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The Magnus embedding is a quasi-isometry

Svetla Vassileva McGill University

Universität Düsseldorf, GAGTA 6, July 2012 IntroductionGeodesics in $A \ge B$ $\bullet \circ \circ$ $\circ \circ \circ$

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Introduction

The Magnus embedding is the main tool for studying *free solvable groups*:

• The n^{th} derived (commutator) subgroup of a group G is

$$G^{(n)} = [G^{(n-1)}, G^{(n-1)}],$$

where $G^{(1)} = G' = [G, G] = \langle [g, g'] | g, g' \in G \rangle$.

• The free solvable group $S_{d,r}$ of degree d and rank r is given by

$$S_{d,r} = \frac{F_r}{F_r^{(d)}}$$

• The Magnus embedding is a map $\phi : S_{d,r} \hookrightarrow \mathbb{Z}^r \wr S_{d-1,r}$.

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Quasi-isometries

• A *quasi-isometric embedding* f between two metric spaces (X, d_X) and (Y, d_Y) is an injective map $f : X \to Y$ such that there exist constants $C_1, \ldots, C_4 > 0$ for which

$$C_1 d_X(x_1, x_2) - C_2 \le d_Y(f(x_1), f(x_2)) \le C_3 d_X(x_1, x_2) + C_4$$

for any $x_1, x_2 \in X$.

- For us, *X*, *Y* are groups and *d_X*, *d_Y* are the corresponding word metrics.
- Wlog, f : G → H is a quasi-isometry if it preserves geodesic length "up to linearity".

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Set-up and notation

•
$$F = \langle x_1, \ldots, x_r \rangle$$

• $N \lhd F$

•
$$N' = [N, N] = \langle [x, y] \mid x, y \in N \rangle$$

•
$$A = \langle a_1, \ldots, a_r \rangle$$
 – free abelian

• $B = F/_N$



• Show that ϕ is a quasi-isometry.

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Wreath products

The *restricted wreath product* is the group:

$$A \wr B = \{ bf \mid b \in B, f \in A^{(B)} \},\$$

with multiplication defined by

$$bf \cdot cg = bc f^c g,$$

where

- $f^{c}(x) = f(xc^{-1})$ for $x \in B$.
- $A^{(B)}$ is the set of all functions from B to A of *finite support*
- Multiplication in $A^{(B)}$ is given by $f \cdot g(x) = f(x)g(x)$
- $1_{A^{(B)}}$ is the function $1: B \to 1_A$.

Remark. *B* acts on $A^{(B)}$, so $A \wr B \simeq B \ltimes A^{(B)}$

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A presentation for $A \wr B$

Let
$$A = \langle X | R_A \rangle$$
, $B = \langle Y | R_B \rangle$. Then

$$A \wr B = \left\langle X \cup Y \mid R_A, R_B, [a_1^{b_1}, a_2^{b_2}] \right\rangle,$$

where $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

• Define $f_{a,b}(x) = \begin{cases} a & \text{if } x = b \\ 1 & \text{otherwise} \end{cases}$

• Then $A \hookrightarrow A \wr B$ (via $a \mapsto f_{a,1}$)

- Any function $f \in A^{(B)}$ can be given as $\{(b_1, a_1), \dots, (b_n, a_n)\}$
- Equivalently, $f = f_{a_1,b_1} \dots f_{a_n,b_n} = f_{a_1,1}^{b_1} \dots f_{a_n,1}^{b_n} \iff a_1^{b_1} \dots a_n^{b_n}$

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Geodesics in $A \wr B$

• Let w be a word in the generators of A and B. Rewrite it as

$$w = b A_1^{B_1} \dots A_k^{B_k},$$

 $A_1, \ldots, A_k \neq 1$ and B_1, \ldots, B_k are distinct.

Theorem (Parry)

$$||w||_{A \wr B} = ||b||_B + \sum_{i=1}^k ||A_i||_A + \mathcal{L}_{\operatorname{Cay}(B)}(B_1, \dots, B_k).$$

\$\mathcal{L}_{\mathcal{Cay}(B)}(B_1, \ldots, B_k)\$ is the length of a *minimum length cycle*: the shortest circuit in Cay(B) passing through \$\{1, B_1, \ldots, B_k\}\$.

