject to the relations

$$W = \langle S \mid s^2 = 1, (st)^{m(s,t)} = 1 \rangle$$

where $m(s, t) \in \mathbb{N} \cup \{\infty\}$ is the order of st. Like in [4], we will only be interested in the case where the set of generators S is finite. State the theorem that finite reflection groups are Coxeter groups [4, §1.9, Thm.] and that the finite Coxeter groups are precisely the finite reflection groups [4, §6.4, Thm.], [1, Cor. 2.68]. Even though all examples of Coxeter groups we consider in this first talk are finite, there are examples of infinite Coxeter groups. These will be featured in Talk 4. Moreover, say a few words about the classification of finite reflection groups [1, §1.3] [4, §2.3, §2.4].

Recall the definition of simplicial complexes in terms of posets and the geometric realization of a finite simplicial complex [1, §A.1.1]. Illustrate this via the following concrete example: Consider the cells [1, §1.4.1] associated to the hyperplane arrangement for a root system of type A_2 [1, Fig. 1.4]. Radially projecting these cells onto the 1-dimensional sphere S^1 yields a triangulation of S^1 by a simplicial complex [1, §1.5.8]. In particular, draw the picture for this example. Furthermore, define chamber complexes [1, Def. A.8], type functions [1, p. 665], and define when a chamber complex is colorable [1, Def. A.10]. Demonstrate these definitions on the concrete example for type A_2, too (cf. [1, Fig. 1.14]).

Define standard parabolic subgroups and standard cosets of a Coxeter group W [1, Def. 2.12]. For later use, state [1, Prop. 2.13] and [1, Prop. 2.16]. Define the Coxeter complex $\Sigma(W, S)$ associated to a Coxeter system (W, S) as the poset of standard cosets [1, Def. 3.1] (cf. [4, §1.15]). Put the example of the simplicial complex associated to the hyperplane arrangement for type A_2 into the context of this definition (cf. [1, Fig. 1.9]). Introduce the terminology of [1, Def. 3.3] as well as the action of W on $\Sigma(W, S)$ by left translation which

[3] The real estate-themed terminology continues with chambers, walls, and galleries.

is simply transitive on the chambers [1, p. 116]. Finally state and prove [1, Thm. 3.5] which says that $\Sigma(W, S)$ is a colorable chamber complex of rank $|S|$ and the action of W on it is type-preserving.

Talk 2: Buildings

From now on, define a Coxeter complex to be any simplicial complex Σ which is isomorphic to $\Sigma(W, S)$, for some Coxeter system (W, S) [1, Def. 3.64]. Sketch the reasoning why one can speak of the unique type (W, S) of such a Coxeter complex Σ. To this end, either sketch how to associate a Coxeter matrix to Σ [1, Def. 3.84, Cor. 3.20], or sketch the relevant parts of (the proof of) the characterization of Coxeter complexes by Tits [1, Thm. 3.65].

Define buildings as simplicial complexes that satisfy certain axioms [1, Def. 4.1]. Briefly discuss the notion of a system of apartments [1, p. 174] and the different conventions concerning terminology [1, Rmk. 4.2]. Show that all apartments of a building are Coxeter complexes of the same type (W, S) [1, Prop. 4.7, Cor. 4.8]. As a concrete example present the thick building associated to the Fano plane ([1, Expl. 4.16, Fig. 4.1], for $m = 3$ over $\mathbb{F}_2$). Define spherical buildings [1, Def. 4.67]. Finally, mention the Solomon-Tits theorem on the homotopy type of buildings [1, Thm. 4.127].

Define when a group G acts strongly transitive on a building Δ, and introduce the subgroups B, T and N [2, V.1A]. Show that in this case $N/T \cong W$ where (W, S) is the type of Δ. Sketch how such an action on a building gives G a Bruhat decomposition $G = \coprod_{w\in W} BwB$ [2, V.1E]. (For background on the retraction $\rho_{\Sigma,C}$ see [1, Prop. 4.33, Def. 4.37, p. 666]. But present this only if time permits.) Moreover, sketch the two other consequences $BsB \cdot BwB \subset BwB \cup BswB$ [2, V.1F], and $sBs \not\subseteq B$ [2, V.1G], for $w \in W$, $s \in S$. Mention that in the next talk we will see the reverse situation: How to associate to a group G with a so called BN-pair a building on which G acts.

Talk 3: Group Actions on Buildings

Define the notion of a BN-pair of a group G [2, p. 110, para. 2]. Define a Tits system (G, B, N, S) and the Weyl group associated to a BN-pair [2, p. 110]. Mention the list of properties (1)–(8) [2, p. 107] if needed. Give an example for BN-pairs, when $G = \mathrm{GL}_n$ [2, pp. 100–101]. Give additional examples if desired.

