AN ALGORITHM FOR FINDING A BASIS OF THE FIXED POINT SUBGROUP OF AN AUTOMORPHISM OF A FREE GROUP (A BRIEF SKETCH)

OLEG BOGOPOLSKI* AND OLGA MASLAKOVA†

Abstract. We prove that for any automorphism α of a free group F of finite rank, one can efficiently compute a basis of the fixed point subgroup $\mathrm{Fix}(\alpha)$.

Let F_n be the free group of finite rank n. For $\alpha \in \mathrm{Aut}(F_n)$ we set

$$\mathrm{Fix}(\alpha) = \{x \in F_n \,|\, \alpha(x) = x\}.$$

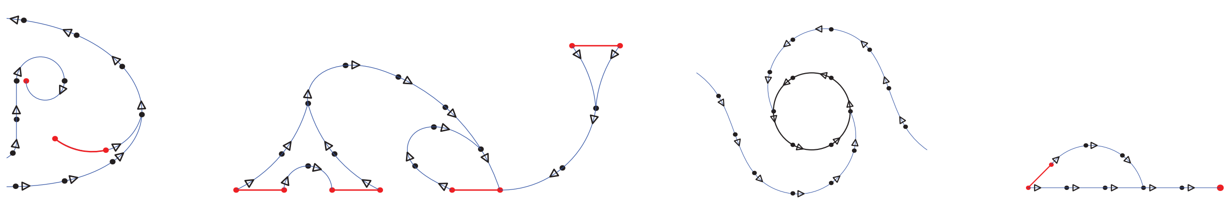
In [1], Bestvina and Handel proved the Scott conjecture that $\mathrm{rk}\,\mathrm{Fix}(\alpha) \leqslant n$. However, the problem of finding a basis of $\mathrm{Fix}(\alpha)$ has been open for almost 20 years. It has been solved in three special cases: for positive automorphisms [3], for special irreducible automorphisms [6, Prop. B], and for all automorphisms of F_2 [2]. Working on the PhD of the second author (2004), we solve this problem in general.

Theorem. Let F_n be the free group of finite rank n. There exists an efficient algorithm which, given an automorphism α of F_n, finds a basis of the fixed point subgroup $\mathrm{Fix}(\alpha)$.

Here we give a sketch of the proof (the complete proof can be found in the arxive). We assume that the reader is familiar with the relative train train technique from [1].

Step 1. Bestvina and Handel proved that the outer class of α can be realized by a relative train track. We prove that α itself can be algorithmically realized by a relative train. This means that one can construct a relative train track $f : (\Gamma, v) \to (\Gamma, v)$ and indicate an isomorphism $i : F \to \pi_1(\Gamma, v)$ such that $i^{-1}\alpha i = f_* : \pi_1(\Gamma, v) \to \pi_1(\Gamma, v)$.

Step 2. Let D_f be the graph of Goldstein and Turner associated with $f : \Gamma \to \Gamma$. This graph may contain infinitely many infinite components.



*Mathematisches Institut, Düsseldorf University, Düsseldorf, Germany
e-mail: *bogopolski@math.uni-duesseldorf.de*.

†Sobolev Institute of Mathematics of SB RAS, Novosibirsk, Russia.
e-mail: *o.s.maslakova@gmail.com*.

The fundamental group of one of the components of D_f (we denote it by $D_f(\mathbf{1}_v)$) can be identified with $\mathrm{Fix}(\alpha)$.

To construct the core of this component, we use two almost inverse flows $\mathcal{F}_1$ and $\mathcal{F}_2$ on D_f. Each vertex μ of D_f determines a μ-river, which starts at μ and follows the first flow as far as possible. The μ-river is either a finite simple path in D_f, or a finite simple path with a cycle at the end, or a ray. Especially important are the main rivers, i.e those rivers which start at the vertices of the so called repelling edges of D_f. Due to a simple observation of Cohen and Lustig, one can find these edges algorithmically.

We prove that if the component $D_f(\mathbf{1}_v)$ is noncontractible, then it is contained in the union of all main rivers together with the repelling edges. The following procedure (not an algorithm) to construct the core of the union is known:

(1) Compute repelling edges.
(2) For each repelling vertex μ determine, whether the μ-river is finite or not.
(3) Compute all elements of all finite μ-rivers from (2).
(4) For each two repelling vertices μ and τ with infinite μ-and τ-rivers determine, whether these rivers intersect.
(5) If the μ-river and the τ-river from (4) intersect, find their first intersection point and compute their initial segments up to this point.

$\boxed{\text{Step 3.}}$ We convert this procedure into an algorithm by finding algorithms which solve the following two problems:

Finiteness problem. Given a vertex μ in D_f, determine whether the μ-river is finite or not.

Membership problem. Given two vertices μ and τ in D_f, decide whether τ belongs to the μ-river.

References

[1] M. Bestvina, M. Handel, *Train tracks and automorphisms of free groups*, Ann. Math. (2), **135**, N 1 (1992), 1–53.

[2] O. Bogopolski, *Classification of automorphisms of the free group of rank 2 by ranks of fixed-point subgroups*, J. Group Theory, **3**, N 3 (2000), 339–351.

[3] M. M. Cohen, M. Lustig, *On the dynamics and the fixed subgroup of a free group automorphism*, Invent. Math., **96**, N 3 (1989), 613–638.

[4] R. Z. Goldstein, E. C. Turner, *Fixed subgroups of homomorphisms of free groups*, Bull. London Math. Soc., **18** (1986), 468–470.

[5] O. S. Maslakova, *Fixed point subgroup of an automorphism of a free group*, Ph.D Thesis (2004) (in Russian).

[6] E. C. Turner, *Finding indivisible Nielsen paths for a train tracks map*, Proc. of a workshop held at Heriot-Watt Univ., Edinburg, 1993 (Lond. Math. Soc. Lect. Note Ser., **204**), Cambridge, Cambridge Univ. Press., 1995, 300-313.