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Two views of Fox derivatives

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The Magnus embedding

- The Magnus embedding was originally defined as φ : F → M, where M is a matrix group with entries in a group ring.
- For a word *w* in generators $X = \{x_1, \ldots, x_r\},\$

$$\phi: F/_{N'} \hookrightarrow A \wr F/_N$$

is given by

$$\phi(w) = \overline{w} \cdot a_1^{\overline{\partial w/\partial x_1}} \dots a_r^{\overline{\partial w/\partial x_r}}$$

• Here, $\frac{\partial w}{\partial x_1}, \ldots, \frac{\partial w}{\partial x_r}$ are the *Fox derivatives* of *w*.

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Fox derivatives

For any $x, y \in X$ the delta-function

$$\frac{\partial y}{\partial x} = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

extends linearly to a derivation $\frac{\partial}{\partial x}$: $\mathbb{Z}F \to \mathbb{Z}F$, called the *Fox partial derivative*.

Properties:

• Product Rule.
$$\frac{\partial uv}{\partial x} = \frac{\partial u}{\partial x} + u \frac{\partial v}{\partial x}$$

• Power Rule. $\frac{\partial u^{-1}}{\partial x} = u^{-1} \frac{\partial u}{\partial x}$

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Example

$$\frac{\partial w}{\partial x_1} = \frac{\partial x_2^{-1}}{\partial x_1} + x_2^{-1} \frac{\partial x_1 x_2 x_1^2 x_2 x_1^{-1} x_2^{-1} x_1^{-1} x_2}{\partial x_1}$$

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Example

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$$= x_2^{-1} \left(\frac{\partial x_1}{\partial x_1} + x_1 \frac{\partial x_2 x_1^2 x_2 x_1^{-1} x_2^{-1} x_1^{-1} x_2}{\partial x_1} \right)$$

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Example

$$\frac{\partial w}{\partial x_1} = \frac{\partial x_2^{-1}}{\partial x_1} + x_2^{-1} \frac{\partial x_1 x_2 x_1^2 x_2 x_1^{-1} x_2^{-1} x_1^{-1} x_2}{\partial x_1}$$
$$= x_2^{-1} \left(\frac{\partial x_1^{-1}}{\partial x_1} + x_1 \frac{\partial x_2 x_1^2 x_2 x_1^{-1} x_2^{-1} x_1^{-1} x_2}{\partial x_1} \right)$$

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Let
$$F = F(x_1, x_2)$$
 and $w = x_2^{-1} x_1 x_2 x_1^2 x_2 x_1^{-1} x_2^{-1} x_1^{-1} x_2$.

$$\begin{aligned} \frac{\partial w}{\partial x_1} &= \frac{\partial x_2^{-1}}{\partial x_1} + x_2^{-1} \frac{\partial x_1 x_2 x_1^2 x_2 x_1^{-1} x_2^{-1} x_1^{-1} x_2}{\partial x_1} \\ &= x_2^{-1} \left(\frac{\partial x_1}{\partial x_1} + x_1 \frac{\partial x_2 x_1^2 x_2 x_1^{-1} x_2^{-1} x_1^{-1} x_2}{\partial x_1} \right) \\ &= x_2^{-1} + x_2^{-1} x_1 \left(\frac{\partial x_2}{\partial x_1} + x_2 \frac{\partial x_1^2 x_2 x_1^{-1} x_2^{-1} x_1^{-1} x_2}{\partial x_1} \right) \end{aligned}$$

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Flows in a Cayley graph

Consider the Cayley graph $\Gamma(G, X)$ as a digraph. Let $p = e_1 \dots e_n$ be a path in Γ and define a *flow* π_p as follows:

 $\pi_p(e) =$ algebraic number of times that *p* traverses *e*.

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Example of flows on $\Gamma(G, X)$

Example. Consider $G = F/F' \simeq \mathbb{Z} \times \mathbb{Z}$ with $X = \{x_1, x_2\}$. Find π_w for $w = x_2^{-1}x_1x_2x_1^2x_2x_1^{-1}x_2^{-1}x_1^{-1}x_2$.



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Geometric interpretation of Fox derivatives

Edges in $\Gamma(F/N, X)$ have the form $e = (g, gx_i)$ for $g \in F/N$ and i = 1, ..., r.