Given a BN-pair, or equivalently a Tits system (G, B, N, S), define the standard parabolic subgroups P and the standard cosets gP [2, V.2][4]. Define $\Delta(G, B)$ as the poset of standard cosets, ordered by reverse inclusion [2, p. 111]. Show that $\Delta(G, B)$ is a thick building on which G acts strongly transitively, and that given any thick building Δ with a strongly transitive G-action, Δ is canonically isomorphic to $\Delta(G, B)$ for an appropriate BN-pair. For this part, state and outline the proof for [1, Thm. 6.56], [2, V.3 Lemma, Thm.].

Note that the fundamental apartment Σ of $\Delta(G, B)$ is the set $\{wP | w \in W\}$ and the fundamental chamber C in Σ corresponds to the the standard subgroup B [2, p. 111]. Moreover, the chambers of $\Delta(G, B)$ correspond to the conjugates of B and the simplicies of $\Delta(G, B)$ to the parabolic subgroups [2, V.3 Rmk. 3.]. Hint at the classification theorem for all finite thick buildings [2, p. 117]. Define the Tits building for a k-split reductive algebraic group G [5, Def. 5.3], and note that this is canonically isomorphic to $\Delta(G(k), P(k))$ [5, Thm. 5.2] (cf. [1, §C.10]). In particular, for a connected k-split reductive algebraic group, explain the relationship between the k-rational Borel subgroups of G and the chambers of the Tits building, as well as the one between the maximal k-split tori and the apartments. If time permits, describe $\Delta(G, B)$, for $G = \mathrm{GL}_n(k)$ and $B \subseteq \mathrm{GL}_n(k)$ the subgroup of all upper triangular matrices, following [2, V.5]. Mention that $\Delta(G, B)$ in this case is isomorphic to the complex of flags of proper nonzero subspaces of k^n.

Finally, let G be a connected reductive group over a finite field k of rank n. Show that the homology group $H_{n-1}(\Delta(G(k), B(k))) \otimes \mathbb{Q}$ affords the Steinberg representation of $G(k)$ [7, Thm. 1 and 2]. Sketch the key steps of the proof.

[4] These are called "special subgroups" resp. "special cosets" in [2].

Talk 4: The Bruhat-Tits Building

Define affine reflections of a real vector space V [1, p. 512]. Introduce affine reflection groups W [1, Def. 10.7], the cells defined by their hyperplanes [1, p. 515], and the set S of reflections with respect to the walls of a fixed chamber [1, p. 515]. State that this gives an affine Coxeter system (W, S) (cf. [1, Thm. 10.8 (i)]) and that $|\Sigma(W, S)| \cong V$ [1, p. 519]. Illustrate all of this with the examples of the infinite dihedral group D_∞ [1, 2.2.2 (ii)] and the affine Coxeter group of type $\widetilde{A}_2$ [1, Expl. 3.7, Fig. 3.1, Expl. 10.14 a)].

Provide some background on the theory of Bruhat-Tits buildings for reductive groups over valued fields, see e.g. [6, §3], [1, p. 693]. Focus then on the example of $G = \mathrm{SL}_n(K)$, for K a field complete with respect to a discrete valuation, with ring of integers A and residue field k, following [1, §6.9.2]. Define B as the inverse image of the group of upper triangular matrices under the reduction homomorphism $\mathrm{SL}_n(A) \to \mathrm{SL}_n(k)$ and N as the group of monomial matrices in $\mathrm{SL}_n(K)$, and sketch that these two subgroups yield an Euclidean BN-pair [1, Def. 11.31]. Explain the description of the Bruhat-Tits building $\Delta(\mathrm{SL}_n(K), B)$ in terms of lattices [1, §6.9.3]. (You can restrict yourself to the case $n = 2$ [1, pp. 355–357] if you prefer.) Discuss the picture for $\mathrm{SL}_2(\mathbb{Q}_p)$ [1, Fig. 6.3, p. 358] which is a building of type D_∞. Possibly hint at the picture for $\mathrm{SL}_3(\mathbb{Q}_p)$. If you want, say a few words about the spherical building at infinity [1, §11.8].

References

[1] P. Abramenko, K. S. Brown, *Buildings, Theory and Applications*, Springer, (2008).
[2] K. S. Brown, *Buildings*, Springer, (1989).
[3] B. Everitt, *A (very short) introduction to buildings*, Expo. Math. **32** (3), (2014), 221–247. `https://eprints.whiterose.ac.uk/90636/1/everitt.pdf`
[4] J. E. Humphreys *Reflection groups and Coxeter groups*, Cambridge University Press, (1990).
[5] J. Tits, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Mathematics, Vol. 386. Springer-Verlag, Berlin-New York, (1974).
[6] B. Rémy, A. Thuillier, A. Werner, *Bruhat-Tits buildings and analytic geometry*, Berkovich Spaces and Applications. Springer Lecture Notes in Mathematics **2119** (2015), 141–202.
[7] L. Solomon, *The Steinberg character of a finite group with BN-pair*, Theory of Finite Groups (Symposium, Harvard Univ., Cambridge, Mass.), (1968).