Abelian Categories

GRK-Workshop on tensor triangulated categories

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Remark: Product and coproduct coincide here; we usually write $\oplus$.

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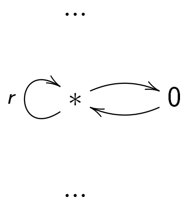

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5. Non-additive categories: e.g. **Set** or **Top**.

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   $\Rightarrow$ does have a cokernel $\mathbb{Z}/2\mathbb{Z}$, but it is not projective!

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A functor is called *exact* if it is left and right exact.

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3. The functor $T \otimes_R - : \mathbf{R\text{-}Mod} \to \mathbf{Ab}$ is right exact, but only left exact if T is flat.

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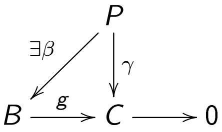

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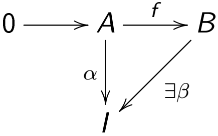

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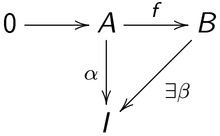

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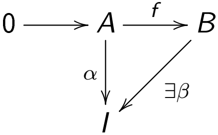

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Follows from reformulation of the definition of projective modules:

"An object P is projective if and only if $Hom(P, -)$ preserves epimorphism" (exercise).

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5. Injective abelian groups: *divisible groups*, e.g. $(\mathbb{Q}, \cdot)$ or $(\mathbb{R}, +)$
6. Noninjective and nonprojective abelian groups: e.g. cyclic groups $\mathbb{Z}/n\mathbb{Z}$ for $n > 1$

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2. Injective, but not projective: group $\mathbb{Q}/\mathbb{Z}$
3. The category **R-Mod** (assuming the axioms of choice) has enough projectives and injectives.

Examples (II)

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Remark: The algebraic K-group $K_0(R)$ "measures" the difference between injective and projective modules.

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If each of the P_i is projective, the resolution is called *projective*.

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Denote by $f_1 : P_1 \to P_0$ the composition.

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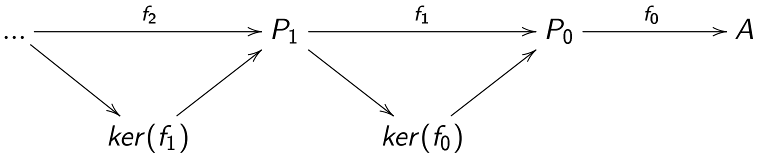

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3. The *Koszul complex* is a free resolution of the quotient ring $R/(x_1, \cdots, x_d)$.

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Collection of (*local*) modules organised in one *global* construction!

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Theorem (1964): Let $\mathcal{A}$ be a small abelian category.Then, there exists a ring R and an exact, fully faithful functor

$$\mathcal{A} \to R - mod$$

embedding $\mathcal{A}$ as a full subcategory in the sense that $Hom_{\mathcal{A}}(M, N) \cong Hom_{R-mod}(M, N)$.

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2. Is the structure of being abelian enough to behave well with a tensor? Can we get/Do we need more?

Thank you for your attention!