AN INTRODUCTION TO MODEL THEORY

by

Pablo Cubides Kovacsics

Contents

Introduction

The present notes grew out of various courses and introductory seminars to model theory I have taught. The content of the present version was given in the SEAMS school Arithmetic, Geometry and Model Theory in the spring of 2019. I would like to gladly thank all the organizers and participants for a very nice school. Special thanks to Quy Thuong Lê who taught the associated tutorial sessions during the course. The tutorials can be found in this here: `http://www.math.uni-duesseldorf.de/~cubides/tutorial_seams.pdf`.

The aim of this course is to introduce students to the basics of model theory with a particular emphasis on algebraic applications. The course is divided into 3 main parts. The first part (Section 1) is a crash course on first-order logic. Basic notions such as languages, structures and formulas will be introduced together with the main proof methods. In the second part (Sections 2-5) we dive directly into basic model theory concepts. Among others, we will define the concepts of theory, logical consequence, elementary classes, completeness, categoricity and quantifier elimination, and prove classical theorems such as the compactness and Löwenheim-Skolem theorems. Some applications to groups and fields will be given. Finally, Section 6 contains a very brief overview of o-minimality. The main goal of this Section is to provide students with a feeling of possible research directions in this area.

Most of the material of these notes is based on classical references such as [1] and [2].

1. Crash course on first-order logic

In the coming sections we will define the three components that characterize first-order logic:

$$\mathcal{M} \models \varphi.$$

The symbol "$\models$" stands for a relation between "structures" on the left-hand side and "formulas" on the right-hand side stating that "the formula φ holds in the structure $\mathcal{M}$", which is already totally intuitive for every mathematician. However, in formalizing such a relation we will gain some inside about mathematical structures and statements. In order to formalize this relation, we need the notion of first-order language, which is the start of this course.

1.1. Languages. — A language $\mathcal{L}$ is the union of three disjoint sets of formal symbols:

(L1) $\mathcal{L}^{\mathfrak{r}}$ the set of *relation symbols* of $\mathcal{L}$,
(L2) $\mathcal{L}^{\mathfrak{f}}$ the set of *function symbols* of $\mathcal{L}$,
(L3) $\mathcal{L}^{\mathfrak{c}}$ the set of *constant symbols* of $\mathcal{L}$,

together with a function $a_{\mathcal{L}}\colon \mathcal{L} \to \mathbb{N}$, called the arity function, which gives the arity of each symbol. We impose that $a_{\mathcal{L}}(R) > 0$ for every $R \in \mathcal{L}^{\mathfrak{r}}$, that $a_{\mathcal{L}}(f) > 0$ for every $f \in \mathcal{L}^{\mathfrak{f}}$ and that $a_{\mathcal{L}}(c) = 0$ for every $c \in \mathcal{L}^{\mathfrak{c}}$. We often omit the index $\mathcal{L}$ in $a_{\mathcal{L}}$ and simply write a when no confusion arises.

1.1.1. Remark. — A language is also called "signature" or even "vocabulary" by some authors.

1.1.2. Examples. —

(1) The empty language $\mathcal{L}_{\emptyset} = \emptyset$ is a language!
(2) The language of ordered sets $\mathcal{L}_{\leqslant} \coloneqq \{\boldsymbol{\leqslant}\}$, where $\boldsymbol{\leqslant}$ is a binary relation symbol.
(3) The language of groups $\mathcal{L}_{\mathrm{g}} = \{\boldsymbol{\cdot}, \boldsymbol{^{-1}}, \boldsymbol{e}\}$, where $\boldsymbol{\cdot}$ is a binary function symbol, $\boldsymbol{^{-1}}$ is a unary function symbol and $\boldsymbol{e}$ is a constant symbol.
(4) The language of rings $\mathcal{L}_{\mathrm{ring}} \coloneqq \{\boldsymbol{+}, \boldsymbol{\cdot}, \boldsymbol{-}, \mathbf{0}, \mathbf{1}\}$, where $\boldsymbol{+}$ and $\boldsymbol{\cdot}$ are binary function symbols, $\boldsymbol{-}$ is a unary function symbol and both $\mathbf{0}$ and $\mathbf{1}$ are constant symbols.

(5) Combining (2) and (3) we have the language of ordered groups

$$\mathcal{L}_{\mathrm{og}} \coloneqq \{\leqslant, \boldsymbol{\cdot}, \boldsymbol{^{-1}}, \boldsymbol{e}\}.$$

(6) Combining (2) and (4) we have the language of ordered rings

$$\mathcal{L}_{\mathrm{or}} \coloneqq \{\leqslant, \boldsymbol{\cdot}, \boldsymbol{+}, \boldsymbol{-}, \boldsymbol{0}, \boldsymbol{1}\}.$$

(7) Let R be a ring. Le language of R-modules is

$$\mathcal{L}_{R\mathrm{mod}} \coloneqq \{\boldsymbol{+}, \boldsymbol{-}, \boldsymbol{0}\} \cup \{\lambda_r \mid r \in R\},$$

where λ_r is a unary function symbol for every $r \in R$.

1.1.3. Definition. — Let $\mathcal{L}$ and $\mathcal{L}'$ be two languages. We say $\mathcal{L}'$ is an *extension* of $\mathcal{L}$ (resp. $\mathcal{L}$ is a sub-language of $\mathcal{L}'$) if $\mathcal{L}^{\mathfrak{r}} \subseteq \mathcal{L}'^{\mathfrak{r}}$, $\mathcal{L}^{\mathfrak{f}} \subseteq \mathcal{L}'^{\mathfrak{f}}$, $\mathcal{L}^{\mathfrak{c}} \subseteq \mathcal{L}'^{\mathfrak{c}}$ and the function $a_{\mathcal{L}'}$ extends $a_{\mathcal{L}}$. We will denote this by $\mathcal{L} \subseteq \mathcal{L}'$.

1.2. $\mathcal{L}$-structures. — Let $\mathcal{L}$ be a language. An $\mathcal{L}$-structure is a tuple

$$\mathcal{M} = (M, (R^{\mathcal{M}})_{R\in\mathcal{L}^{\mathfrak{r}}}, (f^{\mathcal{M}})_{f\in\mathcal{L}^{\mathfrak{f}}}, (c^{\mathcal{M}})_{f\in\mathcal{L}^{\mathfrak{c}}})$$

where M is a non-empty set, for each relation symbol $R \in \mathcal{L}^{\mathfrak{r}}$ with $a_{\mathcal{L}}(R) = m$, the associated set $R^{\mathcal{M}}$ is a subset of M^m, for every function symbol $f \in \mathcal{L}^{\mathfrak{f}}$ of arity $a_{\mathcal{L}}(f) = n$, $f^{\mathcal{M}}$ is a function $f^{\mathcal{M}} \colon M^n \to M$ and for every constant symbol $c \in \mathcal{L}^{\mathfrak{c}}$, $c^{\mathcal{M}} \in M$ is a distinguished element of M. The relation $R^{\mathcal{M}}$ (resp. the function $f^{\mathcal{M}}$ and the element $c^{\mathcal{M}}$) is called the *interpretation of R in $\mathcal{M}$* (resp. the *interpretation of f in $\mathcal{M}$ and c in $\mathcal{M}$*). An $\mathcal{L}$-structure $\mathcal{M}$ will sometimes be written as the pair $(\mathcal{M}, \mathcal{L})$.

1.2.1. Examples. —

– The additive group of the real numbers can be treated as $\mathcal{L}_{\mathrm{g}}$-structure (as defined in (1) of Examples 1.1.2) $\mathcal{M}$ with $M = \mathbb{R}$ and in which one interprets the symbols of $\mathcal{L}_{\mathrm{g}}$ as

$$\begin{cases} \boldsymbol{e}^{\mathcal{M}} = 0 \in \mathbb{R} \\ \boldsymbol{^{-1}}^{\mathcal{M}} : \mathbb{R} \to \mathbb{R}, a \mapsto -a \\ \boldsymbol{\cdot}^{\mathcal{M}} : \mathbb{R}^2 \to \mathbb{R}, (a,b) \mapsto a+b. \end{cases}$$

We will write this $\mathcal{L}_{\mathrm{g}}$-structure as $\mathcal{M} = (\mathbb{R}, +, -, 0)$.

– The multiplicative group of the real numbers is also an $\mathcal{L}_{\mathrm{g}}$-structure $\mathcal{N}$ with $N = \mathbb{R}^\times$ and in which we interpret the symbols in $\mathcal{L}_{\mathrm{g}}$ as

$$\begin{cases} \boldsymbol{e}^{\mathcal{N}} = 1 \in \mathbb{R}^\times \\ \boldsymbol{^{-1}}^{\mathcal{N}} : \mathbb{R}^\times \to \mathbb{R}^\times, a \mapsto a^{-1} \\ \boldsymbol{\cdot}^{\mathcal{N}} : (\mathbb{R}^\times)^2 \to \mathbb{R}^\times, (a,b) \mapsto ab. \end{cases}$$

– Any group can be treated as an $\mathcal{L}_{\mathrm{g}}$-structure!

– Even if it might be at first confusing, there are also $\mathcal{L}_{\mathrm{g}}$-structures which are not groups! In fact, any set X with a distinguished element, a distinguished unary function and a distinguished binary operation is an $\mathcal{L}_{\mathrm{g}}$-structure. For example, we could have defined the $\mathcal{L}_{\mathrm{g}}$-structure $\mathcal{Z}$ with $Z = \mathbb{Z}$ and by interpreting

$$\begin{cases} \boldsymbol{e}^{\mathcal{Z}} = 7 \in \mathbb{Z} \\ \boldsymbol{{}^{-1}}^{\mathcal{Z}} : \mathbb{Z} \to \mathbb{Z}, a \mapsto a+1 \\ \boldsymbol{\cdot}^{\mathcal{Z}} : \mathbb{Z}^2 \to \mathbb{Z}, (a,b) \mapsto a^3 - b^2. \end{cases}$$

– The structure $\mathcal{M}' = (\mathbb{R}; \leqslant, +, \cdot, -, 0, 1)$, is an $\mathcal{L}_{\mathrm{or}}$-structure (with the obvious interpretation). Similarly, every ordered ring can be seen as an $\mathcal{L}_{or}$-structure.

1.2.2. Remark. — Of course we called the language $\mathcal{L}_{\mathrm{g}}$ the "language of groups" because, when studying classes of $\mathcal{L}_{\mathrm{g}}$-structures we will mostly be interested in studying the class of $\mathcal{L}_{\mathrm{g}}$-structures which *are* groups. Indeed, we will distinguish those $\mathcal{L}_{\mathrm{g}}$-structures which are groups from the rest of $\mathcal{L}_{\mathrm{g}}$-structures by declaring that they *satisfy* certain *axioms*. But in order to do this we need to define in the coming sections what "satisfy" and "axioms" mean.

⚠ Most of the time we will use the same letter for both a formal symbol and its interpretation. For example, (and specially on the blackboard!) we will sometimes write the language of groups $\mathcal{L}_{\mathrm{g}} = \{\boldsymbol{\cdot}, \boldsymbol{{}^{-1}}, \boldsymbol{e}\}$ without bold letters as $\mathcal{L}_{\mathrm{g}} = \{\cdot, {}^{-1}, e\}$, or even as $\mathcal{L}_{\mathrm{g}} = \{+, -, 0\}$ and later work in $\mathcal{L}_{\mathrm{g}}$-structures as $(\mathbb{Q}, +, -, 0)$, were technically the symbols '+' in $\mathcal{L}_{\mathrm{g}} = \{+, -, 0\}$ and '+' in $(\mathbb{Q}, +, -, 0)$ are not used in the same way. One has to keep track of this cumbersome notational struggle, which, will become more and more natural with time.

1.3. Sub-structures, embeddings and isomorphisms. — Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{L}$-structures for a given language $\mathcal{L}$. Given $A \subseteq M$, a map $h\colon A \to N$ and $a = (a_1, \ldots, a_k) \in A^k$, we will denote by $h(a)$ the tuple $(h(a_1, \ldots, h(a_k)) \in N^k$.

1.3.1. *Embeddings and isomorphisms.* — An $\mathcal{L}$-*embedding* $h\colon \mathcal{M} \to \mathcal{N}$ is an injective map $h\colon M \to N$ satisfying:

(M1) for every relation symbol $R \in \mathcal{L}^{\mathfrak{r}}$ of arity $a_{\mathcal{L}}(R) = m$ and every $a \in M^m$

$$a \in R^{\mathcal{M}} \Leftrightarrow h(a) \in R^{\mathcal{N}},$$

(M2) for every function symbol $f \in \mathcal{L}^{\mathfrak{f}}$ of arity $a_{\mathcal{L}}(f) = n$ and every $a \in M^n$

$$h(f^{\mathcal{M}}(a) = f^{\mathcal{N}}(h(a)).$$

(M3) for every constant symbol $c \in \mathcal{L}^{\mathfrak{c}}$

$$h(c^{\mathcal{M}}) = c^{\mathcal{N}}.$$

An $\mathcal{L}$-*isomorphism* is a bijective $\mathcal{L}$-embedding. When $M \subseteq N$ and the inclusion is an $\mathcal{L}$-embedding, we say that $\mathcal{M}$ is a *substructure* of $\mathcal{N}$ and we write it like $\mathcal{M} \subseteq \mathcal{N}$. Note that if h is an $\mathcal{L}$-embedding, then $h\colon \mathcal{M} \to h(\mathcal{M})$ is an $\mathcal{L}$-isomorphism. As usual, an $\mathcal{L}$-automorphism of $\mathcal{M}$ is an $\mathcal{L}$-isomorphism $h\colon \mathcal{M} \to \mathcal{M}$ and the set of of $\mathcal{L}$-automorphisms of $\mathcal{M}$, denoted $\mathrm{Aut}_{\mathcal{L}}(\mathcal{M})$, forms a group under composition. We write $\mathcal{M} \cong \mathcal{N}$ if there is an $\mathcal{L}$-isomorphism between $\mathcal{M}$ and $\mathcal{N}$.

