Generalised Triangle Groups

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A generalised triangle group is

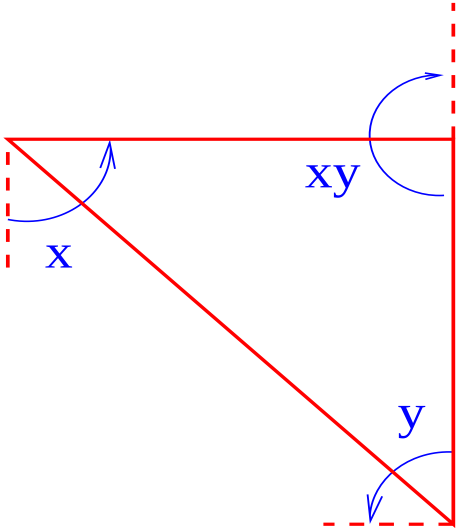
$$G := G(p,q,r,W) := \langle x,y \mid x^p = y^q = W(x,y)^r = 1\rangle,$$

$p,q,r \geq 2$, $W = x^{\alpha(1)}y^{\beta(1)}\cdots x^{\alpha(k)}y^{\beta(k)}$, $k \geq 1$, $0 < \alpha(i) < p$, $0 < \beta(i) < q$.

$G(p,q,r,xy)$ is the triangle group $T(p,q,r) < Isom^+(\Pi)$, generated by rotations about the vertices of a $(\pi/p, \pi/q, \pi/r)$ triangle in the elliptic/Euclidean/hyperbolic plane Π

(depending on sign of $\kappa = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1$).

Example: $T(4,4,2)$



Motivation

Culler/Gordon/Luecke/Shalen (1980's)

If Dehn surgery on $K \subset S^3$ with slope $\frac{a}{b}$ gives $L(p,c)\#L(q,d)$, then $\exists$ W with $G(p,q,b,W)=\{1\}$.

Theorem (Boyer)

Generalised triangle groups are non-trivial.

Proof: Topology of varieties of representations to

$$SO(3) \cong PSU(2) \subset PSL(2,\mathbb{C}) = Isom^+(\mathbb{H}^3)$$

gives existence of an essential rep $\rho : G \to SO(3)$.
($\rho(x), \rho(y), \rho(W)$ have orders p, q, r resp.)

Corollary

Only integer Dehn surgery can give $L(p,c)\#L(q,d)$.

Properties of Generalised Triangle Groups

Theorem (Boyer)

Generalised triangle groups are non-trivial.

Theorem (Baumslag/Morgan/Shalen)

$G = \langle x, y \mid x^p = y^q = W(x,y)^r = 1\rangle$ *is*

- *infinite if* $\kappa = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1 \leq 0$;
- *large if* $\kappa = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1 < 0$.
  $(\exists\ H, K)\ |G : H| < \infty,\ K \triangleleft H,\ H/K \cong F_2$.

Proof. $\exists$ essential rep $\rho : G \twoheadrightarrow \Phi$, $|\Phi| < \infty$.

$$\chi(Presentation(Ker(\rho))) = \kappa|\Phi|.$$

Conjecture (Rosenberger)

Tits alternative: G soluble-by-finite or $G \supset F_2$.

Trace polynomials

$$EssRep(\Gamma) = \{\text{Essential representations } \Gamma \to PSL(2, C)\}$$

is an algebraic variety over $\mathbb{C}$.

$$EssChar(G) := EssRep(G)/PSL(2, \mathbb{C})$$

the essential character variety

$$F = \langle x, y \mid x^p = y^q = 1 \rangle \twoheadrightarrow \langle x, y \mid x^p = y^q = W^r = 1 \rangle = G.$$

$EssChar(F) \supset$ curve $C :=$
$\{(\{\pm X\}, \{\pm Y\})\ ;\ Tr(X) = 2\cos(\pi/p),\ Tr(Y) = 2\cos(\pi/q)\}$.

