On the elementary theory of linear groups.

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GAGTA-6

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First-order logic

First-order language of groups $\mathcal{L}$

- a symbol for multiplication '$\cdot$';
- a symbol for inversion '$^{-1}$';
- and a symbol for the identity '1'.

Formula

Formula Φ with free variables $Z = \{z_1, \ldots, z_k\}$ is

$$Q_1 x_1 Q_2 x_2 \ldots Q_l x_l \, \Psi(X, Z),$$

where $Q_i \in \{\forall, \exists\}$, and $\Psi(X, Z)$ is a Boolean combination of equations and inequations in variables $X \cup Z$. Formula Φ is called a sentence, if Φ does not contain free variables.

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Examples

Using $\mathcal{L}$ one can say that

- A group is (non-)abelian or (non-)nilpotent or (non-)solvable;
- A group does not have p-torsion;
- A group is torsion free;
- A group is a given finite group;
- $\forall x, \forall y, \forall z \; x^k y^l z^m = 1 \to ([x, y] = 1 \wedge [y, z] = 1 \wedge [x, z] = 1)$

Using $\mathcal{L}$ one can **not** say that

- A group is finitely generated (presented) or countable;
- A group is free or free abelian or cyclic.

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Formulas and Sentences

$$\Phi(Z): \quad Q_1 x_1 Q_2 x_2 \ldots Q_l x_l \, \Psi(X, Z),$$

- $\Phi: \ \forall x \forall y \ xyx^{-1}y^{-1} = 1$;
- $\Phi(y): \ \forall x \ xyx^{-1}y^{-1} = 1$.

A truth set of a formula is called *definable*.

Elementary equivalence

The elementary theory $\mathrm{Th}(G)$ of a group is the set of all sentences which hold in G. Two groups G and H are called elementarily equivalent if $\mathrm{Th}(G) = \mathrm{Th}(H)$.

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Classify groups (in a given class) up to elementary equivalence.

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Keislar-Shelah Theorem

An ultrafilter $\mathfrak{U}$ on $\mathbb{N}$ is a 0-1 probability measure. The ultrafilter is non-principal if the measure of every finite set is 0.
Consider the unrestricted direct product $\prod G$ of copies of G.
Identify two sequence (g_i) and (h_i) if they coincide on a set of measure 1. The obtained object is a group called the ultrapower of G.

Theorem (Keislar-Shelah)

Let H and K be groups. The groups H and K are elementarily equivalent if and only if there exists a non-principal ultrafilter $\mathfrak{U}$ so that the ultrapowers H^* and K^* are isomorphic.

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Results of Malcev

Theorem (Malcev, 1961)

Let $G = GL$ (or PGL, SL, PSL), let $n, m \geq 3$, and let K and F be fields of characteristic zero, then $G_m(F) \equiv G_n(K)$ if and only if $m = n$ and $F \equiv K$.

Proof

If $G_m(F) \equiv G_n(K)$, then $G_m^*(F) \simeq G_n^*(K)$. Since $G_m^*(F)$ and $G_n^*(K)$ are $G_m(F^*)$ and $G_n(K^*)$, the result follows from the description of abstract isomorphisms of such groups (which are semi-algebraic, so they preserve the algebraic scheme and the field).

In fact, in the case of GL and PGL the result holds for $n, m \geq 2$.

Classical linear groups over $\mathbb{Z}$

Theorem (Malcev, 1961)

Let $G = GL$ (or PGL, SL, PSL), let $n, m \geq 3$, and let R and S be commutative rings of characteristic zero, then $G_m(R) \equiv G_n(S)$ if and only if $m = n$ and $R \equiv S$.

In the case of GL and PGL the result holds for $n, m \geq 2$.

- Malcev stresses the importance of the case when $R = \mathbb{Z}$, and $n = 2$.

Results of Durnev, 1995

Theorem

The $\forall^2$-theories of the groups $GL(n,\mathbb{Z})$ and $GL(m,\mathbb{Z})$ ($PGL(n,\mathbb{Z})$ and $PGL(m,\mathbb{Z})$, $SL(n,\mathbb{Z})$ and $SL(m,\mathbb{Z})$, or $PSL(n,\mathbb{Z})$ and $PSL(m,\mathbb{Z})$) are distinct, $n > m > 1$. If n is even or n is odd and $m \leq n-2$, then even the corresponding $\forall^1$-theories are distinct.