Theorem (Miasnikov, Roman'kov, Ushakov, Vershik) Let $w \in F$. Then $\frac{\overline{\partial w}}{\partial x} = \sum_{g \in F/N} \pi_w ((g, gx))g.$

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Example of Fox derivatives and flows

$$\frac{\partial w}{\partial x_1} = x_2^{-1} + x_2^{-1} x_1 x_2 + x_2^{-1} x_1 x_2 x_1 - x_2^{-1} x_1 x_2 x_1^2 x_2 x_1^{-1} - x_2^{-1} x_1 x_2 x_1^2 x_2 x_1^{-1} x_2^{-1} x_1^{-1} \frac{\partial w}{\partial x_1} = x_2^{-1} + x_1^{-1} + x_1^2 - x_1^2 x_2 - x_1^{-1} = x_2^{-1} + x_1^2 - x_1^2 x_2$$





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Finding geodesics

Goal. For a word $w \in F/N'$ given as a product of generators *X*, find a geodesic for *w*.

- Read w as a path p_w in Cay(F/N, X). (This is not a typo! N, not N'.)
- This path defines a flow, π_w .
- Consider the subgraph Γ of Cay(F/N, X) which consists of edges of non-zero flow.

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Γ and the minimal forest

- C_1, \ldots, C_l connected components of Γ
- Q minimal forest connecting C_1, \ldots, C_l
- $\Delta = Q \cup C_1 \ldots \cup C_l$



Note. There may be more than one choice for Q.

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From Δ *to* Δ^* *and back*

Consider $\mathbb{Z} \times \mathbb{Z} = \langle x, y \rangle$. $w = yxy^{-1}x^{-1}yxyx^2y^{-1}xyx^{-3}y^{-2}$

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Consider
$$\mathbb{Z} \times \mathbb{Z} = \langle x, y \rangle$$
.
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	Consider $\mathbb{Z} \times \mathbb{Z} = \langle x, y \rangle$.								
	w = y	$yxy^{-1}x^{-1}$	$yxyx^2y^{-1}$	$xyx^{-3}y^{-1}$	-2				
^у									
		Δ							
				-1					
				1	1				
	2			-1	1				
0				1					
2	-1	-2							

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	Consider $\mathbb{Z} \times \mathbb{Z} = \langle x, y \rangle$.								
y	w — .	улу л 	улул у 	лул у 					
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	2		Q	_1	1				
2				1					
_	-1	-2							

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An Euler tour on the vertices of Δ^* corresponds to a geodesic for *w* in $F_{N'}$.

Theorem. (Miasnikov, Roman'kov, Ushakov, Vershik)

$$\|w\|_{F_{N'}} = \sum_{e \in \operatorname{supp}(p_w)} |\pi_w(e)| + 2|E(Q)|.$$

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Main Theorem

Theorem (V)

Let w be an element of $F/_{N'}$ given as a product of generators x_1, \ldots, x_r . Then

$$\frac{1}{2(r+1)} \|w\|_{F_{/N'}} \le \|\phi(w)\|_{A \wr B} \le 3 \|w\|_{F_{/N'}}.$$

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•
$$F = F(x, y), N = F'$$

•
$$F_{N'} \simeq M_2$$
, free metabelian group

•
$$B = {}^{F}/_{N} \simeq \mathbb{Z} \times \mathbb{Z}$$

•
$$A = \langle a_1, a_2 \rangle$$
 - free abelian group

•
$$w = yxy^{-1}x^{-1}yxyx^{2}y^{-1}xyx^{-3}y^{-2}$$

• $\overline{\partial w}_{\partial x} = -1 + 2y + x^{3}y - x^{3}y^{2}, \ \overline{\partial w}_{\partial y} = 2 - 2x - x^{3}y + x^{4}y$
• $\phi(w) = \overline{w} \cdot a_{1}^{\overline{\partial w}_{\partial x}} a_{2}^{\overline{\partial w}_{\partial y}} = x \cdot a_{1}^{-1+2y+x^{3}y-x^{3}y^{2}} a_{2}^{2-2x-x^{3}y+x^{4}y}$
 $= x \cdot (a_{1}^{-1}a_{2}^{2})(a_{1}^{2})^{y}(a_{2}^{-2})^{x}(a_{1}a_{2}^{-1})^{x^{3}y}(a_{1}^{-1})^{x^{3}y^{2}}(a_{2})^{x^{4}y}$

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A crucial lemma

•
$$\phi(w) = \overline{w} \cdot A_1^{B_1} \dots A_k^{B_k}$$
.

