Rank of intersection of free subgroups in free amalgamated products of groups

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Then $H_1 \cap H_2$ is also finitely generated (Howson) and

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(Hanna Neumann conjecture)

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(W.Dicks and S.Ivanov, 2008: more precise estimate).

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Idea of the proof is given further.

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Core graphs

- $\pi : \Psi(H) \to \Gamma(H)$ – the projection:

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- $\Gamma_1(H)$ – the core of $\Gamma(H)$ (the union of all reduced closed paths ending at HT vertex)
- $\Psi_1(H)$ – the (full) inverse image of $\Gamma_1(H)$ under π (a subgraph of $\Psi(H)$ obtained from it by deleting all "unnecessary" edges and vertices)

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- Therefore, the rank of H can be calculated using the graph $\Psi_1(H)$.

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- Suppose that $w_1, ..., w_k$ are all secondary vertices of $\Psi_1(H_1 \cap H_2)$ such that $\eta(w_i) = (v_1, v_2)$, $i = 1...k$, where v_1, v_2 are fixed secondary vertices of $\Psi_1(H_1)$, $\Psi_1(H_2)$ respectively. Then

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- $deg\ w_i \le deg\ v_1, \quad deg\ w_i \le deg\ v_2, \ i = 1...k$ (since subgroups are factor-free)
- $\sum_{i=1}^{k} deg\ w_i \le deg\ v_1 \cdot deg\ v_2$ (since τ is injective)
- After summing over all pairs (v_1, v_2) and using the facts above we obtain the desired estimate.

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Conjecture

Suppose G is a fundamental group of a finite graph of groups X with finite edge groups, and H_1, H_2 are factor-free subgroups of G with finite ranks (a subgroup is factor-free if it intersects trivially with the conjugates to all vertex groups).
Then $H_1 \cap H_2$ also has finite rank, and

$$\bar{r}(H_1 \cap H_2) \leqslant 6n \cdot \bar{r}(H_1)\bar{r}(H_2),$$

where n is the maximum of orders of edge groups of X.

Thank you!