Tits alternatives for graph products

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Background and motivation

Theorem (J. Tits, 1972)

Let H be a finitely generated subgroup of $\mathrm{GL}_n(F)$ for some field F. Then either H is virtually solvable or H contains a non-abelian free subgroup.

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Theorem (Noskov-Vinberg, 2002)

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Recall: a group G is large is there is a finite index subgroup $K \leqslant G$ s.t. K maps onto $\mathbb{F}_2$.

Various forms of Tits Alternative

Definition

Let $\mathcal{C}$ be a class of gps. A gp. G satisfies the Tits Alternative rel. to $\mathcal{C}$ if for any f.g. sbgp. $H \leqslant G$ either $H \in \mathcal{C}$ or H contains a copy of $\mathbb{F}_2$.

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The thm. of Noskov-Vinberg claims that Coxeter gps. satisfy the Strong Tits Alternative rel. to $\mathcal{C}_{vab}$.

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$$[a, b] = 1 \ \forall a \in G_u, \forall b \in G_v \text{ whenever } (u, v) \in E\Gamma.$$

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If $A \subseteq V\Gamma$ and Γ_A is the full subgraph of Γ spanned by A then $\mathfrak{G}_A := \{G_v \mid v \in A\}$ generates a special subgroup G_A of $G = \Gamma\mathfrak{G}$ which is naturally isomorphic to $\Gamma_A\mathfrak{G}_A$.

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Theorem A (Antolín-M.)

Let $\mathcal{C}$ be a class of gps. with (P0)–(P4). Then a graph product $G = \Gamma\mathfrak{G}$ satisfies the Tits Alternative rel. to $\mathcal{C}$ iff each G_v, $v \in V\Gamma$, satisfies this alternative.

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Evidently the conditions (P0)–(P4) are necessary.

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Corollary

If all vertex gps. are linear then $G = \Gamma\mathfrak{G}$ satisfies the Tits Alternative rel. to $\mathcal{C}_{vsol}$.

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(P5) is necessary, b/c if $L \neq \{1\}$ has no proper f.i. sbgps., then $L * L$ cannot be large.

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Corollary

Suppose $\mathcal{C} = \mathcal{C}_{sol-m}$ for some $m \geq 2$ or $\mathcal{C} = \mathcal{C}_{vsol-n}$ for some $n \geq 1$. Let G be a graph product of gps. from $\mathcal{C}$. Then any f.g. sbgp. of G either belongs to $\mathcal{C}$ or is large.

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Any residually free gp. satisfies the Strongest Tits Alternative rel. to $\mathcal{C}_{ab}$.

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Observe that if $L * L$ maps onto $\mathbb{F}_2$ then L must have an epimorphism onto $\mathbb{Z}$.

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Combining with a result of Lyndon-Schützenberger we also get

Corollary

If G is a RAAG and $a, b, c \in G$ *satisfy* $a^m b^n = c^p$, *for* $m, n, p \geq 2$, *then* a, b, c *pairwise commute.*