• $\sum_{i} ||A_i||_A$ is the sum of absolute values of the coefficients in the Fox derivatives

Lemma

$$\sum_i \|A_i\|_A = \sum_{e \in \operatorname{supp}(p_w)} |\pi_w(e)|.$$

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Proof by example

•
$$w = yxy^{-1}x^{-1}yxyx^{2}y^{-1}xyx^{-3}y^{-2}$$

• $\overline{\partial w}/_{\partial x} = -1 + 2y + x^{3}y - x^{3}y^{2}$,
• $\overline{\partial w}/_{\partial y} = 2 - 2x - x^{3}y + x^{4}y$
• $\phi(w) = x \cdot \underbrace{(a_{1}^{-1}a_{2}^{2})}_{A_{1}^{B_{1}}} \underbrace{(a_{2}^{-2})^{x}}_{A_{2}^{B_{2}}} \underbrace{(a_{1}a_{2}^{-1})^{x^{3}y}}_{A_{4}^{B_{4}}} \underbrace{(a_{1}^{-1})^{x^{3}y^{2}}}_{A_{5}^{B_{5}}} \underbrace{(a_{2})^{x^{4}y}}_{A_{6}^{B_{6}}}$
• $\sum_{i=1}^{6} ||A_{i}||_{A} = (1+2) + 2 + 2 + (1+1) + 1 + 1 = \sum_{e \in \text{supp}(p_{w})} |\pi_{w}(e)|$

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Comparing graphs – first look



•
$$g \in F_N$$
 such that $\begin{cases} \pi_w(g, gx) \neq 0 \\ \text{or} \\ \pi_w(g, gy) \neq 0 \end{cases} \longleftrightarrow B_i \text{ for some } i.$

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 $\begin{aligned} \text{fuction} \quad & \text{Geodesics in } A \wr B \\ & \text{OOO} \quad & \text{Geodesics in } F/N' \\ & \|\phi(w)\|_{A \wr B} \leq 3 \|w\|_{F_{N'}} \\ & \|w\|_{F_{N'}} = \sum_{l} |\pi_w(e)| + 2|E(Q)| \quad \|\phi(w)\|_{A \wr B} = \|\overline{w}\|_{F_{N}} + \sum \|A_i\| + |\mathcal{T}| \end{aligned}$

$$\|w\|_{F_{/N'}} = \sum_{i=1}^{l} |\pi_w(e)| + 2|E(Q)| \quad \|\phi(w)\|_{A \wr B} = \|\overline{w}\|_{F_{/N}} + \sum_{i=1}^{l} \|A_i\| + \|\mathcal{T}\|$$
$$= \sum_{i=1}^{l} |E(C_i)| + 2|E(Q)|$$

• Any tour on $V(\Delta)$ is longer than a minimal tour \mathcal{T} on $\{1, B_1, \dots, B_k\}$, so $|\mathcal{T}| \leq ||w||_{F_{/N'}}$

•
$$\sum ||A_i|| = \sum |\pi_w(e)|$$
, so

$$\sum \|A_i\| \le \|w\|_{F_{/N'}}$$

• $\|\overline{w}\|_{F_{/N}} \le \|w\|_{F_{/N'}}$ • $\Longrightarrow \|\phi(w)\|_{A \wr B} \le 3 \|w\|_{F_{/N'}}$

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 $\|w\|_{F_{N'}} \le (2r+1) \|\phi(w)\|_{A \setminus B}$



 $\begin{aligned} \|w\|_{F_{N'}} &= \sum |\pi_w(e)| + 2|E(Q)| & \|\phi(w)\|_{A \wr B} = \|\overline{w}\|_{F_N} + \sum \|A_i\| + |\mathcal{T}| \\ &= \sum_{i=1}^l |E(C_i)| + 2|E(Q)| & = \sum \pi_w(e) + |\mathcal{T}| \end{aligned}$

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Geodesics in F/N'0000

 $\|w\|_{F_{N'}} \le (2r+1)\|\phi(w)\|_{A \wr B}$



$$\|w\|_{F_{N'}} = \sum |\pi_w(e)| + 2|E(Q)| \qquad \|\phi(w)\|_{A \wr B} = \|\overline{w}\|_{F_{N}} + \sum \|A_i\| + |\mathcal{T}|$$

= $\sum_{i=1}^l |E(C_i)| + 2|E(Q)| \qquad = \sum \pi_w(e) + |\mathcal{T}|$

Geodesics in $A \wr B$ 000 Two views of Fox derivatives

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