The reader can check that, if $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}_{\mathrm{g}}$-structures which are in addition groups (resp. $\mathcal{L}_{\mathrm{ring}}$-structures which are rings), then an $\mathcal{L}_{\mathrm{g}}$-embedding (resp. an $\mathcal{L}_{\mathrm{ring}}$-embedding) is an embedding of groups (resp. rings) in the usual sense. When the language $\mathcal{L}$ is clear from the context we will often omit $\mathcal{L}$ and simply say embedding and isomorphism, if no confusion arises. Similarly, we write $\mathrm{Aut}(\mathcal{M})$ instead of $\mathrm{Aut}_{\mathcal{L}}(\mathcal{M})$ if no confusion arises.

1.4. Syntax. — In this section we introduce the *syntax* (variables, terms, formulas, sentences) of first-order logic. For the remaining of Section, we let $\mathcal{M}$ and $\mathcal{N}$ denote $\mathcal{L}$-structures.

1.4.1. *Variables and terms.* — We let $\mathrm{Var}_{\mathcal{L}}$ be an infinite set of formal symbols called the $\mathcal{L}$-*variables* which we suppose different from every other symbol in $\mathcal{L}$. A *multivariable* of $\mathcal{L}$ is simply a tuple $x = (x_1, \dots, x_k)$ of distinct variables in $\mathrm{Var}_{\mathcal{L}}$. The length of x is the length of the tuple, and will be usually denoted by $|x|$. Two multivariables $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_m)$ are said to be disjoint if $x_i \neq y_j$ for every $i \in \{1, \dots, k\}$ and every $j \in \{1, \dots, m\}$. In what follows x and y will denote multivariables of $\mathcal{L}$, unless otherwise stated.

We define the $\mathcal{L}$-*terms* as the smallest collection of words over the alphabet $\mathcal{L} \cup \mathrm{Var}_{\mathcal{L}}$ such that

(T1) if $x \in \mathrm{Var}_{\mathcal{L}}$, then x is an $\mathcal{L}$-term;
(T2) if c is a constant symbol in $\mathcal{L}$, then c is an $\mathcal{L}$-term;
(T3) if $t_1, \dots, t_n$ are $\mathcal{L}$-terms and $f \in \mathcal{L}^{\mathfrak{f}}$ is such that $a(f) = n$, then $ft_1 \cdots t_n$ is an $\mathcal{L}$-term.

We will omit $\mathcal{L}$ in $\mathcal{L}$-term when the language is clear from the context. The following lemma will guarantee an induction procedure for $\mathcal{L}$-terms.

1.4.2. Example. — Consider the language of groups $\mathcal{L}_g$. Then the following are $\mathcal{L}_g$-terms:

$$\cdot \cdot exy \qquad\qquad \cdot \cdot {}^{-1}yyx.$$

Be aware of the prefix notation! We will later adopt a standard infix notation for binary operations together with the usual conventions for functions and unary operations such as $^{-1}$. The benefit of the prefix (or Polish) notation is purely syntactical. Thus, we will later write $(e \cdot x) \cdot y$ for the first term above, and $(y^{-1} \cdot y) \cdot x$ for the second one (see Convention 1.4.4). Note that as $\mathcal{L}_g$-terms, $(y^{-1} \cdot y) \cdot x$ and x are different terms! Although in any group both terms will define the same function, as $\mathcal{L}_g$-terms they are different. We will soon define the function associated to an $\mathcal{L}$-term in an $\mathcal{L}$-structure.

***1.4.3. Lemma* (Unique readability of terms).** — *Every $\mathcal{L}$-term is either an $\mathcal{L}$-variable, a constant symbol from $\mathcal{L}$, or equal to a word $ft_1 \cdots t_n$ for a unique tuple $(f, t_1, \ldots, t_n)$ with $f \in \mathcal{L}^{\mathfrak{f}}$ of arity $n > 0$ and each t_i an $\mathcal{L}$-term for $i \in \{1, \ldots, n\}$.*

The previous lemma allows us to do definitions and proofs by induction on the complexity of terms. Let us give an example. Given an $\mathcal{L}$-term t, we define the set $V(t)$ of "variables occurring in t" by induction on the complexity of t as follows:

1. if t is a variable x from $\mathcal{L}$, then $V(t) = \{x\}$.
2. if t is a constant symbol c from $\mathcal{L}$, then $V(t) = \emptyset$.
3. if $t = ft_1 \cdots t_n$ with $f \in \mathcal{L}^{\mathfrak{f}}$ of arity $n > 0$ and $t_1, \ldots, t_n$ terms, then $V(t) = \bigcup_{i=1}^{n} V(t_i)$.

1.4.4. Convention. — We will often write $f(t_1, \ldots, t_n)$ instead of $ft_1 \cdots t_n$ to maintain the intuition of notation. Moreover, in the language of rings $\mathcal{L}_{\text{ring}}$, we will write $(x + (-y)) \cdot z$ instead of $\cdot + x - yz$; and even, if there is no confusion, we will write $(x - y)z$ knowing that $-$ is not a binary function symbol in $\mathcal{L}_{\text{ring}}$ but of arity 1.

Let $x = (x_1, \ldots, x_n)$ be a multivariable. An $(\mathcal{L}, x)$-term is an $\mathcal{L}$-term t such that every variable in $V(t)$ is a variable of the multivariable x. We write $t(x)$ to state that t is an $(\mathcal{L}, x)$-term. Note that it is not required that every variable in x occurs in $V(t)$.

1.4.5. Definition. — Let $x = (x_1, \ldots, x_n)$ be a multivariable and $t(x)$ be an $\mathcal{L}$-term (so an $(\mathcal{L}, x)$-term). We define a function $t^{\mathcal{M}} \colon M^n \to M$ associated to $t(x)$ as follows: for $a = (a_1, \ldots, a_n) \in M^n$

1. if $t = x_i$, then $t^{\mathcal{M}}(a) \coloneqq a_i$;
2. if $t = c$ is a constant symbol, then $t^{\mathcal{M}}(a) \coloneqq c^{\mathcal{M}}$;
3. if $t = ft_1 \cdots t_n$ with $f \in \mathcal{L}^{\mathfrak{f}}$ of arity $n > 0$ and $t_i = t_i(x)$ an $(\mathcal{L}, x)$-term for $i = 1, \ldots, n$, then
$$t^{\mathcal{M}}(a) \coloneqq f^{\mathcal{M}}(t_1^{\mathcal{M}}(a), \ldots, t_n^{\mathcal{M}}(a)) \in M.$$

1.4.6. Examples. — Let R be a commutative ring treated as an $\mathcal{L}_{\text{ring}}$-structure. Let $t(x, y, z)$ be the $\mathcal{L}_{\text{ring}}$-term $(x - y)z$. The function $t^R \colon R^3 \to R$ is simply the function sending $(a, b, c) \in R$ to $(a - b)c$. In fact, for every $\mathcal{L}$-term $t(x_1, \ldots, x_n)$ there is a (unique) polynomial $P^t(X_1, \ldots, X_n)$ with integer coefficients such that for every (commutative) ring R we have that $t^R(a) = P^t(a)$ for every $a \in R^n$. Conversely, for each polynomial $P \in \mathbb{Z}[X_1, \ldots, X_n]$ there is an $\mathcal{L}_{\text{ring}}$-term $t(x_1, \ldots, x_n)$ such that $t^R(a) = P(a)$ for every commutative ring R and every $a \in R^n$. Note that in general not every polynomial in $R[x_1, \dots, x_n]$ can be seen as an $\mathcal{L}_{\mathrm{ring}}$-term, as we might not have constants for the coefficients. For example, the polynomial $\pi x_1 \in \mathbb{R}[x_1]$ has no $\mathcal{L}_{\mathrm{ring}}$-term counterpart.

1.5. Logical symbols. — We now fix the "logical symbols" of first-order logic:

$$\top \quad \bot \quad = \quad \neg \quad \wedge \quad \exists$$

which are called *True*, *False*, *equality*, *negation*, *conjunction* and *existential quantifier*, respectively. These symbols are suppose to be distinct of every symbol from $\mathcal{L}$ and from the set of variables $\mathrm{Var}_{\mathcal{L}}$ (for every language $\mathcal{L}$). The symbols $\neg, \wedge$ are called *connectives* (or logic connectives) and $\exists$ is called a *quantifier.*

1.5.1. *Atomic $\mathcal{L}$-formulas.* — The *atomic $\mathcal{L}$-formulas* are words on the alphabet

$$\mathcal{L} \cup \mathrm{Var}_{\mathcal{L}} \cup \{\top, \bot, =\},$$

and are defined as the smallest collection of words such that

(A1) $\top$ and $\bot$ are atomic $\mathcal{L}$-formulas;
(A2) for $R \in \mathcal{L}^{\mathfrak{r}}$ of arity m and $\mathcal{L}$-terms $t_1 \dots, t_m$, the word $Rt_1 \dots t_m$ is an atomic $\mathcal{L}$-formula;
(A3) for $\mathcal{L}$-terms t_1, t_2, the word $t_1 = t_2$ is an atomic $\mathcal{L}$-formula.

***1.5.2. Lemma* (Unique readability of atomic formulas)**

Every atomic $\mathcal{L}$-formula is either $\top$, $\bot$, or an atomic $\mathcal{L}$-formula of the form $Rt_1 \dots t_m$ for unique tuple $(R, t_1, \dots, t_m)$ where $R \in \mathcal{L}^{\mathfrak{r}}$ with $a(R) = m$ and $t_1, ldots, t_m$ are $\mathcal{L}$-terms, or of the form $t_1 = t_2$ with unique $\mathcal{L}$-terms t_1 and t_2.

1.5.3. *Formulas.* — The set of *$\mathcal{L}$-formulas* corresponds to the set of words over the alphabet

$$\mathcal{L} \cup \mathrm{Var}_{\mathcal{L}} \cup \{\top, \bot, =, \neg, \wedge, \exists\},$$

and are defined as the smallest collection of words such that

(F1) all atomic $\mathcal{L}$-formulas are $\mathcal{L}$-formulas;
(F2) if φ and ψ are $\mathcal{L}$-formulas, then so are $\neg\varphi$ and $\wedge\varphi\psi$;
(F3) if φ is an $\mathcal{L}$-formula and $x \in \mathrm{Var}_{\mathcal{L}}$, then $\exists x\varphi$ is also an $\mathcal{L}$-formula.

As for $\mathcal{L}$-terms, we will often omit $\mathcal{L}$ when there is no confusion and simply say 'formula'. Given a formula $\varphi = a_1 \cdots a_m$, a sub-formula of φ is a sub-word $a_i \cdots a_k$ with $1 \leq i \leq k \leq m$ which is also a formula.

***1.5.4. Lemma* (Unique readability of formulas).** — *Every $\mathcal{L}$-formula is either an atomic formula, or a formula of the form $\neg\varphi$, $\wedge\varphi\psi$ or $\exists x\varphi$ for unique $\mathcal{L}$-formulas φ, ψ and a unique variable x.*

1.5.5. Remark. —

(1) As for $\mathcal{L}$-terms, given formulas φ and ψ, we will use $\varphi \wedge \psi$ instead of $\wedge\varphi\psi$ for an more natural reading of formulas. We will also allow the usual parenthesis use in order to clarify the hidden under the polish or prefix notation. Analogously, we will write $t_1 \neq t_2$ instead of $\neg t_1 = t_2$.
(2) Similarly, given two formulas φ and ψ, we will use classic notation to denote disjunction $\varphi \vee \psi$, implication $\varphi \to \psi$, and double implication $\varphi \leftrightarrow \psi$ and universal quantification $\forall x\varphi$, which are abbreviations of the formulas $\neg(\neg\varphi \wedge \neg\psi)$, $\neg(\varphi \wedge \neg\psi)$, $(\neg(\varphi \wedge \neg\psi)) \wedge (\neg(\psi \wedge \neg\varphi))$ and $\neg\exists x\neg\varphi$, respectively.

1.5.6. Definition. — Given an $\mathcal{L}$-formula φ, we define by induction on φ the set $FV(\varphi)$ of *free variables of* φ as follows. For φ an atomic $\mathcal{L}$-formula:

1. if $\varphi = \top$ or $\varphi = \bot$, then $FV(\varphi) = \emptyset$;
2. if $\varphi = Rt_1 \cdots t_n$ with $R \in \mathcal{L}^{\mathfrak{r}}$ of arity n and t_i an $\mathcal{L}$-term for each $i \in \{1, \ldots, n\}$, then $FV(\varphi) = \bigcup_{i=1}^{n} V(t_i)$.
3. if φ is $= t_1t_2$, with t_1, t_2 $\mathcal{L}$-terms, then $FV(\varphi) = V(t_1) \cup V(t_2)$.

For the inductive step we suppose FV has been defined for $\mathcal{L}$-formulas φ and ψ.

1. $FV(\neg\varphi) = FV(\varphi)$;
2. $FV(\varphi \wedge \psi) = FV(\varphi \vee \psi) = FV(\varphi) \cup FV(\psi)$;
3. $FV(\exists x\varphi) = FV(\varphi) \setminus \{x\}$.

1.5.7. Examples. — Let's consider the $\mathcal{L}_{\mathrm{ring}}$-formula $\exists x(x \cdot y = 1) \wedge x = 0$, which written in prefix form corresponds to $\wedge\exists x = \cdot xy1 = x0$. The set of free variables of this formula is $\{x, y\}$ even if the formula contains $\exists x$. In contrast, the free variables of of the subformula $\exists x(x \cdot y = 1)$ is the singleton set $\{y\}$.

1.5.8. Definition. — An occurrence of a variable x in a formula φ is said to be *free* if it does not belong to a sub-formula of φ which starts by $\exists x$.

