New examples of totally disconnected locally compact groups

Murray Elder, George Willis

GACGTA 2012, Düsseldorf
A topological space $X$ is

**Hausdorff** if for each $x \neq y$ there are disjoint open sets, one containing $x$ and the other $y$

**locally compact** if for each $x$ and each open set $U$ containing $x$ there is a compact open set $V \subseteq U$ containing $x$

**connected** if it is not the disjoint union of two open sets

**totally disconnected** if for each $x \neq y$, $X$ is the disjoint union of open sets, one containing $x$ and the other $y$
G is a topological group if

G is a group and a topological space such that \((x, y) \mapsto xy^{-1}\)
is a continuous map (from \(G \times G\) to \(G\))

**Lem:** Let \(G\) be a locally compact group and \(G_0\) the connected component containing the identity. Then \(G_0\) is an open normal subgroup and \(G/G_0\) is **totally disconnected**.

In other words, to understand locally compact groups you just need to understand the **connected** and **totally disconnected** cases.
Understanding totally disconnected locally compact groups

Any (abstract) group $G$ with the *discrete topology* is totally disconnected (and locally compact).

**Question:** What other (tdlc) topologies can you *put on* $G$?
If $G$ is finitely generated, let $\mathcal{T}$ be the topology on $\text{Aut}(\text{Cay}(G))$ with basis

$$N(x, F) = \{ y \in \text{Aut}(\text{Cay}(G)) \mid x.f = y.f \quad \forall f \in F \}$$

where $F$ is a finite set of vertices of $\text{Cay}(G)$. 

**Aut(Cay(G))**
In some cases this topology is nondiscrete (e.g. nonabelian free groups)

However, the subspace topology on $G$, or even the closure of $G$ in $\text{Aut}(\text{Cay}(G))$, is discrete

(for each $\alpha \neq e \in \text{Aut}(\text{Cay}(G))$ there is some $v$ so that $\alpha \notin N(e, \{v\})$ so the intersection of $N(e, \{v\})$ over all $v$ is just $\{e\}$).

Instead, here is a trick with \textit{commensurated subgroups} that sometimes makes a nondiscrete tdlc group in which $G$ embeds densely.
Commensurability and commensurated subgroups

**Defn:** Let $G$ be a group, and $H$, $K$ subgroups. $H$ and $K$ are *commensurable* if $H \cap K$ is finite index in both $H$ and $K$.

**Lem:** Commensurability is an equivalence relation
Commensurability and commensurated subgroups

**Defn:** H is *commensurated by G* if \( gHg^{-1} \) is commensurable with H for all \( g \in G \).

**Lem:** If G is finitely generated, it suffices to check \( gHg^{-1} \) is commensurable with H just for the generators.
Example 1: Baumslag-Solitar groups

\[ \text{BS}(m, n) = \langle a, t \mid ta^m t^{-1} = a^n \rangle \]

the cyclic subgroup \( \langle a \rangle \) is commensurated
Example 2: tdlc groups

Every tdlc group $G$ has a **compact open subgroup** (van Dantzig).

An **automorphism** of a topological group $\alpha : G \to G$ is a group isomorphism that is also a homeomorphism ($\alpha$ and $\alpha^{-1}$ are continuous).

If $V$ is a compact open subgroup of $G$, then $\alpha(V)$ is also compact and open, and $\alpha(V) \cap V$ is open, so its cosets in $V$ are an open cover, its index is finite

(i.e. $\alpha(V) \cap V$ is commensurated by $V$)
Scale

**Defn:** \[ s(\alpha) = \min_{V \text{ compact open}} \{ [V : \alpha(V) \cap V] \} \]

is the **scale** of the automorphism \( \alpha \).

A subgroup that realises this minimum for a group element is called **minimizing**.
Scale

In the case that $\alpha$ is the inner automorphism $x \mapsto gxg^{-1}$, the scale is a function $s : G \rightarrow \mathbb{Z}^+$ which satisfies some useful properties:

- $s$ is continuous

- $s(x^n) = s(x)^n$

- $s(gxg^{-1}) = s(x)$

- the number of prime factors of the scales of a (compactly generated) tdlc group is finite
Recipe

Let $G$ be an abstract group with a \textbf{commensurated} subgroup $H$, and suppose $H$ has \textbf{no subgroup that is normal in $G$}.

Then $G$ acts (faithfully) on $G/H$ by permuting cosets, so $G \leq \text{Sym}(G/H)$.

if $x \not\in H$ then $xH \neq H$

if $x \in H$ and $xgH = gH$ for all $g \in G$ then $x \in \bigcap_{g \in G} gHg^{-1}$ which is normal so must be $\{e\}$
Recipe

Let $\mathcal{T}$ be the topology on $\text{Sym}(G/H)$ with basis

$$N(x, F) = \{ y \in \text{Sym}(G/H) \mid y(gH) = x(gH) \ \forall \ (gH) \in F \}$$

for each $x \in \text{Sym}(G/H)$ and each finite subset $F$ of $G/H$. 
Recipe

Take the *closure* of \( G \) in \( \text{Sym}(G/H) \)

which is the intersection of all closed subsets of \( \text{Sym}(G/H) \) that contain \( G \).

We denote the closed subgroup by \( G//H \).

\( (G \text{ is dense in } G//H) \)
Locally compact

Since $H$ is commensurated, the orbits of cosets under $H$ are finite,

$$\text{Stab}_H(gH) = N(e,gH) = H \cap gHg^{-1}$$
so the orbit $HgH$ is $H/\text{Stab}_H$ which is finite when $H$ is commensurated

so $H$ acts on $G/H$ by permuting cosets in finite blocks,

so $H \leq \prod \text{Sym}(HgH)$ which is compact by Tychonov’s theorem.

The closure of $H$ is also a subgroup of this compact group, so is compact. It is open since it is equal to $N_{G//H}(e,H)$.

It follows that $G//H$ is locally compact since each point lies in a translate of $\overline{H}$. 
Totally disconnected

Since the action of $G$ on $G/H$ is faithful,

for each $x \neq y \in G$ there is a coset $gH$ with $xgH \neq ygH$.

$N_{G//H}(x, gH)$ is an open set containing $x$, and its complement

$$\bigcup_{z \in N_{G//H}(x, gH)} N_{G//H}(z, gH)$$

is open and contains $y$.

So $G//H$ is a tdlc group.
New examples

So given a group $G$, a subgroup $H$

- having no subgroups normal in $G$
- and commensurated by $G$

the recipe produces a ready-made tdlc group

Since $\langle a \rangle$ is commensurated by $\text{BS}(m,n)$, and when $|m| \neq |n|$ has no subgroup that is normal in $\text{BS}(m,n)$,

we get a (nondiscrete) topology on $\text{BS}(m,n)$.

(i.e. we have a tdlc group in which $\text{BS}(m,n)$ is dense)
Scales of $\text{BS}(m,n) / \langle a \rangle$

**Thm (E, Willis):** The set of scales for $\text{BS}(m,n) / \langle a \rangle$ for all $m, n \neq 0$ is

$$\left\{ \left( \frac{\text{lcm}(m,n)}{m} \right)^k, \left( \frac{\text{lcm}(m,n)}{n} \right)^k : k \in \mathbb{N} \right\}$$

Since $\text{BS}(m,n)$ is dense in its closure, and $s: \text{BS}(m,n) / \langle a \rangle \to \mathbb{Z}$ is continuous, if we show that scales of elements in $\text{BS}(m,n)$ take only these values, the result for $\text{BS}(m,n) / \langle a \rangle$ follows.

See our paper (on arxiv very soon) for more details
Thanks and References

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