Theorem

There exists m so that for every $n \geq 3$, the $\forall^2\exists^m$-theory of $GL(n,\mathbb{Z})$ is undecidable. Similarly, for every $n \geq 3$, $n \neq 4$, the $\forall^2\exists^m$-theory of $SL(n,\mathbb{Z})$ is undecidable.

That is, there exists m so that for any n there is no algorithm that, given a $\forall^2\exists^m$-sentence decides whether or not it is true in $GL(n,\mathbb{Z})$ (or $SL(n,\mathbb{Z})$)

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Lifting elementary equivalence

- Let $1 \to N \to G \to Q \to 1$ be a group extension.
- Use Q and N to understand $Th(G)$.
- Suppose that we know which groups are elementarily equivalent to N and Q.
- Then if the action of Q on N can be described using first-order language and if N is definable in G, then we may be able to describe groups elementarily equivalent to G.

Example

- Linear groups.
- Soluble groups.
- Nilpotent groups.

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Finitely generated groups elementarily equivalent to $PSL(2,\mathbb{Z}), SL(2,\mathbb{Z}), GL(2,\mathbb{Z})$ and $PGL(2,\mathbb{Z})$

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Finitely generated groups elementarily equivalent to $PSL(2, \mathbb{Z})$

$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generate $SL(2, \mathbb{Z})$.
S has order 4, ST has order 6, $S^2 = (ST)^3 = -I_2$,

$SL(2, \mathbb{Z}) \simeq \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ and $PSL(2, \mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3 = {}^{SL(2,\mathbb{Z})}/_{Z(SL(2,\mathbb{Z}))}$.

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Theorem (Sela, 2011)

A finitely generated group G is elementary equivalent to $PSL(2,\mathbb{Z})$ if and only if G is a hyperbolic tower (over $PSL(2,\mathbb{Z})$).

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- $1 \to F_2 = PSL(2,\mathbb{Z})' \to PSL(2,\mathbb{Z}) \to \mathbb{Z}_2 \times \mathbb{Z}_3 \to 1$
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Hyperbolic towers over $PSL(2,\mathbb{Z})$

- Induction on height of tower.
- Any hyperbolic tower T^0 of height 0 is a free product of $PSL(2,\mathbb{Z})$ with some (possibly none) free groups and fundamental groups of hyperbolic surfaces of Euler characteristic at most -2.
- A hyperbolic tower T^n is built from a tower T^{n-1} by taking free product of T^{n-1} with free groups and surface groups and then attaching finitely many hyperbolic surface groups or punctured 2-tori along boundary subgroups in such a way that T^n retracts to T^{n-1} and the restriction of this retraction onto any of the surfaces has nonabelian image in T^{n-1}

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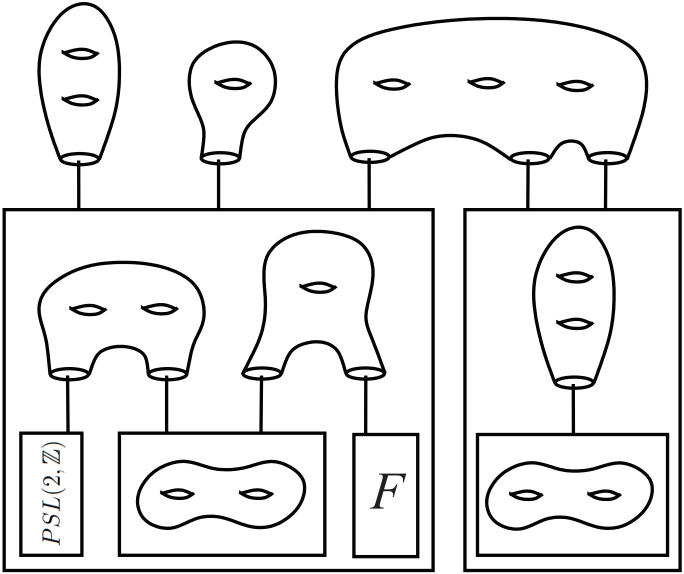