For example, the first two occurrences of x in the formula given in the previous example are not free but the third one is. The only occurrence of the variable y is a free occurrence. Let's note that the definition of free occurrence of variable is only applied to the prefix writing of a formula.

Similarly to $\mathcal{L}$-terms, given a multivariable x we define an $(\mathcal{L}, x)$-formula as an $\mathcal{L}$-formula φ such that every free variable of φ is among the variables in x. We also write $\varphi(x)$ to indicate that φ is an $(\mathcal{L}, x)$-formula. An *$\mathcal{L}$-sentence* is a $\mathcal{L}$-formula which has no free variables. An *atomic $\mathcal{L}$-sentence* is an $\mathcal{L}$-sentence which is also an atomic $\mathcal{L}$-formula.

⚠ The reader should keep in mind the different ways in which the symbol '='; is used. Sometimes it is used as one of the logical symbols, but we also use it to denote the usual identity between mathematical objects. The context should always tell you which is which!

1.6. Substitution. — Let $x = (x_1, \dots, x_n)$ be a multivariable, $t(x)$ be an $\mathcal{L}$-term, $\varphi(x)$ be an $\mathcal{L}$-formula and $s_1, \dots s_n$ be $\mathcal{L}$-terms. We define

- $t(s)$: the word obtained by simultaneously replacing each occurrence of the variable x_i by s_i in $t(x)$;
- $\varphi(s)$: the word obtained by simultaneously replacing each free occurrence of the variable x_i by s_i in $\varphi(x)$.

We leave to the reader to show that $t(s)$ is indeed an $\mathcal{L}$-term and that $\varphi(s)$ is indeed an $\mathcal{L}$-formula. Note in particular that if no term s_i contains variables, then $\varphi(s)$ is an $\mathcal{L}$-sentence.

1.7. Satisfiability. — We can finally define the truth or satisfiability relation. Given an $\mathcal{L}$-formula $\varphi(x)$ and $a \in M^{|x|}$, we define the relation

$$\mathcal{M} \models \varphi(a),$$

by induction on formulas. For atomic $\mathcal{L}$-formulas we have:

(S1) $\mathcal{M} \models \top$ and $\mathcal{M} \not\models \bot$;

(S2) for $R \in \mathcal{L}^{\mathfrak{r}}$ of arity m and $\mathcal{L}$-terms $t_1(x), \dots, t_m(x)$

$$\mathcal{M} \models Rt_1(a) \dots t_m(a) \text{ if and only if } (t_1^{\mathcal{M}}(a), \dots, t_m^{\mathcal{M}}(a)) \in R^{\mathcal{M}},$$

(S3) for $\mathcal{L}$-terms $t_1(x)$, $t_2(x)$

$$\mathcal{M} \models t_1(a) = t_2(a) \text{ if and only if } t_1^{\mathcal{M}}(a) = t_2^{\mathcal{M}}(a).$$

Given $\mathcal{L}$-formulas $\varphi(x), \psi(x)$ and $\theta(x, y)$ for which $\models$ has been already defined,

(S4) $\mathcal{M} \models \neg\varphi(a)$ if and only if $\mathcal{M} \not\models \varphi(a)$,

(S5) $\mathcal{M} \models \varphi(a) \wedge \psi(a)$ if and only if $\mathcal{M} \models \varphi(a)$ and $\mathcal{M} \models \psi(a)$.

(S6) $\mathcal{M} \models \exists y\theta(x, y)$ if and only if there is $b \in M$ such that $\mathcal{M} \models \theta(a, b)$.

1.7.1. Examples. —

1. Let $\mathcal{G} = (G, \cdot, ^{-1}, e)$ be a group in the language $\mathcal{L}_{\mathrm{g}}$. The $\mathcal{L}_{\mathrm{g}}$-sentence $\varphi \coloneqq \forall x\exists y(x \cdot y = e)$ is satisfied by $\mathcal{G}$, that is $\mathcal{G} \models \varphi$. Consider now the $\mathcal{L}_{\mathrm{g}}$-sentence $\psi \coloneqq \forall x\forall y(x \cdot y = y \cdot x)$. Clearly, $\mathcal{G}$ satisfies ψ if and only if G is an abelian group.
2. Which $\mathcal{L}$-sentences can one form when $\mathcal{L}$ is the empty language $\mathcal{L}_\emptyset$? One can express the property of having at least k elements, and also of having exactly k elements, as show the following $\mathcal{L}_\emptyset$-sentences

$$\exists x_1 \cdots \exists x_k (\bigwedge_{i \neq j} x_i \neq x_j)$$

$$\exists x_1 \cdots \exists x_k (\bigwedge_{i \neq j} x_i \neq x_j \wedge \forall y (\bigvee_{i=1}^{k} y = x_i)).$$

1.7.2. Definition. — Two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are *elementary equivalent*, in symbols $\mathcal{M} \equiv \mathcal{N}$, if the satisfy the same $\mathcal{L}$-sentence, that is, for every $\mathcal{L}$-sentence φ (without free variables),

$$\mathcal{M} \models \varphi \text{ if and only if } \mathcal{N} \models \varphi.$$

We finish this section with a notation concerning languages that will be used in the coming sections.

1.7.3. Notation. — Let $\mathcal{M}$ be an $\mathcal{L}$-structure a A be a subset of M. We define the language $\mathcal{L}(A)$ as the language $\mathcal{L}$ together with a set of new constant symbols $\{c_a \mid a \in A\}$. The structure $\mathcal{M}$ is naturally an $\mathcal{L}(A)$-structure by interpreting each constant symbol c_a as a. In practice, we never write c_a, but directly a. For example, given a tuple of variables x, every $\mathcal{L}(A)$-formula $\varphi(x)$ is of the form $\psi(x, a)$ for some $\mathcal{L}$-formula $\psi(x, y)$ and some tuple $a \in A^{|y|}$. One often says that $\varphi(x)$ is an *$\mathcal{L}$-formula with parameters from A*.

2. Elementary substructures and Löwenheim-Skolem theorems

Throughout this section we let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures for some language $\mathcal{L}$.

2.1. Equivalence. — Given a multivariable $x = (x_1, \ldots, x_n)$ and an $\mathcal{L}$-formula $\varphi(x)$, we write $\exists x\varphi$ and $\forall x\varphi$ (or even $\exists x\varphi(x)$ and $\forall x\varphi(x)$) as abbreviations for the formulas

$$\exists x_1 \cdots \exists x_n\varphi \text{ and } \forall x_1 \cdots \forall x_n\varphi,$$

respectively. The satisfaction relation in $\mathcal{M}$ is extended to $\mathcal{L}$-formulas $\varphi(x)$ by defining

$$\mathcal{M} \models \varphi(x) \text{ if and only if } \mathcal{M} \models \forall x\varphi.$$

We write $\models \varphi$ if $\mathcal{M} \models \varphi$ for every $\mathcal{L}$-structure $\mathcal{M}$. Two formulas $\varphi(x), \psi(x)$ (with the same distinguished multivariable x) are *equivalent* if $\models \varphi \leftrightarrow \psi$. For example, it is not difficult to see that the formulas $\neg\forall x\varphi$ and $\exists x\neg\varphi$ are equivalent for every formula φ. One can also state that $\wedge$ and $\vee$ are associative and commutative by writing for arbitrary formulas φ, ψ, θ

$$\begin{cases} \models (\varphi \wedge (\psi \wedge \theta)) \leftrightarrow ((\varphi \wedge \psi) \wedge \theta) \\ \models (\varphi \wedge \psi) \leftrightarrow (\psi \wedge \varphi), \end{cases}$$

and similarly by replacing $\wedge$ by $\vee$. This allows us to write conjunctions and disjunctions without parentheses. Given formulas $\varphi_1, \ldots, \varphi_n$ we will use the following notation for conjunctions and disjunctions:

$$\bigwedge_{i=1}^{n} \varphi_i \text{ abbreviates } \varphi_1 \wedge \cdots \wedge \varphi_n \text{ and}$$

$$\bigvee_{i=1}^{n} \varphi_i \text{ abbreviates } \varphi_1 \vee \cdots \vee \varphi_n.$$

2.1.1. Definition. — A formula $\varphi(x)$ is said to be in prenex form if it is of the form

$$Q_1 y_1 \cdots Q_m y_m \psi(x, y)$$

where $Q_i \in \{\exists, \forall\}$ and $\psi(x, y)$ is a quantifier free formula.

2.1.2. Lemma. — *Every formula φ is equivalent to a formula in prenex form.* □

2.1.3. Definition. — A formula $\varphi(x)$ is said to be an existential formula (resp. an universal formula) if is it equivalent to a formula in prexex form which only contains existential (resp. universal) quantifiers.

2.2. Maps preserving formulas. — Let $A \subseteq M$ and $h \colon A \to N$. Let $\varphi(x)$ be an $\mathcal{L}$-formula with $x = (x_1, \ldots, x_n)$. We say that h *preserves the formula* $\varphi(x)$ if for every $a = (a_1, \ldots, a_n) \in A^n$

$$\mathcal{M} \models \varphi(a) \Rightarrow \mathcal{N} \models \varphi(h(a)).$$

2.2.1. Lemma. — *Let $h \colon M \to N$ be a function. Then h is an $\mathcal{L}$-embedding if and only if h preserves all quantifier free $\mathcal{L}$-formulas.*

Proof. — Let $h \colon \mathcal{M} \to \mathcal{N}$ be an embedding, $\varphi(x)$ be a quantifier-free $\mathcal{L}$-formula and $a \in M^{|x|}$. We show by induction on φ than

$$(*) \qquad \mathcal{M} \models \varphi(a) \Leftrightarrow \mathcal{N} \models \varphi(h(a)).$$

Suppose first φ is atomic. The result is trivial for $\top$ and $\bot$. Suppose first that $\varphi(x)$ is of the form $Rt_1(x) \cdots t_n(x)$, for $R \in \mathcal{L}^{\mathfrak{r}}$ of arity n and $t_i(x)$ an $\mathcal{L}$-term for $i = 1, \ldots, n$. We have

$$\begin{aligned}
\mathcal{M} \models \varphi(a) &\Leftrightarrow (t_1^{\mathcal{M}}(a), \ldots, t_n^{\mathcal{M}}(a)) \in R^{\mathcal{M}} \\
&\Leftrightarrow (h(t_1^{\mathcal{M}}(a)), \ldots, h(t_n^{\mathcal{M}}(a))) \in R^{\mathcal{N}} \text{ (def. of embedding)} \\
&\Leftrightarrow (t_1^{\mathcal{N}}(h(a)), \ldots, t_n^{\mathcal{N}}(h(a))) \in R^{\mathcal{N}} \text{ (Exercise 1 in Tutorial 1)} \\
&\Leftrightarrow \mathcal{N} \models Rt_1(h(a)) \cdots t_n(h(a)) \Leftrightarrow \mathcal{N} \models \varphi(h(a)).
\end{aligned}$$

The remaining case is $\varphi(x)$ is of the form $t_1(x) = t_2(x)$ for t_1, t_2 two $\mathcal{L}$-terms. We have

$$\begin{aligned}
\mathcal{M} \models \varphi(a) &\Leftrightarrow h(t_1^{\mathcal{M}}(a)) = h(t_2^{\mathcal{M}}(a)) \text{(since } h \text{ is injective)} \\
&\Leftrightarrow t_1^{\mathcal{N}}(h(a)) = t_2^{\mathcal{N}}(h(a)) \text{ (Exercise 1 in Tutorial 1)} \\
&\Leftrightarrow \mathcal{N} \models t_1(h(a)) = t_2(h(a)) \Leftrightarrow \mathcal{N} \models \varphi(h(a)).
\end{aligned}$$

This shows $(*)$ for atomic formulas. Now suppose h satisfies $(*)$ for the quantifier-free formulas $\psi(x)$ and $\theta(x)$. If φ is $\neg\psi(x)$ then

$$\begin{aligned}\mathcal{M} \models \varphi(a) &\Leftrightarrow \mathcal{M} \not\models \psi(a)\\ &\Leftrightarrow \mathcal{N} \not\models \psi(h(a))\\ &\Leftrightarrow \mathcal{N} \models \varphi(h(a)).\end{aligned}$$

Finally, if $\varphi(x)$ is $\psi(x) \wedge \theta(x)$ we have

$$\begin{aligned}\mathcal{M} \models \varphi(a) &\Leftrightarrow \mathcal{M} \models \psi(a) \wedge \theta(a)\\ &\Leftrightarrow \mathcal{M} \models \psi(a) \text{ and } \mathcal{M} \models \theta(a) \text{ (by (S5))}\\ &\Leftrightarrow \mathcal{N} \models \psi(h(a)) \text{ and } \mathcal{N} \models \theta(h(a)) \text{ (induction hypothesis)}\\ &\Leftrightarrow \mathcal{N} \models \varphi(h(a)) \text{ (by (S5))} .\end{aligned}$$

For the converse, suppose $h\colon M \to N$ is a map preserving quantifier-free $\mathcal{L}$-formulas. Injectivity follows from the preservation of the quantifier free formula $x \neq y$. To show condition (M1) in the definition of embedding (see Section 1.3.1), let $R \in \mathcal{L}^{\mathfrak{r}}$ of arity n and $a \in M^n$. Then, since h preserves both the formula $Rx_1 \cdots x_n$ and $\neg Rx_1 \cdots x_n$ we have

$$\begin{aligned}a \in R^{\mathcal{M}} &\Leftrightarrow \mathcal{M} \models Ra\\ &\Leftrightarrow \mathcal{N} \models R(h(a))\\ &\Leftrightarrow h(a) \in R^{\mathcal{N}}\end{aligned}$$