$EssChar(G) \cap C$ finite – solutions of $\tau_W(\lambda) = 2\cos(m\pi/r)$ with $(m, r) = 1$, $\lambda := Tr(XY)$, $\tau_W(\lambda) \in \mathbb{C}[\lambda]$, $deg(\tau_W) = k$.

$\tau_W(\lambda)$ is the trace polynomial

Theorem (JH/Metaftsis/Thomas)

G finite generalised triangle group $\Rightarrow$ G one of short list (modulo two unknown cases).

Proof: $\rho(G)$ finite ($\forall\ \rho : G \to PSL(2,\mathbb{C})$) means the trace equation

$$\tau_W(\lambda) = 2\cos(m\pi/r)$$

has only finitely many solutions.
Multiple solutions mean G infinite.
($Ker(PSL(2,\mathbb{C}[X]/\langle\langle X^2\rangle\rangle) \twoheadrightarrow PSL(2,\mathbb{C}))$ torsion-free abelian.)
Hence bound on $length(W) = deg(\tau_W)$.
Computer search plus ad-hoc arguments.

Theorem (Lévai/Rosenberger/Souvignier)

Unknown case 1 infinite. Unknown case 2:

$$\langle x, y \mid x^2 = y^3 = (xyxyxyxy^2xy^2xyxy^2xy^2)^2 = 1\rangle$$

is finite of order $2^{20} \cdot 3^4 \cdot 5 = 424,673,280.$

Known Results on Rosenberger Conjecture

$$G = \langle x, y \mid x^p = y^q = W(x,y)^r = 1 \rangle$$

$$W = x^{\alpha(1)} y^{\beta(1)} \cdots x^{\alpha(k)} y^{\beta(k)}$$

Conjecture (Rosenberger)

$G \supset F_2$ or G soluble-by-finite.

The Rosenberger Conjecture holds for G if:

- $\kappa = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1 < 0$ (Baumslag/Morgan/Shalen)
- $r > 2$ (Fine/Levin/Rosenberger) idea: ≥ 2 allowable values for $tr(W) \Rightarrow$ allowable values of λ need to distribute over them.
- $k \leq 4$ (Fine/Levin/Rosenberger) – case-by-case analysis.
- $k \leq 6$ (Williams, modulo few open cases) – ditto.

Known results continued

The Rosenberger Conjecture holds for G if:

- $\kappa = 0$ (JH) – multiple roots $\Rightarrow$ $(\exists\ H)\ |G : H| < \infty$, H^{ab} high rank, $def(H) = 1$. Apply BNS invariant and commutative algebra to $[H, H]^{ab}$.
- $p = 2$, $q > 5$ (Benyash-Krivets, B-K/Barkovitch, JH/Williams) – mixture of tricks.
- $(p, q, r) = (3, 4, 2)$ (Benyash-Krivets for k odd; JH/Williams for general case).
- $(p, q, r) = (2, 4, 2)$ and k odd. (Benyash-Krivets).

Remaining open cases

- $(p, q, r) = (2, 3, 2)$. $\qquad (p, q, r) = (2, 5, 2)$.
- $(p, q, r) = (2, 4, 2)$, k even.
- $(p, q, r) = (3, 3, 2)$. $\qquad (p, q, r) = (3, 5, 2)$.

The $(3,3,2)$ case

$G = \langle x, y \mid x^3 = y^3 = W(x,y)^2 = 1 \rangle$.

$\rho : x \mapsto \pm X$, $y \mapsto \pm Y$ with $tr(X) = 1 = tr(Y)$.

$\rho(G) \supset F_2$ unless

(i) $\rho(G) = A_4$ (roots $0, 1$), or

(ii) $\rho(G) = A_5$ (roots $(1 \pm \sqrt{5})/2$).

So

$$\tau_W(\lambda) = \lambda^a (\lambda - 1)^b (\lambda^2 - \lambda - 1)^c.$$

$\lambda \in [-1, 2] \Rightarrow$ unitary rep $\Rightarrow |\tau(\lambda)| \leq 2$.