Finitely generated groups elementarily equivalent to $SL(2, \mathbb{Z})$

- We have $1 \to \mathbb{Z}_2 \to SL(2, \mathbb{Z}) \to PSL(2, \mathbb{Z}) \to 1$.
- Let $G \equiv SL(2, \mathbb{Z})$, then $Z(G) \equiv Z(SL(2, \mathbb{Z}))$, hence $Z(G) = \mathbb{Z}_2$.
- Since $Z(G)$ is definable and G is f.g., $Q = {}^{G}/_{Z(G)} \equiv PSL(2, \mathbb{Z})$ is a hyperbolic tower.
- Hence, G is a central extension of a tower by $\mathbb{Z}_2$.
- Central extensions are described using the second cohomology group $H^2(Q, Z(G))$.

1. Use the explicit description of towers and compute the cohomology.
2. Do a trick.

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- $Z(G^*) \simeq Z(G)^*$ and G^* is the central extension of Q^* by $Z(G)^*$. The corresponding cocycle $f^* : Q^* \times Q^* \to A^*$ is defined coordinate-wise, i.e. $f^* = (f)$.
- The cocycle $h : PSL(2,\mathbb{Z}) \times PSL(2,\mathbb{Z}) \to \mathbb{Z}_2$ satisfies: $h(x,x) = 1$ for all x of order 2, and $h(y,z) = 0$ otherwise.
- By the properties of ultrafilters, the same holds the cocycle $h^* = (h)$ which defines $SL(2,\mathbb{Z})^*$ as the extension of $PSL(2,\mathbb{Z})^*$.

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Finitely generated groups elementarily equivalent to $SL(2, \mathbb{Z})$

Theorem

A finitely generated group G is elementarily equivalent to $SL(2, \mathbb{Z})$ if and only if G is the central extension of a hyperbolic tower over $PSL(2, \mathbb{Z})$ by $\mathbb{Z}_2$ with the cocycle $f : PSL(2, \mathbb{Z}) \times PSL(2, \mathbb{Z}) \to \mathbb{Z}_2$, where $f(x, x) = 1$ for all $x \in PSL(2, \mathbb{Z})$ of order 2 and $f(x, y) = 0$ otherwise.

Conjecture

There are commutative rings R and S so that $R \equiv S$, but $SL(2, R) \not\equiv SL(2, S)$

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Baumslag-Solitar groups

Recall that

$$BS(m, n) = \langle a, b \mid a^{-1} b^m a = b^n \rangle$$

Baumslag-Solitar groups

1. In $BS(1,n)$, one has $C(b) = BS(1,n)'$ is a normal, abelian n-divisible subgroup (and contains $BS(1,n)'$).
2. It follows that if $G \equiv BS(1,n)$, then there is $A \triangleleft G$, $A \equiv BS(1,n)'$ and $Q = {}^{G}/_{A} \equiv {}^{BS(1,n)}/_{BS(1,n)'}$.
3. G is f.g. iff Q is f.g. and A is f.g. as Q-module.
4. Using Szmielew's theorem and the structure theorem for divisible abelian groups, we get: $Q \simeq \mathbb{Z}$ and $A \simeq \mathbb{Z}[\frac{1}{n}]$.
5. It is now left to understand the action of Q on A. The corresponding groups are classified and one can exhibit a formula that distinguishes $BS(1,n)$ from any other such group.

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Nilpotent groups: elementary equivalence

Free nilpotent group $UT_3(\mathbb{Z})$ of class 2 and rank 2:

$$1 \to \mathbb{Z} = Z(UT_3(\mathbb{Z})) \to UT_3(\mathbb{Z}) \to \mathbb{Z}^2 \to 1$$

Theorem (Oger)

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Groups elementarily equivalent to $UT_3(R)$

Theorem (Belegradek)

$G \equiv UT_3(R)$ iff $G \simeq UT_3(S, f_1, f_2)$ and $S \equiv R$.