For condition (M2), let f be a function symbol of arity m, $x = (x_1, \ldots, x_m)$ and y be a single variable. Consider the quantifier-free formula $f(x) = y$. Then, for every $a \in M^m$ and $b \in M$ we have

$$\begin{aligned}f^{\mathcal{M}}(a) = b &\Leftrightarrow \mathcal{M} \models f(a) = b\\ &\Leftrightarrow \mathcal{N} \models f(h(a)) = h(b)\\ &\Leftrightarrow f^{\mathcal{N}}(h(a)) = h(b)\\ &\Leftrightarrow h(f^{\mathcal{M}}(a)) = f^{\mathcal{N}}(h(a)).\end{aligned}$$

Condition (M3) are proven similarly using the formula $x = c$. ☐

In the particular case where $\mathcal{M} \subseteq \mathcal{N}$ we obtain :

2.2.2. Corollary. — *Suppose $\mathcal{M} \subseteq \mathcal{N}$. Then*

1. *if $\varphi(x)$ is quantifier free, then $\mathcal{M} \models \varphi(a) \Leftrightarrow \mathcal{N} \models \varphi(a)$ for all $a \in M^{|x|}$;*
2. *if $\varphi(x)$ is an existential formula, then $\mathcal{M} \models \varphi(a) \Rightarrow \mathcal{N} \models \varphi(a)$ for all $a \in M^{|x|}$;*
3. *if $\varphi(x)$ is an universal formula, then $\mathcal{N} \models \varphi(a) \Rightarrow \mathcal{M} \models \varphi(a)$ for all $a \in M^{|x|}$.*

Proof. — Part (1) follows from Lemma 2.2.1 since the inclusion is an embedding. For (2), let $\varphi(x)$ be an existential formula. This means that there is a quantifier free formula $\psi(x,y)$ such that $\varphi(x)$ is equivalent to $\exists y\psi(x,y)$. Therefore

$$\begin{aligned}\mathcal{M}\models\varphi(a) &\Leftrightarrow \mathcal{M}\models\exists y\psi(a,y)\\ &\Leftrightarrow \mathcal{M}\models\psi(a,b) \text{ for some } b\in M^{|y|}\\ &\Leftrightarrow \mathcal{N}\models\psi(a,b) \text{ by part } (i)\\ &\Rightarrow \mathcal{N}\models\exists y\psi(a,y)\\ &\Leftrightarrow \mathcal{N}\models\varphi(a).\end{aligned}$$

Point (3) is proved similarly. □

2.2.3. Definition. — We say that a map $h\colon A\to N$ is an $\mathcal{L}$-*elementary embedding* if it preserves all $\mathcal{L}$-formulas. When $\mathcal{M}\subseteq\mathcal{N}$ and the inclusion map is elementary, we say that $\mathcal{M}$ is an *elementary substructure of* $\mathcal{N}$ (respectively $\mathcal{N}$ is an *elementary extension of* $\mathcal{M}$) and we denote it by $\mathcal{M}\preceq\mathcal{N}$.

Note that by Lemma 2.2.1, every elementary map $h\colon M\to N$ is an embedding, so in particular injective. Moreover, since it preserves negation we have that for every $\mathcal{L}$-formula $\varphi(x)$ and every $a\in M^{|x|}$

$$\mathcal{M}\models\varphi(a)\Leftrightarrow\mathcal{N}\models\varphi(h(a)).$$

As we have seen from the Tutorial 1 (Exercise 2), isomorphisms are particular instances of elementary maps. The following proposition gives us a criterion to check that a given substructure is an elementary substructure. In fact we prove a bit more.

***2.2.4. Proposition* (Tarski-Vaught Test).** — *Let $\mathcal{N}$ be an $\mathcal{L}$-structure and $A\subseteq N$. Suppose that for every $\mathcal{L}(A)$-formula $\varphi(x)$ with $|x|=1$,*

$$(*)\qquad \mathcal{N}\models\exists x\varphi(x)\Rightarrow\mathcal{N}\models\varphi(a) \textit{ for some } a\in A.$$

Then A is the universe of an elementary substructure $\mathcal{M}\preceq\mathcal{N}$.

Proof. — Let us first prove that A is the universe of a substructure of $\mathcal{N}$. Note that since $N\neq\emptyset$, $A\neq\emptyset$ since $\mathcal{N}\models\exists x(x=x)$. It remains to show that A is closed under the functions associated to function symbols in $\mathcal{L}$. Let f be a function symbol in $\mathcal{L}$ or arity n and $a\in A^n$. Consider the formula $\varphi(x):=f(a)=x$. Since $\mathcal{N}$ is an $\mathcal{L}$-structure, $\mathcal{N}\models\exists x\varphi(x)$, and by $(*)$ we have $f^{\mathcal{N}}(a)\in A$.

Let us now show by induction on an $\mathcal{L}(A)$-formula $\varphi(x)$ that

$$\mathcal{M}\models\varphi(a)\Leftrightarrow\mathcal{N}\models\varphi(a) \text{ for all } a\in A^{|x|}.$$

If φ is an atomic formula, the result follows by Corollary 2.2.2. Assuming the result for $\varphi(x),\psi(x)$, the result follows immediately for $\neg\varphi(x)$ and for $\varphi(x)\wedge\psi(x)$. Finally, suppose the result holds for $\psi(x,y)$ and that $\varphi(x)$ is of the form $\exists y\psi(x,y)$. Then

$$\begin{aligned}\mathcal{A}\models\exists y\psi(a,y) &\Leftrightarrow \mathcal{A}\models\psi(a,b) \text{ for some } b\in A\\ &\Leftrightarrow \mathcal{N}\models\psi(a) \text{ for some } a\in A\\ &\Leftrightarrow \mathcal{N}\models\exists y\psi(a,y),\end{aligned}$$

where the right-to-left arrow in the last line follows by the assumption of the proposition. □

2.2.5. ***Theorem*** **(Downwards Löwenheim-Skolem).** — *Let $\mathcal{N}$ be an $\mathcal{L}$-structure. Let $S\subseteq N$ be a set and κ be an infinite cardinal such that*

$$\max\{|S|,|\mathcal{L}|\}\leq\kappa\leq|N|.$$

Then, there is an elementary substructure $\mathcal{M}\preceq\mathcal{N}$ such that $S\subseteq M$ and $|M|=\kappa$.

Proof. — Without loss of generality we may suppose $|S|=\kappa$. We let $M=\bigcup_{i<\omega}S_i$ where the S_i are defined inductively as follows:

- $S_0 \coloneqq S$;
- if S_i has been defined, list all $\mathcal{L}(S_i)$-formulas $\varphi(x)$ with $|x|=1$ which are satisfiable in $\mathcal{N}$, that is,

  $$A_i\coloneqq\{\varphi(x)\mid\mathcal{N}\models\exists x\varphi(x),\ \varphi(x) \text{ an } \mathcal{L}(S_i)\text{-formula}\},$$

  and for each $\varphi\in A_i$, let $a_\varphi\in N$ be such that $\mathcal{N}\models\varphi(a_\varphi)$. Then set $S_{i+1}\coloneqq\{a_\varphi\mid\varphi\in A_i\}$.

Note that by Tarski-Vaught test, we have that M is the universe of an elementary substructure $\mathcal{M}\preceq\mathcal{N}$ and it contains S. It remains to show that $|M|\leqslant\kappa$. It suffices to show that $|S_i|\leqslant\kappa$ for all $i<\omega$. Indeed, we have that

$$|M|=|\bigcup_{i<\omega}S_i|\leqslant\aleph_0\kappa=\kappa.$$

We show that $|S_i|\leqslant\kappa$ by induction on n. For $n=0$, this follows from the assumption on κ. Suppose that $|S_i|\leqslant\kappa$. Since the number of $\mathcal{L}(S_i)$-formulas corresponds to $\max\{\aleph_0,|S_i|\})$, we have

$$|S_{i+1}|\leqslant|A_i|=\max\{\aleph_0,|S_i|\})=S_i\leqslant\kappa,$$

which completes the result. □

2.2.6. ***Theorem*** **(Upwards Löwenheim-Skolem).** — *Let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure. Let κ be a cardinal such that*

$$\max\{|M|,|\mathcal{L}|\}\leq\kappa.$$

Then, there is an elementary extension $\mathcal{M}\preceq\mathcal{N}$ such that $|N|=\kappa$.

The proof will be postponed until we introduce the Compactness theorem in Section 4.

3. Theories and models

We introduce in this section the notions of theory and model of a theory. In what follows, unless otherwise stated, t will denote an $\mathcal{L}$-term, φ, ψ and θ will denote $\mathcal{L}$-formulas. We will omit the prefix $\mathcal{L}$ for $\mathcal{L}$-term, $\mathcal{L}$-formula, *etc.*, when no confusion arises.

3.1. Theories. — An $\mathcal{L}$-*theory* T is simply a set of $\mathcal{L}$-sentences. We say that an $\mathcal{L}$-structure $\mathcal{M}$ is a *model of* T, noted $\mathcal{M} \models T$, if $M \models \varphi$ for every sentence $\varphi \in T$. A theory T is said to be *satisfiable* or *consistent*, if it has at least one model. For example, the theory $T = \{\forall x(x \neq x)\}$ is not satisfiable. We say that φ is a *logical consequence* of T, noted $\Sigma \models \varphi$, if $\mathcal{M} \models \varphi$ for every model $\mathcal{M}$ of T. We will also write $\psi \models \varphi$ instead of $\{\psi\} \models \varphi$.

3.1.1. Examples. — (1) The $\mathcal{L}_\emptyset$-theory of the infinite set T_∞ is axiomatized by the following axioms

$$\{\exists x_1 \cdots \exists x_n (\bigwedge_{i \neq j} x_i \neq x_j) \mid n < \omega\}.$$

(2) The theory of groups is axiomatized by the following $\mathcal{L}_{\mathrm{g}}$-axioms

$$\text{Groups} \coloneqq \begin{cases} \forall x(x \cdot e = x) \\ \forall x(x \cdot x^{-1} = e) \\ \forall x \forall y \forall z((x \cdot y) \cdot z = x \cdot (y \cdot z)) \end{cases}$$

The theory Th(Groups) is the set of $\mathcal{L}_{\mathrm{g}}$-sentences which is satisfied by all groups.

(3) The theory of abelian groups, often axiomatized in the additive language of groups $\{+, -, 0\}$ is axiomatized by

$$\text{AG} \coloneqq \begin{cases} \forall x(x + 0 = x) \\ \forall x(x + (-x) = 0) \\ \forall x \forall y \forall z((x + y) + z = x + (y + z)) \\ \forall x \forall y(x + y = y + x) \end{cases}$$

(4) The theory of non-trivial torsion-free abelian groups is axiomatized by

$$\text{TFAG} \coloneqq \text{AG} \cup \{\forall x(nx = 0 \to x = 0) : n > 0\} \cup \{\exists x(x \neq 0)\}.$$

And the theory of non-trivial torsion-free divisible abelian groups by

$$\text{TFDAG} \coloneqq \text{TFAG} \cup \{\forall x \exists y(nx = y) : n > 0\}.$$

(5) Partially ordered sets are $\mathcal{L}_{\leqslant}$-structures axiomatized by

$$\text{Poset} \coloneqq \begin{cases} \forall x(x \leqslant x) \\ \forall x\forall y\forall z((x \leqslant y \wedge y \leqslant z) \to x \leqslant z) \\ \forall x\forall y((x \leqslant y \wedge y \leqslant x) \to x = y) \end{cases}$$

(6) If we abbreviate $x \leqslant y \wedge x \neq y$ by $x < y$, dense ordered linear orders without endpoints (DLO) are axiomatized by

$$\text{DLO} \coloneqq \begin{cases} \text{Poset}\cup \\ \{\forall x\forall y(x < y \vee y < x \vee x = y)\}\cup \\ \{\forall x\forall y\exists z(x < y \to (x < z \wedge z < y), \forall x\exists y\exists z(y < x \wedge x < z)\}. \end{cases}$$

(7) Ordered abelian groups are $\mathcal{L}_{\text{og}}$-structures axiomatized by

$$\text{OAG} \coloneqq \text{Poset} \cup \text{AG} \cup \{\forall x\forall y\forall z(x \leq y \to x + z \leq y + z)\}.$$

(8) Rings are $\mathcal{L}_{\text{ring}}$-structures axiomatized by

$$\text{Rings} \coloneqq \text{AG} \cup \begin{cases} \forall x\forall y\forall z((x \cdot y) \cdot z = x \cdot (y \cdot z)) \\ \forall x(x \cdot 1 = x) \\ \forall x(1 \cdot x = x) \\ \forall x\forall y\forall z(x \cdot (y + z) = (x \cdot y) + (x \cdot z)) \\ \forall x\forall y\forall z((x + y) \cdot z = (x \cdot z) + (y \cdot z)) \end{cases}$$

(9) Fields are $\mathcal{L}_{\text{ring}}$-structures axiomatized by

$$\text{Fields} \coloneqq \text{Rings} \cup \{\forall x\forall y(x \cdot y = y \cdot x), 1 \neq 0, \forall x(x \neq 0 \to \exists y(x \cdot y = 1))\}.$$

(10) Let p be a prime number. Fields of characteristic p are axiomatized by

$$\text{Fields}_p \coloneqq \text{Fields} \cup \{p1 = 0\},$$

and fields of characteristic 0 by

$$\text{Fields}_0 \coloneqq \text{Fields} \cup \{n1 \neq 0 : n > 1\}.$$

(11) Algebraically closed fields are axiomatized by

$$\text{ACF} \coloneqq \text{Fields} \cup \{\forall y_0 \cdots \forall y_{n-1}\exists x(x^n + y_{n-1}x^{n-1} + \cdots + y_0 = 0) : n > 1\}.$$

For p either a prime number or $p = 0$, algebraically closed fields of characteristic p are axiomatized by

$$\text{ACF}_p \coloneqq \text{Fields}_p \cup \text{ACF}.$$

(12) Ordered rings $\mathcal{L}_{\text{or}}$-structures axiomatized by

$$\text{OrdRings} \coloneqq \text{OAG} \cup \text{Rings} \cup \{\forall x\forall y(0 \leq x \wedge 0 \leq y \to 0 \leq xy)\},$$

(13) Ordered fields are $\mathcal{L}_{\text{or}}$-structures axiomatized by

$$\text{OrdFields} \coloneqq \text{OrdRings} \cup \text{Fields}.$$

3.1.2. Definition. — Let T be an $\mathcal{L}$-theory.

- The class $\mathrm{Mod}(T)$ corresponds to the class of all models of T.
- A class of $\mathcal{L}$-structures $\mathcal{C}$ is an *elementary class* if there is a theory Σ such that $\mathcal{C} = \mathrm{Mod}(\Sigma)$.
- A subset $T_0 \subseteq T$ is an *axiomatization* of T if $\mathrm{Mod}(T_0) = \mathrm{Mod}(T)$.
- We say that T is closed under logical consequence if $\varphi \in T$ whenever $T \models \varphi$.
- For an $\mathcal{L}$-structure $\mathcal{M}$, *the theory of* $\mathcal{M}$ is the set of $\mathcal{L}$-sentences which holds in $\mathcal{M}$, namely,
$$\mathrm{Th}(\mathcal{M}) \coloneqq \{\varphi : \mathcal{M} \models \varphi, \varphi \text{ an } \mathcal{L}\text{-sentnce}\}.$$
- If $\mathcal{C}$ is a class of $\mathcal{L}$-structures, *the common theory of* $\mathcal{C}$ is the set of $\mathcal{L}$-sentences which holds in every element if $\mathcal{C}$, namely,
$$\mathrm{Th}(\mathcal{C}) \coloneqq \{\varphi : \mathcal{M} \models \varphi, \text{ for every } \mathcal{M} \in \mathcal{C}\}.$$
- A theory T is *complete* if for every sentence φ either $T \models \varphi$ or $T \models \neg\varphi$. A complete theory containing a set of sentences Σ is called a *completion of* Σ.
- Let κ be a cardinal. Suppose T has models of cardinality κ. We say T is κ-categorical if every two models of T of cardinality κ are isomorphic.