- $\tau_W(2) = 2^a \Rightarrow a \leq 1$; $\tau_W(-1) = \pm 2^b \Rightarrow b \leq 1$;
- $\tau_W(\frac{1}{2}) = \pm \left(\frac{1}{2}\right)^{a+b} \left(\frac{5}{4}\right)^c \Rightarrow c \leq 3(a + b + 1) \leq 9$

The $(3,3,2)$ case

$G = \langle x, y \mid x^3 = y^3 = W(x,y)^2 = 1\rangle$.

$W = x^{\alpha(1)} y^{\beta(1)} \dots x^{\alpha(k)} y^{\beta(k)}$

$$\tau_W(\lambda) = \lambda^a (\lambda - 1)^b (\lambda^2 - \lambda - 1)^c$$

$a \leq 1$, $b \leq 1$, $c \leq 3(a+b+1) \leq 9$. So: $k = a + b + 2c \leq 20$.

Computer search: up to equivalence, 19 words have correct τ_W.

Of these, 12 have small cancellation property $\Rightarrow G \supset F_2$.

The other 7 have $k \leq 6 \Rightarrow$ result already known.

Theorem

The Rosenberger Conjecture holds for generalised triangle groups of type $(3,3,2)$.

The $(3,5,2)$ case

$G = \langle x, y \mid x^3 = y^5 = W(x,y)^2 = 1 \rangle$.

$\rho : x \mapsto \pm X$, $y \mapsto \pm Y$ with $tr(X) = 1$, $tr(Y) = (1+\sqrt{5})/2$.

$\rho(G) \supset F_2$ unless $\rho(G) = A_5$ (roots $0, 1, (\pm 1 + \sqrt{5})/2)$.

$Aut(\mathbb{Z}_5)$ permutes these 4 roots, so equivalent.

If no multiple roots, then $k \leq 4$ so OK by Fine/Rosenberger.

So assume 0 a multiple root, ie $\lambda^2 | \tau_W(\lambda)$.

Theorem

If $\lambda^2 | \tau_W(\lambda)$, then $G \supset F_2$.

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Idea of proof.

$\Lambda = \mathbb{C}[\lambda]/\langle\langle \lambda^2 \rangle\rangle$. $\phi : G \to PSL(2, \Lambda)$. $\rho : G \to A_5$.

$$1 \to A \to Im(\phi) \to A_5 \to 1,$$

where $A \cong \mathbb{Z}^6$ (spanned by $V(Icosahedron)$: $\mathbb{Q}A = \mathbb{Q}[\sqrt{5}]^3 \subset \mathbb{R}^3$).

$C := \rho^{-1}(\mathbb{Z}_2) = \langle x_1, \ldots, x_{15} | r_1, \ldots, r_{14}, s_1^2, s_2^2 \rangle$. $C \twoheadrightarrow \mathbb{Z}^2 \oplus \mathbb{Z}_2^4$.

$\Rightarrow H_1([C, C], \mathbb{Z}_2)$ infinite.

$\Rightarrow$ BNS-invariant $\Delta := \Delta(C, \mathbb{Z}_2) \neq S^1$.

$K = \rho^{-1}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ induces -1 on $H^1(C)$, so $-\Delta = \Delta$.

$\Delta \cup -\Delta = \Delta \neq S^1$.

$G \supset C \supset F_2$.

Remaining cases $(2, m, 2)$, $m = 3, 4, 5$

$m = 5$: Theory restricts τ_W, but no length bound.
Computer search succeeds up to length ~ 15. (Short words by Fine-Rosenberger, Williams or ad-hoc. Longer words small cancellation.)

$m = 4$: Odd length by Benyash-Krivets. For even length, theory strongly restricts τ_W, but still no length bound.
Computer search succeeds up to length ~ 50. (No words of length > 16 or in $[9, 15]$. Length 16 small-cancellation. Length 8 ad-hoc. Shorter words by Fine-Rosenberger and Williams.)

$m = 3$: Even length similar to $(3, 3, 2)$ case. OK except for small number of unknown cases (6, reduced by Button to 2).

Odd length - who knows? Computer searches find non-small-cancellation examples at all lengths.