$UT_3(R) = \left\{ \begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \right\}$, with the multiplication:

$$(\alpha, \beta, \gamma)(\alpha', \beta', \gamma') = (\alpha + \alpha', \beta + \beta', \gamma + \gamma' + \alpha\beta').$$

Let $f_1, f_2 : R^+ \times R^+ \to R$ be two symmetric 2-cocycles. New operation on $UT_3(R)$:

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The ring R inside $UT_3(R)$

- As a set $Z(UT_3(R)) = R$.
- If $c_1, c_2 \in Z(UT_3(R))$, then we can "interpret" addition in R as: "$c_1 + c_2 = c_1 \cdot c_2$".
- Furthermore, we can "interpret" multiplication in R as: $z_1 \times z_2 = [x_1, x_2]$, where $[x_1, a] = z_1$, $[x_2, b] = z_2$.
- 0_R is 1 and 1_R is $[a, b]$.

Theorem (Malcev)

R is interpretable in $UT_3(R)$. It follows that the elementary theory of $UT_3(\mathbb{Z})$ (=free 2-nilpotent 2-generated) is undecidable.

$$\begin{array}{ccccc}
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Lie ring/algebra of a nilpotent group

Let G be t.f. nilpotent. Define $Lie(G)$ as follows:

- $Lie(G) = \oplus_{i=1}^{\infty} \Gamma_i / \Gamma_{i+1}$, as an abelian group;
- Let $x = \sum_{i=1}^{\infty} x_i \Gamma_{i+1}$ and $y = \sum_{i=1}^{\infty} y_i \Gamma_{i+1}$, where $x_i, y_i \in \Gamma_i$ be elements of $Lie(G)$. Define a product $\circ$ on $Lie(G)$ by

$$x \circ y = \sum_{k=2}^{\infty} \sum_{i+j=2}^{k} [x_i, y_j] \Gamma_{i+j+1}.$$

Since Γ_i are definable in G, understanding groups $\equiv$ to G is closely related to understanding rings $\equiv$ to $Lie(G)$.

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R-groups

Example

- For a free nilpotent group, $Lie(G)$ is a free nilpotent Lie ring.
- For a nilpotent pc group, $Lie(G)$ is a pc nilpotent Lie algebra.
- Consider R-algebra and "go back" to the group.

If we are to understand groups $\equiv$ to an R-group G, we should understand rings $\equiv$ to the Lie R-algebra $Lie(G)$.

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If we are to understand groups $\equiv$ to an R-group G, we should understand rings $\equiv$ to the Lie R-algebra $Lie(G)$.

Nilpotent groups and R-groups

Let R be an associative domain. The ring R gives rise to the category of R-groups. Enrich the language $\mathcal{L}$ with new unary operations $f_r(x)$, one for any $r \in R$. For $g \in G$ and $\alpha \in R$ denote $f_\alpha(g) = g^\alpha$.

Definition

An structure G of the language $\mathcal{L}(R)$ is an R-group if:

- G is a group;
- $g^0 = 1, g^{\alpha+\beta} = g^\alpha g^\beta, g^{\alpha\beta} = g^{\alpha\beta}$.

As the class of R-groups is a variety, so one has R-subgroups, R-homomorphisms, free R-groups, nilpotent R-groups etc.

Example

R-modules are R-groups.

Hall R-groups

P. Hall introduced a subclass or R-groups, so called Hall R-groups.

Definition

Let R be a *binomial* ring. A nilpotent group G of a class m is called a Hall R-group if for all $x, y, x_1, \ldots, x_n \in G$ and any $\lambda, \mu \in R$ one has:

- G is a nilpotent R-group of class m;
- $(y^{-1}xy)^{\lambda} = (y^{-1}xy)^{\lambda}$;
- $x_1^{\lambda} \cdots x_n^{\lambda} = (x_1 \cdots x_n)^{\lambda} \tau_2(x)^{C_2^{\lambda}} \cdots \tau_m(x)^{C_m^{\lambda}}$, where $\tau_i(x)$ is the i-th Petrescu word defined in the free group $F(x)$ by
$$x_1^i \cdots x_n^i = \tau_1(x)^{C_1^{\lambda}} \tau_2(x)^{C_2^{\lambda}} \cdots \tau_i(x)^{C_i^{\lambda}}.$$

Proposition (Hall)

Let R be a binomial ring. Then the unitriangular group $UT_n(R)$ and, therefore, all its subgroups are Hall R-groups.