Note that for an $\mathcal{L}$-structure $\mathcal{M}$, the theory $\mathrm{Th}(\mathcal{M})$ is complete. By definition, Mod(Groups) is the class of all groups, so the class of groups is an elementary class. Similarly, Mod(Fields) is the class of fields, etc. An example of an $\mathcal{L}$-sentence of which is a logical consequence of Groups is
$$\forall x((\forall y(xy = y)) \to x = e),$$
since it is a logical consequence of Groups. Is the theory of groups complete? Is the theory of fields complete? In general to determine if a given theory is complete is a difficult problem. The following lemma gives an nice criterion.

3.1.3. Lemma. — *A theory T is complete if and only if all of its models are elementarily equivalent.*

Proof. — Suppose T is complete and let $\mathcal{M}$ and $\mathcal{N}$ be two models of T. Let φ be an $\mathcal{L}$-sentence and suppose that $\mathcal{M} \models \varphi$. This implies that $T \not\models \neg\varphi$ (why?). Since T is complete, we must have that $T \models \varphi$. Therefore, since $\mathcal{N}$ is also a model of T, $\mathcal{N} \models \varphi$. Now, φ was arbitrary, which shows that $\mathcal{M} \equiv \mathcal{N}$. For the converse, we prove the contrapositive, so suppose there is an $\mathcal{L}$-sentence φ such that $T \not\models \varphi$ and $T \not\models \neg\varphi$. The former implies that there is a model $\mathcal{M}$ of T such that $M \not\models \varphi$, and the latter that there is a model $\mathcal{N}$ of T such that $\mathcal{N} \not\models \neg\varphi$, or equivalently, that $\mathcal{N} \models \varphi$. But this shows that $\mathcal{M}$ and $\mathcal{N}$ are not elementarily equivalent. $\square$

***3.1.4. Theorem* (Vaught's test).** — *Let κ be an infinite cardinal and suppose $|\mathcal{L}| \leqslant \kappa$. If T is κ-categorical then T is complete.*

Proof. — By Lemma 3.1.3, it suffices to show that any two models of T are elementarily equivalent. So let $\mathcal{M}$ and $\mathcal{N}$ be two models of T. If $|M| = |N| = \kappa$, then $\mathcal{M} \cong \mathcal{N}$ and therefore $\mathcal{M} \equiv \mathcal{N}$. The remaining three remaining cases all having a very similar proof. We sketch two cases:

Case 1: Suppose $|M| \leqslant \kappa \leqslant |N|$. Then

$$\begin{tikzcd} & \mathcal{N}' \arrow[r, "\prec"] \arrow[d, "\cong"] & \mathcal{N} \\ \mathcal{M} \arrow[r, "\prec"] & \mathcal{M}' & \end{tikzcd} \Rightarrow \mathcal{M} \equiv \mathcal{M}' \equiv \mathcal{N}' \equiv \mathcal{N},$$

where the two horizontal arrows are inclusions and obtained by Löwenheim-Skolem theorems (both ascending and descending) to obtain $\mathcal{M}'$ and $\mathcal{N}'$ such that $|M'| = |N'| = \kappa$. The vertical arrow exists by the assumption that T is κ-categorical.

Case 2: Suppose $|M|, |N| \leqslant \kappa$. The same idea as in the previous case applies now to the following diagram

$$\begin{tikzcd} \mathcal{N} \arrow[r, "\prec"] & \mathcal{N}' \arrow[d, "\cong"] \\ \mathcal{M} \arrow[r, "\prec"] & \mathcal{M}' \end{tikzcd} \Rightarrow \mathcal{M} \equiv \mathcal{M}' \equiv \mathcal{N}' \equiv \mathcal{N}.$$

□

3.1.5. Theorem. — *The theory T_∞ of the infinite set is κ-categorical in every infinite cardinal κ. It is a complete theory.*

Proof. — Follows directly by Vaught's test. □

***3.1.6. Theorem* (Cantor).** — *Any two countably infinite dense linear orders are isomorphic.*

3.1.7. Theorem. — *The theory DLO of dense linear orders without end-points is $\aleph_0$-categorical and hence complete.*

Proof. — Follows by Vaught's test using Cantor's theorem. □

3.1.8. Theorem. — *The theory T of non-trivial torsion-free divisible abelian groups is κ-categorical in every uncountable cardinal κ. It is a complete theory.*

Proof. — The last statement follows from the first and Vaught's test. We will show the first part by a series of observations:

Obs 1: There is a one-to-one correspondence between models of T and $\mathbb{Q}$-vector spaces. Indeed, if $(G, +, -, 0)$ is a model of T, then $\mathbb{Q}$ acts on G as follows. For every $g \in G$ and every $n \in \mathbb{N}^*$ there is some $h \in G$ such that $nh = g$. We let g/n denote h.

Note that this is indeed well-defined since if $nh_1 = nh_2 = g$, then $n(h_1 - h_2) = 0$, and since G is torsion-free this implies that $h_1 - h_2 = 0$, therefore $h_1 = h_2$. We defined the action

$$\frac{m}{n} \cdot g \mapsto m(\frac{g}{n}).$$

It is not hard to check that G is a $\mathbb{Q}$-vector space under this action and that every $\mathbb{Q}$-vector space arises in this way in a unique way up to isomorphism.

Obs 2: Two $\mathbb{Q}$-vector spaces $(V_1, +, 0)$ and $(V_2, +, 0)$ are isomorphic if and only if they have the same dimension (i.e., if B_1 and B_2 are bases of V_1 and V_2 respectively, and $|B_1| = |B_2|$, then any bijection between B_1 and B_2 induces an isomorphism between V_1 and V_2).

Obs 3: Two models G_1, G_2 of T are $\mathcal{L}_{\mathrm{g}}$-isomorphic if and only if their corresponding $\mathbb{Q}$-vector spaces are isomorphic (as $\mathbb{Q}$-vector spaces). (Exercise).

Obs 4: If G is a model of T, and its associated $\mathbb{Q}$-vector space has dimension λ, then $|G| = \max\{\lambda, \aleph_0\}$. In particular, if $|G| > \aleph_0$, then it must have a basis of cardinality $|G|$.

As a consequence of these observations, we obtain that if κ is an uncountable cardinal (i.e. $\kappa > \aleph_0$), and G_1, G_2 are two models of T such that $|G_1| = |G_2| = \kappa$, then they must have both a basis of cardinality κ. By the second observation, they must be isomorphic as $\mathbb{Q}$-vector spaces, and hence they are $\mathcal{L}_{\mathrm{g}}$-isomorphic. □

3.1.9. Corollary. — *The following groups are elementary equivalent:* $(\mathbb{R}, +), (\mathbb{R} \oplus \mathbb{R}, +), (\mathbb{Q}, +), (\mathbb{C}, +)$.

***3.1.10. Theorem* (Steinitz).** — *Two uncountable algebraically closed fields of the same characteristic and the same cardinality are isomorphic.*

3.1.11. Theorem. — *Let p be either a prime number or 0. The theory* ACF_p *of algebraically closed fields of characteristic p is κ-categorical in every uncountable cardinal κ. It is a complete theory.*

Proof. — This follows by Stenitz theorem and Vaught's test. □

4. The compactness theorem

This section contains one of most (if not the most) fundamental theorems of model theory, the *compactness theorem*:

***4.0.1. Theorem* (Compactness).** — *Let Σ be an $\mathcal{L}$-theory. Then* $\mathrm{Mod}(\Sigma) \neq \emptyset$ *if and only if* $\mathrm{Mod}(\Sigma_0) \neq \emptyset$ *for every finite subset $\Sigma_0 \subseteq \Sigma$. In words, Σ is satisfiable if and only if every finite subset of Σ is satisfiable.*

The following is an equivalent formulation (exercise).

***4.0.2. Theorem* (Compactness).** — *Let Σ be an $\mathcal{L}$-theory. Then*

$$\Sigma \models \varphi \text{ if and only if } \Sigma_0 \models \varphi \text{ for a finite subset } \Sigma_0 \subseteq \Sigma.$$

Before proving this theorem, we will look at various applications.

4.1. Applications of compactness. —

***4.1.1. Theorem* (Upward Löwenheim-Skolem).** — *Let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure and $\kappa \geq \max(|\mathcal{L}|, |M|)$ be a cardinal. There is an elementary extension $\mathcal{N}$ of $\mathcal{M}$ such that $|N| = \kappa$.*

Proof. — Let $\mathcal{L}(M)$ be the extension of $\mathcal{L}$ by a new constant symbol c_a for each element in $a \in M$. Let $\mathcal{M}_M$ be the $\mathcal{L}(M)$-structure in which each such constant symbol c_a is interpreted as a. Let C be a set of cardinality κ of yet new constant symbols. Consider the set of formulas

$$\Sigma = \mathrm{Th}(\mathcal{M}_M) \cup \{c \neq d : c, d \in C \text{ different constant symbols }\}.$$

It suffices to show that Σ is satisfiable. Indeed, if $\mathcal{N}$ is a model of Σ, the inclusion is an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$ since $\mathcal{N} \models Th(\mathcal{M}_M)$ (exercise). On the other hand, every model of Σ must have at least κ many distinct elements, those of which are the interpretation of the constant symbols from C. Let us show that Σ is satisfiable by compactness, that is, let us show that every finite subset Σ_0 of Σ has a model. The set Σ_0 can only contain a finite amount of constant symbols from C. Since M is infinite, we can always interpret those constants as different elements in $\mathcal{M}$. Therefore, $\mathcal{M}$ is always a model of Σ_0 once we interpret the constant appearing in Σ_0 by distinct elements of M. This gives us an elementary extension $\mathcal{N}$ of $\mathcal{M}$ of cardinality at least κ. To obtain an elementary extension of cardinality exactly κ we apply the downwards Löwenheim-Skolem theorem to $\mathcal{N}$. $\square$

4.1.2. Proposition. — *Let $\mathcal{C}$ be an elementary class of $\mathcal{L}$-structures. Suppose $\mathcal{C}$ has structures of arbitrarily large finite cardinality. Then $\mathcal{C}$ has infinite models.*

Proof. — Suppose for a contradiction that $\mathcal{C}$ has only finite models. Since $\mathcal{C}$ is an elementary class we have that $\mathcal{C} = \mathrm{Mod}(T)$ for some $\mathcal{L}$-theory T. Now consider the following set of $\mathcal{L}$-sentences:

$$\Sigma := T \cup \{\exists x_1 \cdots \exists x_n (\bigwedge_{i \neq j} x_i \neq x_j) : n \in \mathbb{N}^*\}.$$

For simplicity, let us denote the formula $\exists x_1 \cdots \exists x_n (\bigwedge_{i \neq j} x_i \neq x_j)$ by φ_n. We show by compactness that Σ is satisfiable. Let Σ_0 be a finite subset of Σ. Then there is some n_0 such that if $\varphi_n \in \Sigma_0$, then $n < n_0$. Now, since $\mathcal{C}$ has structures of arbitrarily large cardinality, let $\mathcal{M} \in \mathcal{C}$ be such that M has more than n_0 elements. Then clearly $\mathcal{M} \models \Sigma_0$. By compactness, there is an $\mathcal{L}$-structure $\mathcal{N} \models \Sigma$. Since $\mathcal{N} \models T$, then $\mathcal{N} \in \mathcal{C}$. But also $\mathcal{N} \models \varphi_n$ for all $n \in \mathbb{N}$, so $\mathcal{N}$ is infinite, a contradiction. $\square$

4.1.3. Proposition. — *The following theories are not finitely axiomatizable.*

1. *the theory T_∞ of the infinite set.*
2. *The theory of torsion-free abelian groups.*
3. *the theory of algebraically closed fields.*

Proof. — We sketch the idea for the first one (the proof of the other two is similar, using a bit of algebra). Suppose T_∞ was finite axiomatizable, say by a finite set of $\mathcal{L}_\emptyset$-sentences Σ. In particular, each sentence in Σ must be a logical consequence of T, that is $T \models \Sigma_0$. By compactness (second version), there is a finite subset of T_∞, say T_0, such that $T_0 \models \Sigma$. This implies that T_∞ is already axiomatized by T_0. Now, since T_0 is finite, there is some n_0 such that T_0 is a subset of

$$\{\exists x_1 \cdots \exists x_n (\bigwedge_{i \neq j} x_i \neq x_j) : n < n_0\}.$$

But then, any finite set of cardinality bigger than n_0 is a model of T_0, and hence of T_∞, a contradiction. □

***4.1.4. Theorem* (Lefschetz Principle).** — *Let φ be an $\mathcal{L}_{\mathrm{ring}}$-sentence. The following are equivalent*

1. $\mathrm{ACF}_0 \models \varphi$;
2. $(\mathbb{C}, \cdot, +, -, 0, 1) \models \varphi$;
3. $\mathrm{ACF}_p \models \varphi$ *for all but finitely many primes p;*
4. $\overline{\mathbb{F}_p}^a \models \varphi$ *for all but finitely many primes p.*

Proof. — (1) → (2) This follow from the fundamental theorem of algebra as $\mathbb{C}$ is algebraically closed.

(2) → (3) Assume (2). Then, since ACF_0 is complete, we have that $\mathrm{ACF}_0 \models \varphi$. Now by compactness, there is a finite subset Σ of ACF_0 such that $\Sigma_0 \models \varphi$. Now looking at the axioms, this means that for all but finitely many primes ACF_p contains Σ. Therefore, $\mathrm{ACF}_p \models \varphi$ for all but finitely many primes.

(3) → (4) This follows again because of the completeness of ACF_p.

(4) → (3) We show the contrapositive, so suppose $\mathrm{ACF}_0 \models \neg\varphi$. The same argument as in (1) → (2) → (3), shows that $\mathrm{ACF}_p \models \neg\varphi$ for all but finitely many primes. This shows the result. □

4.2. Filters, ultrafilters and ultraproducts. —

4.2.1. Definition. — Let X be a non empty set. A *filter on* X is a non-empty family $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying the following properties:

(F1) $\emptyset \notin \mathcal{F}$;
(F2) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
(F3) if $A \in \mathcal{F}$ and $A \subseteq B$ for $B \in \mathcal{F}$, then $B \in \mathcal{F}$.

4.2.2. Examples. —

1. The *trivial filter* on X is $\mathcal{F} = \{X\}$.
2. For $A \in \mathcal{P}(X)$ non-empty, the filter generated by A is the filter $\mathcal{F}_A := \{B \in \mathcal{P}(X) : A \subseteq B\}$. When $A = \{a\}$, we called such filter *principal* and denote it by $\mathcal{F}_a$.
3. For X infinite, *Fréchet's filter* on X, denoted $\mathcal{F}_{Fr}$, is the family of cofinite subsets of X. Note that this filter is not generated by a subset of X.

It is easy to check that every filter $\mathcal{F}$ on X satisfies the *finite intersection property*:

(FIP) Every finite intersection of elements in $\mathcal{F}$ is non-empty.

One can ask if a subset $S \subseteq \mathcal{P}(X)$ can generate a filter. Let $\mathcal{F}(S)$ denote the smallest family containing S which is closed by conditions (F2) and (F3). Note that

$$\mathcal{F}(S) := \bigcup_{A \in S'} \mathcal{F}_A,$$

where S' is the closure of S under finite intersections.

4.2.3. Lemma. — *Let $S \subseteq \mathcal{P}(X)$. Then $\mathcal{F}(S)$ is a filter if and only if S has the FIP.*

Proof. — From right-to-left, the result is trivial as every filter has the FIP and $\mathcal{F}(S)$ contains S. From left-to-right, the FIP implies that $\emptyset \notin \mathcal{F}(S)$ and by construction $\mathcal{F}(S)$ satisfies conditions (F2) and (F3). □

Hereafter, X will denote an infinite set and all filters will be filters on X unless otherwise stated.

4.2.4. Definition. — A filter $\mathcal{F}$ is an *ultrafilter* if it is maximal with respect to inclusion.

The following fact is left as an exercise:

4.2.5. Fact. —

1. *Every principal filter is an ultrafilter.*
2. *If $\mathcal{F}$ is an non-principal ultrafilter, then it contains Fréchet's filter.*

Existence of ultrafilters is equivalent to the following weak version of the axiom of choice.

Ultrafilter's Lemma. *Every filter $\mathcal{F}$ is contained in an ultrafilter.* □

4.2.6. Proposition. — *Let $\mathcal{F}$ be a filter. The following are equivalent:*

1. *$\mathcal{F}$ is an ultrafilter;*
2. *if $A \cup B \in \mathcal{F}$, then $A \in \mathcal{F}$ or $B \in \mathcal{F}$;*
3. *for every $A \subseteq X$, $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.*

Proof. —

(1) $\Rightarrow$ (2): we show the contrapositive, so suppose there are $A, B \subseteq X$ such that $A \cup B \in \mathcal{F}$ but neither A nor B belong to $\mathcal{F}$. It suffices to show that either $S_1 = \mathcal{F} \cup \{A\}$ or $S_2 = \mathcal{F} \cup \{A\}$ have the FIP. Indeed, if one of them has the FIP, then by Lemma 4.2.3 and the Ultrafilter's Lemma, there would be an ultrafilter containing S_i, which shows that $\mathcal{F}$ is not an ultrafilter. Suppose for a contradiction that neither S_1 nor S_2 has the FIP. Then, there are $C_1, C_2 \in \mathcal{F}$ such that $A \cap C_1 = \emptyset$ and $B \cap C_2 = \emptyset$. But $C = C_1 \cap C_2 \in \mathcal{F}$ and $(A \cup B) \cap C = \emptyset$, which contradicts that $A \cup B \in \mathcal{F}$.
(2) $\Rightarrow$ (3): this follows directly since $A \cup (X \setminus A) = X \in \mathcal{F}$.
(3) $\Rightarrow$ (1): we show the contrapositive, so suppose that there is an filter $\mathcal{G}$ strictly containing $\mathcal{F}$. Consider $A \in \mathcal{G} \setminus \mathcal{F}$. We cannot have $X \setminus A \in G$ since $\mathcal{G}$ is a filter, hence $X \setminus A \notin \mathcal{F}$. This shows that neither A nor $X \setminus A$ belong to $\mathcal{F}$, but their union is X which belong to every filter. □

4.2.7. *Ultraproducts.* —

4.2.8. Definition. — Let $(\mathcal{M}_i)_{i\in I}$ be a family of $\mathcal{L}$-structures. We define $\mathcal{M}$ the *direct product structure* on the cartesian product $M = \prod_{i\in I} M_i$ as follows:

- for a function symbol f of arity n, and $a_1, \ldots, a_n \in M$
$$f^{\mathcal{M}}(a_1, \ldots, a_n) = (f^{\mathcal{M}_i}(a_1(i), \ldots, a_n(i)))_{i\in I}$$
- for a relation symbol R of arity n and $a_1, \ldots, a_n \in M$
$$(a_1, \ldots, a_n) \in R^{\mathcal{M}} \Leftrightarrow (a_1(i), \ldots, a_n(i)) \in R^{\mathcal{M}_i} \text{ for all } i \in I.$$

We denote the structure $\mathcal{M}$ by $\prod_{i\in I} \mathcal{M}_i$.

4.2.9. Lemma. — *Let $(\mathcal{M}_i)_{i\in I}$ be a family of $\mathcal{L}$-structures.*

1. *Let $\varphi(x)$ be an atomic formula with $|x| = n$ and $a_1, \ldots, a_n \in \prod_i M_i$. Then,*
$$\prod_{i\in I} \mathcal{M}_i \models \varphi(a_1, \ldots, a_n) \Leftrightarrow \mathcal{M}_i \models \varphi(a_1(i), \ldots, a_n(i)) \textit{ for all } i \in I.$$
2. *For each $i \in I$, the projection $\prod_{i\in I} \mathcal{M}_i \to \mathcal{M}_i$ preserves all atomic formulas.*

Proof. — This is an easy exercise. □

4.2.10. Lemma. — *Let I be a non-empty set, $\mathcal{F}$ a filter on I and $(M_i)_{i\in I}$ a family of non-empty sets. The relation on $\prod M_i$ defined by*
$$a \sim_{\mathcal{F}} b \Leftrightarrow \{i \in I : a(i) = b(i)\} \in \mathcal{F},$$
is an equivalence relation.

Proof. — It is reflexive since the set I belongs to every filter on I. It is trivially symmetric. for transitivity, suppose $a \sim_{\mathcal{F}} b$ and $b \sim_{\mathcal{F}} c$. Then, $A = \{i \in I : a(i) = b(i)\} \in \mathcal{F}$ and $B = \{i \in I : b(i) = c(i)\} \in \mathcal{F}$. Since $A \cap B \in \mathcal{F}$ and $A \cap B \subseteq \{i \in I : a(i) = c(i)\}$, this shows that $a \sim_{\mathcal{F}} c$. □

For $a \in \prod M_i$ we denote $a/\mathcal{F}$ its equivalence class by $\sim_{\mathcal{F}}$. Given $a = (a_1, \ldots, a_n) \in (\prod_{i\in I} \mathcal{M}_i)$ we write $a/\mathcal{F}$ for the tuple $((a_1)/\mathcal{F}, \ldots, (a_n)/\mathcal{F})$. Similarly, for $i \in I$, $a(i)$ will denote the tuple $(a_1(i), \ldots, a_n(i))$.

4.2.11. Definition. — Let I be non-empty, $\mathcal{F}$ be a filter on I and $(\mathcal{M}_i)_{i\in I}$ be a family of $\mathcal{L}$-structures. We define the *reduced product over* $\mathcal{F}$ $\mathcal{M}$ on the set $\prod_{i\in I} M_i/\sim_{\mathcal{F}}$ by :

- for a function symbol f of arity n, and $a_1, \ldots, a_n \in M$

$$f^{\mathcal{M}}(a/\mathcal{F}) = (f^{\mathcal{M}_i}(a(i)))/\mathcal{F}$$

- for a relation symbol R of arity n and $a_1, \ldots, a_n \in M$

$$a/\mathcal{F} \in R^{\mathcal{M}} \Leftrightarrow \{i \in I : a(i) \in R^{\mathcal{M}_i}\} \in \mathcal{F}.$$

We denote $\mathcal{M}$ by $\prod_{i\in I} \mathcal{M}_i/\mathcal{F}$ or $\prod_{\mathcal{F}} \mathcal{M}_i$. When $\mathcal{F}$ is an ultrafilter, we call the reduced product an *ultraproduct.*

4.2.12. Lemma. — *The reduced product is well-defined.*

Proof. — We need to show that the previous definition does not depend on the representatives. Let $a, b \in (\prod_{i\in I} \mathcal{M}_i)^n$ such that $a_j \sim_{\mathcal{F}} b_j$ pour $j = 1, \ldots, n$. Then, $A_j = \{i \in I : a_j(i) = b_j(i)\} \in \mathcal{F}$ for each $j = 1, \ldots, n$. Their intersection $\bigcap_{j=1}^n A_j$ is also an element of $\mathcal{F}$, and therefore:

$$\bigcap_{j=1}^{n} A_j \subseteq \{i \in I : a_j(i) = b_j(i), \text{ pour tout } j \in \{1, \ldots, n\}\} \in \mathcal{F}.$$

This shows that

$$\begin{aligned} f^{\mathcal{M}}((a_1)_{\mathcal{F}}, \ldots, (a_n)_{\mathcal{F}}) &= (f^{\mathcal{M}_i}(a_1(i), \ldots, a_n(i)))_{\mathcal{F}} \\ &= (f^{\mathcal{M}_i}(b_1(i), \ldots, b_n(i)))_{\mathcal{F}} \\ &= f^{\mathcal{M}}((b_1)_{\mathcal{F}}, \ldots, (b_n)_{\mathcal{F}}), \end{aligned}$$

so $f^{\mathcal{M}}$ is well-defined. Similarly,

$$\begin{aligned} ((a_1)_{\mathcal{F}}, \ldots, (a_n))_{\mathcal{F}}) \in R^{\mathcal{M}} &\Leftrightarrow \{i \in I : (a_1(i), \ldots, a_n(i)) \in R^{\mathcal{M}_i}\} \in \mathcal{F} \\ &\Leftrightarrow \{i \in I : (b_1(i), \ldots, b_n(i)) \in R^{\mathcal{M}_i}\} \in \mathcal{F} \\ &\Leftrightarrow ((b_1)_{\mathcal{F}}, \ldots, (b_n))_{\mathcal{F}}) \in R^{\mathcal{M}}. \end{aligned}$$

$\square$

4.2.13. **Lemma.** — *Let I, $\mathcal{F}$ and $(\mathcal{M}_i)_{i\in I}$ as in the previous definition, $\mathcal{M} = \prod_{i\in I}\mathcal{M}_i/\mathcal{F}$ the reduced product, $t(x)$ a term with $|x| = n$ and $a = (a_1, \dots, a_n) \in (\prod_{i\in I} M_i)^n$. Then,*

$$t^{\mathcal{M}}(a/\mathcal{F}) = (t^{\mathcal{M}_i}(a(i)))/\mathcal{F}.$$

Proof. — This is a routine argument by induction on terms. □

4.2.14. **Lemma.** — *Let I, $\mathcal{F}$ and $(\mathcal{M}_i)_{i\in I}$ as in the previous definition, $\varphi(x)$ an atomic formula with $|x| = n$ and $a = (a_1, \dots, a_n) \in (\prod_{i\in I} M_i)^n$. Then,*

$$\prod_{i\in I}\mathcal{M}_i/\mathcal{F} \models \varphi(a/\mathcal{F}) \Leftrightarrow \{i \in I : \mathcal{M}_i \models \varphi(a(i))\} \in \mathcal{F}.$$

Proof. — Cases $\top$ and $\perp$ are trivial. If $\varphi(x)$ is of the form $Rx_1, \dots x_n$, the result follows by definition. We are left with the case when $\varphi(x)$ is of the form $t_1(x) = t_2(x)$ for t_1, t_2 $\mathcal{L}$-terms. By Lemma 4.2.13,

$$\begin{aligned}\prod_{i\in I}\mathcal{M}_i/\mathcal{F} \models t_1(a/\mathcal{F}) = t_2(a/\mathcal{F}) &\Leftrightarrow t_1^{\mathcal{M}}(a/\mathcal{F}) = t_2\mathcal{M}(a/\mathcal{F}) \\ &\Leftrightarrow (t_1^{\mathcal{M}_i}(a(i))/\mathcal{F} = (t_2^{\mathcal{M}_i}(a(i)))/\mathcal{F} \\ &\Leftrightarrow \{i \in I : \mathcal{M}_i \models t_1^{\mathcal{M}_i}(a(i)) = t_2^{\mathcal{M}_i}(a(i))\}.\end{aligned}$$

□

Using ultrafilters we can now generalize to arbitrary sentences.