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Idea of Miasnikov (late 1980's)

1. With an R-algebra A, associate a nice bilinear map $f_A : A/Ann(A) \times A/Ann(A) \to A^2$.
2. A ring $P(f_A) \supseteq R$, and the $P(f_A)$-modules A^2 and $A/Ann(A)$ are interpretable in A in the language of rings.

Algebras elementarily equivalet to well-structured algebras

Let A be well-structured and $Ann(A) = A^2$. Let B be a ring $\equiv$ to A.

$$\begin{array}{lll} 1 \to A^2 & \to A \to & A/A^2 \to 1 \\ 1 \to B^2 & \to B \to & B/B^2 \to 1 \\ 1 \to {A^2}^* & \to A^* \to & A/{A^2}^* \to 1 \end{array}$$

Well-structured algebras

Definition

A is called *well-structured* if

- $R = P(f_A)$ and $Ann(A) < A^2$;
- the modules A^2, $A/_{Ann(A)}$, $Ann(A)$, $A/_{A^2}$ and $A^2/_{Ann(A)}$ are free; in this case, the algebra A, as an R-module, admits the following decomposition
$$A \simeq A/_{A^2} \oplus A^2/_{Ann(A)} \oplus Ann(A);$$
- Let $U = \{u_1, \dots, u_k\}$, $V = \{v_1, \dots, v_l\}$ and $W = \{w_1, \dots, w_m\}$ be basis of the free modules $A/_{A^2}$, $A^2/_{Ann(A)}$ and $Ann(A)$, respectively. Then the structural constants of A in the basis $U \cup V \cup W$ are integer. In other words,
$$xy = \sum_{s=1}^{k} \alpha_{xys} u_s + \sum_{s=1}^{l} \beta_{xys} v_s + \sum_{s=1}^{m} \gamma_{xys} w_s,$$
where $x, y \in U \cup V \cup W$ and $\alpha_{xys}, \beta_{xys}, \gamma_{xys} \in \mathbb{Z}$.

Characterisation theorem for well-structured algebras

Theorem (Casals-Ruiz, Fernandez-Alcober, K., Remeslennikov)
Let A be a well structured R-algebra and B be a ring. Then

$$B \equiv A \text{ if and only if } B \simeq QA(S, \mathfrak{s})$$

for some ring S, $S \equiv R$ and some symmetric 2-cocycle $\mathfrak{s} \in S^2({}^{QA}/_{QA^2}, Ann(QA))$.

Abelian deformations

Definition

Let A be a well-structured $P(f_A)$-algebra. Define the ring $QA = QA(S, \mathfrak{s})$, called *abelian deformation of A*, as follows.

- Let S be a commutative unital ring of characteristic zero. Let K, L, and M be free S-modules of ranks $rank(A/A^2)$, $rank(A^2/Ann(A))$ and $rank(Ann(A))$, respectively.
- The ring QA, as an abelian group, is defined as an abelian extension of M by $K \oplus L$ via a symmetric 2-cocycle: let $x_1, y_1 \in K$, $x_2, y_2 \in L$, $x_3, y_3 \in M$ and $\mathfrak{s} \in S^2(K, M)$, set

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + \mathfrak{s}(x_1, y_1)).$$

- The multiplication in QA is defined on the elements of the basis of K, L and M using the structural constants of A and extended by linearity to the ring QA.

Lie algebras of some groups

Theorem

Let R *be an integral domain of characteristic zero. And let* G *be one of the following groups:*

- *free nilpotent* R*-group;*
- $UT(n, R)$*;*
- *directly indecomposable partially commutative nilpotent* R*-group.*

Then $Lie(G)$ *is well-structured.*

Characterisation theorem for groups

Theorem (Casals-Ruiz, Fernandez-Alcober, K., Remeslennikov)

Let G and R be as above and let H be a group, $H \equiv G$. Then H is $QG(S)$ over some ring S such that $S \equiv R$ as rings.