4.2.15. **Theorem (Łoś).** — *Let I, $\mathcal{F}$ and $(\mathcal{M}_i)_{i\in I}$ as in the previous definition and suppose $\mathcal{F}$ is an ultrafilter. Let $\varphi(x)$ be a formula with $|x| = n$ and $a = (a_1, \dots, a_n) \in (\prod_{i\in I}\mathcal{M}_i)^n$. Then,*

$$\prod_{i\in I}\mathcal{M}_i/\mathcal{F} \models \varphi(a/\mathcal{F}) \Leftrightarrow \{i \in I : \mathcal{M}_i \models \varphi(a(i))\} \in \mathcal{F}.$$

In particular, if φ is a sentence,

$$\prod_{i\in I}\mathcal{M}_i/\mathcal{F} \models \varphi \Leftrightarrow \{i \in I : \mathcal{M}_i \models \varphi\} \in \mathcal{F}.$$

Proof. — We proceed by induction on $\varphi(x)$. The atomic case is already given by Lemma 4.2.14. Suppose $\varphi(x)$ is of the form $\neg\psi(x)$. Then, by Proposition 4.2.6 and the induction hypothesis

$$\begin{aligned}\prod_{i\in I}\mathcal{M}_i/\mathcal{F} \models \neg\varphi(a/\mathcal{F})) &\Leftrightarrow \prod_{i\in I}\mathcal{M}_i/\mathcal{F} \not\models \varphi(a/\mathcal{F}) \\ &\Leftrightarrow \{i \in I : \mathcal{M}_i \models \varphi(a(i))\} \notin \mathcal{F}. \\ &\Leftrightarrow I \setminus \{i \in I : \mathcal{M}_i \models \varphi(a(i))\} \in \mathcal{F}. \\ &\Leftrightarrow \{i \in I : \mathcal{M}_i \models \neg\varphi(a(i))\} \in \mathcal{F}.\end{aligned}$$

Now suppose $\varphi(x)$ is of the form $(\psi \wedge \theta)(x)$. Here we use simply the fact that $\mathcal{F}$ is a filter and the proof is a routine exercise. It remains the case where $\varphi(x)$ is of the form $\exists y\psi(x,y)$. In this case,

$$\begin{aligned}\prod_{i\in I}\mathcal{M}_i/\mathcal{F} \models \exists y\psi(a/\mathcal{F}), y) &\Leftrightarrow \prod_{i\in I}\mathcal{M}_i/\mathcal{F} \models \psi(a/\mathcal{F}), b/\mathcal{F}) \text{ pour un } b \in \prod_{i\in I}\mathcal{M}_i \\ &\Leftrightarrow A = \{i \in I : \mathcal{M}_i \models \psi(a(i), b(i))\} \in \mathcal{F}. \\ &\Leftrightarrow A \subseteq \{i \in I : \mathcal{M}_i \models \exists y\psi(a(i), y)\} \in \mathcal{F}.\end{aligned}$$

□

Proof of the Compactness Theorem:. — Let I be the set of finite subsets of Σ. By hypothesis, for every $\Delta \in I$, there is an $\mathcal{L}$-structure $\mathcal{M}_\Delta$ such that $\mathcal{M}_\Delta \models \Delta$. For each sentence $\varphi \in \Sigma$, let

$$U_\varphi \coloneqq \{\Delta \in I : \varphi \in \Delta\}.$$

Take $S = \{U_\varphi : \varphi \in \Sigma\}$. Let us show that S has the FIP. Indeed, if $\varphi_1, \ldots, \varphi_n \in \Sigma$, then for each $1 \leq i \leq n$

$$\{\varphi_1, \ldots, \varphi_n\} \in U_{\varphi_i}$$

since it contains φ_i, and hence

$$\{\varphi_1, \ldots, \varphi_n\} \in U_{\varphi_1} \cap \cdots \cap U_{\varphi_n}.$$

This shows that S has the FIP. By Lemma 4.2.3, $\mathcal{F}(S)$ is a filter and by the Ultrafilter's Lemma there is $\mathcal{F}$ an ultrafilter containing $\mathcal{F}(S)$, and hence containing S. Let us show that $\mathcal{M} \coloneqq \prod_{\Delta\in I}\mathcal{M}_\Delta/\mathcal{F}$ is a model of Σ. For $\varphi \in \Sigma$ and the fact that $\mathcal{M}_\Delta \models \Delta$,

$$U_\varphi = \{\Delta \in I : \varphi \in \Delta\} \subseteq \{\Delta \in I : \mathcal{M}_\Delta \models \varphi\}.$$

Since $U_\varphi \in \mathcal{F}$, then $\{\Delta \in I : \mathcal{M}_\Delta \models \varphi\} \in \mathcal{F}$, and by Łoś theorem we obtain that $\mathcal{M} \models \varphi$. □

5. Quantifier elimination

5.0.1. Definition. — Let T be a satisfiable $\mathcal{L}$-theory. We say two $\mathcal{L}$-formulas $\varphi(x)$ and $\psi(x)$ are equivalent modulo T if

$$T \models \forall x(\varphi(x) \leftrightarrow \psi(x)).$$

If $\mathcal{M}$ is an $\mathcal{L}$-structure, we say that $\varphi(x)$ and $\psi(x)$ are equivalent in $\mathcal{M}$ if

$$\mathcal{M} \models \forall x(\varphi(x) \leftrightarrow \psi(x)).$$

5.0.2. Definition. — Let T be a satisfiable $\mathcal{L}$-theory. We say T has quantifier elimination (in short QE) if every $\mathcal{L}$-formula is equivalent modulo T to a quantifier free $\mathcal{L}$-formula.

5.0.3. Remark. — Note that if $\mathcal{L}$ has no constant symbols, the only quantifier free formulas are $\top$ and $\bot$. Thus, if T has quantifier elimination it has to be complete.

5.0.4. Proposition. — *Suppose T has quantifier elimination and let $\mathcal{M}$ and $\mathcal{N}$ be models of T. Then*

1. *if $\mathcal{A}$ be a common substructure of both $\mathcal{M}$ and $\mathcal{N}$, then for every $\mathcal{L}$-formula $\varphi(x)$ and every $a \in A^{|x|}$,*

$$\mathcal{M} \models \varphi(a) \Leftrightarrow \mathcal{N} \models \varphi(a).$$

   In other words, $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent in $\mathcal{L}(A)$.

2. *if $\mathcal{M} \subseteq \mathcal{N}$ then $\mathcal{M} \prec \mathcal{N}$.*

Proof. — For (1), suppose that $\mathcal{M} \models \varphi(a)$. Since T has quantifier elimination, let $\psi(x)$ be a quantifier-free $\mathcal{L}$-formula such that $T \models \forall x(\varphi(x) \leftrightarrow \psi(x)$. Thus, $\mathcal{M} \models \psi(a)$, since $\mathcal{M}$ is a model of T. By Lemma 2.2.1, this implies that $\mathcal{A} \models \psi(a)$, and again by the same lemma, $\mathcal{N} \models \psi(a)$. Since $\mathcal{N}$ is a model of T, we conclude that $\mathcal{N} \models \varphi(a)$.

For (2), we use Tarski-Vaught (Proposition 2.2.4), so let $\varphi(x, y)$ be a quantifier-free $\mathcal{L}$-formula with y a single variable and suppose $\mathcal{N} \models \exists y \varphi(a, b)$ for $a \in M^{|x|}$. By quantifier elimination, there is an $\mathcal{L}$-formula $\psi(x)$ such that $T \models \forall x(\exists y(\varphi(x, y)) \leftrightarrow \psi(x))$. Then, $\mathcal{N} \models \psi(a)$. By Lemma 2.2.1, $\mathcal{M} \models \psi(a)$, and therefore, since $\mathcal{M}$ is a model of T, $\mathcal{M} \models \exists y \varphi(a, y)$. Then there is some $b \in M$ such that $\mathcal{M} \models \varphi(a, b)$, and since φ was quantifier-free, again by Lemma 2.2.1 we have that $\mathcal{N} \models \varphi(a, b)$. $\square$

5.0.5. Lemma. — *Let T be a satisfiable $\mathcal{L}$-theory. Suppose that for every quantifier free formula $\varphi(x, y)$ with $|y| = 1$, there is a quantifier free formula $\psi(x)$ such that*

$$T \models \forall x(\exists y \varphi(x, y) \leftrightarrow \psi(x)).$$

Then T has quantifier elimination.

Proof. — We show that for every $\mathcal{L}$-formula $\varphi(x)$ there is a quantifier-free $\mathcal{L}$-formula $\psi(x)$ such that $T \models \forall x(\varphi(x) \leftrightarrow \psi(x))$ by induction on formulas. If $\varphi(x)$ is an atomic formula, then it is quantifier-free and there is nothing to show. Suppose the result holds for $\theta_1(x), \theta_2(x)$ and for $\theta(x, y)$. We have three cases:

Case 1: if $\varphi(x)$ is $\neg\theta_1(x)$, then by induction hypothesis there is $\psi_1(x)$ quantifier-free such that $T \models \forall x(\theta_1(x) \leftrightarrow \psi_1(x))$. Then setting $\psi(x)$ to be $\neg\psi_1(x)$ shows the result.

Case 2: if $\varphi(x)$ is $\theta_1(x) \wedge \theta_2(x)$, then by induction hypothesis there are $\psi_i(x)$ quantifier-free for $i = 1, 2$ such that $T \models \forall x(\theta_i(x) \leftrightarrow \psi_i(x))$. Then setting $\psi(x)$ to be $\psi_1(x) \wedge \psi_2(x)$ shows the result.

Case 3: if $\varphi(x)$ is $\exists y \theta(x, y)$, then by induction hypothesis there is $\xi(x, y)$ quantifier-free such that $T \models \forall x \forall y(theta(x, y) \leftrightarrow \xi(x, y))$. Therefore, $\varphi(x)$ is equivalent modulo T to $\exists y \xi(x, y)$, and the result follows by the assumption of the lemma. $\square$

We will use the following model-theoretic criterion for quantifier elimination.

5.0.6. Proposition. — *Let T be a satisfiable $\mathcal{L}$-theory. The following are equivalent:*

1. *T has quantifier elimination.*
2. *Let $\mathcal{M}, \mathcal{N}$ be models of of T and $\mathcal{A}$ be a common substructure. Let $x = (x_1, \ldots, x_n)$ be a tuple of variables, y be a single variable and $\varphi(x, y)$ be a quantifier-free $\mathcal{L}$-formula. If there are $a \in A^n$ and $b \in N$ such that $\mathcal{N} \models \varphi(a, b)$, then there is $b' \in M$ such that $\mathcal{M} \models \varphi(a, b')$.*

Proof. — The implication from (1) to (2) follows from Proposition 5.0.4 part (i). For the converse, assume (2) holds. By Lemma 5.0.5, it suffices to show that given a quantifier-free $\mathcal{L}$-formula $\varphi(x, y)$ with $|y| = 1$, the formula $\exists y \varphi(x, y)$ is equivalent modulo T to a quantifier-free $\mathcal{L}$-formula. We split in three cases:

Case 1: Suppose that $T \models \forall x \exists \varphi(x, y)$. In this case, $\exists y \varphi(x, y)$ is equivalent modulo T to $\top$.

Case 2: Suppose that $T \models \forall x \neg \exists \varphi(x, y)$. In this case, $\exists y \varphi(x, y)$ is equivalent modulo T to $\perp$.

Case 3: Suppose that both $T \cup \{\exists \varphi(x, y)\}$ and $T \cup \{\neg \exists \varphi(x, y)\}$ are satisfiable. Consider the set of $\mathcal{L}$-formulas

$$\Sigma(x) \coloneqq \{\psi(x) : \psi(x) \text{ is quantifier-free and } T \models \forall x(\exists \varphi(x, y) \leftrightarrow \psi(x))\}.$$

Let $c = (c_1, \ldots, c_n)$ be an n-tuple of new constant symbols and consider the set of $\mathcal{L}(c)$-sentences $T \cup \Sigma(c)$.

5.0.7. Claim. — *It holds that $T \cup \Sigma(c) \models \exists y \varphi(c, y)$.*

Let us complete the proof assuming the claim holds. By compactness, there are $\psi_1, \ldots, \psi_k(x) \in \Sigma(x)$, such that

$$T \models \forall x (\bigwedge_{i=1}^{k} \psi_k(x) \to \exists y \varphi(x, y).$$

But since each $\psi_i(x) \in \Sigma(x)$, it also holds that

$$T \models \forall x (\bigwedge_{i=1}^{k} \psi_k(x) \leftrightarrow \exists y \varphi(x, y).$$

Since the conjuction is quantifier-free, we have the result.

Suppose for a contradiction the claim does not hold, this means that there is an $\mathcal{L}(c)$-structure $\mathcal{M}$, such that $\mathcal{M} \models T \cup \Sigma(c) \cup \neg \exists \varphi(c, y)$. Let $\mathcal{A}$ be the substructure generated by c. Note that $\mathcal{A} \models \Sigma(c)$. Consider the set

$$T_0 \coloneqq \{\theta(c) : \mathcal{A} \models \theta(c), \theta(c) \text{ quantifier-free}\}.$$

We consider one last set of formulas, in the language $\mathcal{L}(c)$

$$H \coloneqq T \cup T_0 \cup \{\exists y \varphi(c, y)\}.$$

Let us first show that H is satisfiable. For if not, by compactness, there are $\theta_1(c), \ldots, \theta_\ell(x) \in T_0$ such that

$$T \models \forall x(\bigwedge_{i=1}^{\ell} \theta_i(c) \to \neg\exists\varphi(c,y)),$$

which is equivalent to

$$T \models \forall x(\exists\varphi(c,y) \to \bigvee_{i=1}^{\ell} \neg\theta_i(c)).$$

Since $\theta(x)$ is quantifier-free, this shows that $\bigvee_{i=1}^{\ell} \neg\theta_i(x)$ belongs to $\Sigma(x)$, which contradicts that $\mathcal{A} \models \Sigma(c)$. Threrefore, H is satisfiable. Let $\mathcal{N}$ be a model of H. Without loss of generality, we may assume that $\mathcal{A}$ is a substructure of $\mathcal{N}$ since $\mathcal{N} \models T_0$. Now since $\mathcal{N} \models \exists y\varphi(c,y)$, there is $b \in N$ such that $\mathcal{N} \models \varphi(c,b)$. By the assumption this implies that there is $b' \in M$ such that $\mathcal{M} \models \varphi(c,b')$, a contradiction. □

5.0.8. Examples. — The following theories have quantifier elimination:

1. The theory DLO of dense linear orders without endpoints.
2. The theory DOAG of divisible ordered abelian groups.
3. The theory ACF_p of algebraically closed fields of characteristic p.
4. The theory of the real field in the language of ordered rings, $\mathrm{Th}(\mathbb{R}, \leqslant, \cdot, +, -, 0, 1)$.
5. The theory of the p-adic field $\mathbb{Q}_p$ in the following language $(\mathbb{Q}_p, \cdot, +, -, 0, 1, (P_n)_{n>0})$, where P_n is a unary relation symbol interpreted as the set of n-powers
$$P_n \coloneqq \{x \in \mathbb{Q}_p : \exists y x = y^n\}.$$

As an application we can provide a proof of the Nullstellensatz (usually referred as weak Nullstellensatz).

5.0.9. Theorem. — *Let K be any algebraically closed field. For $x = (x_1, \ldots, x_n)$, let $f_1(x), \ldots, f_m \in K[x]$ be such that $(f_1, \ldots, f_n) \neq K[x]$. Then $f_1, \ldots, f_n$ have a common zero in K.*

Proof. — Let I be a maximal ideal containing $(f_1, \ldots, f_m)$. By Kronecker's extension theorem, the extension $F \coloneqq K[x]/I$ contains a common zero of $f_1, \ldots, f_n$. Let L be an algebraic closure of F. Clearly, L also contains a common zero of $f_1, \ldots, f_n$. But this is expressible by a formulas in the language of rings, namely,

$$L \models \exists x_1 \cdots \exists x_n(\bigwedge_{i=1}^{m} f_i(x) = 0).$$

Now $K \subseteq L$ and they are both algebraically closed fields. By quantifier elimination of ACF and Proposition 5.0.4, we have that $K \prec L$. Therefore,

$$K \models \exists x_1 \cdots \exists x_n(\bigwedge_{i=1}^{m} f_i(x) = 0).$$

$\square$

6. A brief introduction to o-minimality

6.1. O-minimal structures. — Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Recall that a subset $X \subseteq M^n$ is called $\emptyset$-*definable* if there is an $\mathcal{L}$-formula $\varphi(x)$ with $x = (x_1, \ldots, x_n)$ such that

$$X = \varphi(M) \coloneqq \{a \in M^n : \mathcal{M} \models \varphi(a)\}.$$

For $A \subseteq M$, the set X is A-*definable* if there is $\mathcal{L}$-formula $\psi(x, y)$ and $b \in A^{|y|}$ such that

$$X = \psi(M, b) \coloneqq \{a \in M^n : \mathcal{M} \models \psi(a, b)\}.$$

We may sometimes say $\mathcal{L}$-definable for $\emptyset$-definable subsets, and $\mathcal{L}(A)$-definable for A-definable subsets. When A is non-empty, we also say that X is *definable with parameters from* A. Some books use definable for M-definable. A function $f\colon M^n \to M^m$ is said to be definable if its graph is definable.

6.1.1. Definition. — Let $\mathcal{L}$ be a language containing $\{\leqslant\}$. An $\mathcal{L}$-structure $\mathcal{M}$ is *o-minimal* if $\leqslant$ is interpreted as a dense total order without endpoints on M and every $\mathcal{L}$-definable (possibly with parameters) subset $X \subseteq M$ is a finite union of intervals and points. An $\mathcal{L}$-theory T is o-minimal if every model of T is o-minimal.

6.1.2. Examples. —

1. The structures $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ are o-minimal. In fact, by quantifier elimination of DLO, every dense linear order without endpoints is o-minimal.
2. The structures $(\mathbb{Q}, <, +, 0)$ and $(\mathbb{R}, <, +, 0)$ are o-minimal. This follows by quantifier elimination of DOAG
3. The structure $(\mathbb{R}, <, +, \cdot, 0, 1)$ is o-minimal. This follows by quantifier elimination (more in the next section).
4. The structure $(\mathbb{R}, <, \mathbb{Z}, +, \cdot, 0, 1)$ is not o-minimal.
5. The structure $(\mathbb{R}, <, \sin, +, \cdot, 0, 1)$ is not o-minimal.
6. The structure $(\mathbb{Q}, <, +, \cdot, 0, 1)$ is not o-minimal (look at the set of squares).

***6.1.3. Theorem* (Pillay-Steinhorn-Knight).** — *Let $\mathcal{R}$ be an o-minimal structure. If $\mathcal{R} \equiv \mathcal{R}'$, then $\mathcal{R}'$ is also o-minimal.*

6.2. Semi-algebraic sets and quantifier elimination. —

6.2.1. Definition. — Let R be an ordered ring. A semi-algebraic subset of R^n is a finite union of sets of the form

$$\{x \in R^n : f(x) = 0, g_1(x) > 0, \ldots, g_k(x) > 0\},$$

with $f, g_1, \ldots, g_k \in R[x_1, ..., x_n]$.

Given this definition, most of the time it is difficult to determine whether a set is semi-algebraic or not. For example, if $X \subseteq \mathbb{R}^n$ is semi-algebraic, is the closure $\overline{X}$ also semi-algebraic? The following theorem will help us in giving an alternative description of semi-algebraic sets.

6.2.2. Definition. — An ordered field R is a *real closed field* if it satisfies the intermediate value theorem for polynomials (i.e., given a polynomial $P(x)$, if $a < b$ and $P(a) < c < P(b)$, then there is $u \in R$ such that $P(u) = c$.).

***6.2.3. Theorem* (Tarski-Seidenberg).** —

1. *Let R be a real closed field. The collection of all semi-algebraic sets on R is closed under projections.*
2. *The theory of real closed fields* RCF *has quantifier elimination in the language $\mathcal{L}_{\mathrm{or}}$ of ordered rings.*
3. *A subset $X \subseteq \mathbb{R}^n$ is semi-algebraic if and only if it is $\mathcal{L}_{\mathrm{or}}$-definable (with parameters).*

As a consequence of the previous result, we obtain that every real closed field is o-minimal. In particular, the field of real numbers is an o-minimal structure.

What other structures on the reals are o-minimal?

***6.2.4. Theorem* (Wilkie).** — *The structure $\mathbb{R}_{\exp} := (\mathbb{R}, \exp, \cdot, +, -, 0, 1)$ is o-minimal.*

Given a power series $f \in \mathbb{R}[\![x_1, \dots, x_n]\!]$ with radius of convergence bigger than 1, its associated restricted analytic function $\tilde{f} \colon \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\tilde{f}(a) := \begin{cases} f(a) & a \in [-1,1]^n \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{F}_n$ denote the set of all power series in $\mathbb{R}[\![x_1, \dots, x_n]\!]$ with radius of convergence bigger than 1 and $\mathcal{F} := \bigcup_{n>0} \mathcal{F}_n$.

***6.2.5. Theorem* (Denef-van den Dries).** — *The structure*

$$\mathbb{R}_{\mathrm{an}} := (\mathbb{R}, \cdot, +, -, 0, 1, (\tilde{f})_{f \in \mathcal{F}})$$

is o-minimal.

***6.2.6. Theorem* (van den Dries-Miller).** — *The structure*

$$\mathbb{R}_{\mathrm{an},\exp} := (\mathbb{R}, \exp, \cdot, +, -, 0, 1, (\tilde{f})_{f \in \mathcal{F}})$$

is o-minimal.

6.3. Monotonicity theorem and cell-decomposition. — Let $\mathcal{R} = (R, <, \cdots)$ be an o-minimal structure. A function $f: (a, b) \to R$ (a possibly $-\infty$ and b possibly $+\infty$) is said to be *strictly monotone* if it is either constant, strictly increasing or strictly decreasing.

***6.3.1. Theorem* (Monotonicity).** — *Let $f: (a, b) \to R$ be a definable function. Then there exist elements $a_0 = a < a_1 < \cdots < a_k = b$ in R such that $f|(a_i, , a_{i+1})$ is continuous and strictly monotone.*

Note that this shows that the behaviour at infinity of every definable function $f: R \to R$ is either constant, tends to $+\infty$ or to $-\infty$. Similarly, every continuous definable function $f: [a, b] \to R$ attains a maximum and minimum on $[a, b]$.

Let $X \subseteq R^n$ be a definable set and $f, g: X \to R$ be two definable continuous functions or the constant function $-\infty$ or $+\infty$. Suppose moreover that $f < g$, i.e., $f(x) < g(x)$ for all $x \in X$. We let

$$(f, g)_X := \{(x, y) \in X \times R : f(x) < y < g(x)\}.$$

Let us now define what cells are.

6.3.2. Definition. — We inductively define $(i_1, \ldots, i_n)$-cells of R^n, where $i_j \in 0, 1$, as follows:

- a (0)-cell of R is a singleton $\{a\} \subseteq R$ and a (1)-cell of R is an open interval (a, b) of R (possibly $(-\infty, +\infty)$;
- assuming that $(i_1, \ldots, i_n)$-cells of R^n have been defined, a $(i_1, \ldots, i_n, 0)$-cell of R^{n+1} is the graph of a continuous definable function $f: C \to R$ where C is a $(i_1, \ldots, i_n)$-cell of R^n, and a $(i_1, \ldots, i_n, 1)$-cell of R^{n+1} is a set of the form $(f, g)_C$ where C is a $(i_1, \ldots, i_n)$-cell of R^n, and $f, g: C \to R$ are continuous definable functions (or the constant functions $-\infty, +\infty$) such that $f < g$.

A subset $X \subseteq R^n$ is a cell if it is a $(i_1, \ldots, , i_n)$-cell of R^n for some tuple $(i_1, \ldots, , i_n)$. Note that a $(1, \ldots, 1)$-cell is an open subset of R^n.

6.3.3. Remark. — One can show that cells are locally closed, and isomorphic by a coordinate projection to open cells. Moreover, if X is a $(i_1, \ldots, , i_n)$-cell, the sum of the digits in the tuple $(i_1, \ldots, , i_n)$ equals the topological dimension of X. Recall that the topological dimension of a subset $X \subseteq R^n$ is the minimal k for which there is a projection $\rho: R^n \to R^k$ such that $\rho(X)$ has non-empty interior.

A cell decomposition $\mathcal{C}$ of R^n is a finite partition of R^n where each set $C \in \mathcal{C}$ is a cell. Given subsets $X_1, \ldots, X_m \subseteq R^n$, we say that a cell decomposition $\mathcal{C}$ preserves $X_1, \ldots, X_m$ if for every $C \in \mathcal{C}$ either $C \subseteq X_i$ or $C \cap X_i = \emptyset$ for each $i \in \{1, \ldots, m\}$.

The following cell decomposition is one of the key theorems in o-minimality and shows the resemblance between semi-algebraic sets and definable sets in o-minimal structures.

***6.3.4. Theorem* (Cell decomposition).** — *For every $n > 0$ it holds that*

1. *for definable sets $X_1, \ldots, X_m \subseteq R^n$, there is a cell decomposition $\mathcal{C}$ which preserves $X_1, \ldots, X_m$;*
2. *for a definable set $X \subseteq R^n$ and a definable function $f\colon X \to R$, then there exists a cell decomposition $\mathcal{C}$ of R^n preserving X such that $f|C$ is continuous for each $C \in \mathcal{C}$ contained in X.*

Some consequences of cell decomposition:

1. for definable sets $X, Y \subseteq R^n$, $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$.
2. if $X \subseteq R^n$, $Y \subseteq R^m$ and $f\colon X \to Y$ is a definable bijection, then $\dim(X) = \dim(Y)$.
3. In every o-minimal structure over the reals, every definable set has finite many connected components.

References

[1] D. Marker – *Model theory*, Graduate Texts in Mathematics, vol. 217, Springer-Verlag, New York, 2002, An introduction.

[2] K. Tent & M. Ziegler – *A course in model theory*, Lecture Notes in Logic, vol. 40, Association for Symbolic Logic, La Jolla, CA; Cambridge University Press, Cambridge, 2012.

Pablo Cubides Kovacsics