Lindström’s Theorem

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Theorem (Lindström)

There is no logic that is more expressive than classical first order logic and that satisfies both the Compactness and the Löwenheim-Skolem properties.

II. The proof

following Ebbinghaus/Flum/Thomas, Introduction to mathematical logic, Chapters XII/XIII
**1st outline of Lindström’s proof:** Let $\mathcal{L}$ be a regular logic satisfying LöSko($\mathcal{L}$) and Comp($\mathcal{L}$). Assume that $\mathcal{L}_{\omega\omega} < \mathcal{L}$.

Then there exists a $\psi \in L(S)$ not equivalent to any first order sentence.

1. Show that for all $m \in \mathbb{N}$ there exist $S$-structures $A, B$ with $A \models_{\mathcal{L}} \psi$, $B \models_{\mathcal{L}} \neg \psi$ and $A \approx_m B$.

2. Using Comp($\mathcal{L}$) we get $p$-isomorphic $S$-structures $A, B$ with $A \models_{\mathcal{L}} \psi$, $B \models_{\mathcal{L}} \neg \psi$.

3. By LöSko($\mathcal{L}$) we can assume w.l.o.g. that $A$ and $B$ are countable. Then we have $A \cong B$ but $A \models_{\mathcal{L}} \psi$, $B \models_{\mathcal{L}} \neg \psi$. This contradicts the isomorphism invariance of abstract logics!
Step 1. of the proof of Lindström’s theorem

Let $\mathcal{L}$ be a regular logic satisfying LöSko($\mathcal{L}$) and Comp($\mathcal{L}$). Assume that $\mathcal{L}_{\omega} < \mathcal{L}$.

Then there exists a $\psi \in L(S)$ not equivalent to any first order sentence.

By Repl($\mathcal{L}$) (allowing to replace function symbols with relation symbols) we can assume w.l.o.g. that $S$ contains only relation symbols. Remember that we want to prove the following:

**Proposition** (Step 1): For all $m \in \mathbb{N}$ and all finite $S_0 \subseteq S$ there exist $S$-structures $\mathcal{A}$, $\mathcal{B}$ with $\mathcal{A} \models \mathcal{L} \psi$, $\mathcal{B} \models \mathcal{L} \neg \psi$ and $\mathcal{A}|_{S_0} \cong^m \mathcal{B}|_{S_0}$.

(Note that we had to pass to a finite subsignature for our $m$-isomorphism. This will not be a problem in the course of the proof of Lindström’s theorem)
**Proposition (Step 1):** For all $m \in \mathbb{N}$ and all finite $S_0 \subseteq S$ there exist $S$-structures $\mathcal{A}$, $\mathcal{B}$ with $\mathcal{A} \models \mathcal{L} \psi$, $\mathcal{B} \models \mathcal{L} \neg \psi$ and $\mathcal{A}|_{S_0} \cong_m \mathcal{B}|_{S_0}$.

**Proof:** Let $S_0 \subseteq S$ be finite and $m \in \mathbb{N}$. Define $\varphi := \bigvee \{ \varphi^m_{\mathcal{A}|_{S_0}, \emptyset} \mid \mathcal{A} \models \psi \}$ ("this structure is $m$-isomorphic to $\mathcal{A}|_{S_0}$ for an $\mathcal{A}$ with $\mathcal{A} \models \psi$")

There is an $\mathcal{A}$ with $\mathcal{A} \models \psi$, because otherwise $\psi$ would be logically equivalent to a contradiction (this is what it means to have no models), and hence a first order formula. So the above disjunction is non-empty. From the definition of the $\varphi^m_{\mathcal{A}|_{S_0}, \emptyset}$ one can also see that the disjunction is finite, so $\varphi$ is a first order sentence.

Clearly $\psi \rightarrow \varphi$ is a valid formula: If an $S$-structure $\mathcal{A}$ satisfies $\mathcal{A} \models \psi$, then it occurs in the disjunction and its $S_0$-reduction is $m$-isomorphic to itself.

On the other hand $\varphi \rightarrow \psi$ (i.e. $\neg \varphi \lor \psi$) is not a valid formula, otherwise $\psi$ would be equivalent to the first order formula $\varphi$. Hence its negation $\varphi \land \neg \psi$ is satisfiable, i.e. there exists an $S$-structure $\mathcal{B}$ such that $\mathcal{B} \models \varphi$ and $\mathcal{B} \models \neg \psi$.

The first part, $\mathcal{B} \models \varphi$, means exactly that this $\mathcal{B}|_{S_0}$ is $m$-isomorphic to $\mathcal{A}|_{S_0}$ for an $\mathcal{A}$ satisfying $\psi$. \qed
Internalizing Step 1

For Step 2, the passage from an \(m\)-isomorphism to a \(p\)-isomorphism, we have to express the statement of Step 1, i.e. (for given \(S_0 \subseteq S\))

\[
\exists S - \text{str. } A, B \text{ s.t. } A \models \psi, \ B \models \neg \psi, \ A|_{S_0} \cong_m B|_{S_0}
\]

internally in the language of our abstract logic.

More precisely we will construct a signature \(S^+\) and an \(S^+\)-sentence \(\gamma \in L(S^+)\) such that an \(S^+\)-structure satisfying \(\gamma\) consists of two \(S\)-structures, one satisfying \(\psi\), the other \(\neg \psi\), and an \(m\)-isomorphism between them.
Define $S^+ := S \cup \{U, V, W, P, <, I, G, f, c\}$ where $U, V, W, P$ are unary relation symbols, $<, I$ are binary relations symbols, $G$ is a ternary relation symbol, $f$ a unary function symbol and $c$ a constant symbol.

The statement of Step 1 gives us, for all $m \in \mathbb{N}$, an $S^+$-structure $\mathcal{K}_m$:

The underlying set is $K_m := A \uplus B \uplus \{1, \ldots, m\} \uplus P$, where $A, B$ are the underlying sets of $\mathfrak{A}, \mathfrak{B}$, $P := \bigcup_{n=1}^{m} I_n$ (with $I_n$ the set of $n$ times extendable partial isomorphisms $\mathfrak{A} \to \mathfrak{B}$).

– We interpret the symbols $U$ as the subset $A$, $V$ as the subset $B$, $W$ as the subset $\{1, \ldots, m\}$ and $P$ as the subset which we already called $P$ above.
– We interpret the binary symbol $<$ as the order relation on $W = \{1, \ldots, m\}$ and $I \subseteq W \times P$ as the relation $I(n, p) :\iff p \in I_n$
– We interpret the ternary symbol $G$ as $G \subseteq P \times A \times B$ where $G(p, a, b) :\iff a \in dom(p), p(a) = b$
– We interpret the function symbol $f$ as the predecessor function on $W$ and the constant symbol $c$ as the maximal element $m$ of $W$. 
The $S^+$-structure $\mathcal{R}_m$ satisfies the following first order sentences:

- $(\mathcal{W}, <, f, c)$ is a total order with maximal element $c$ and predecessor function $f$ (where we set $f(0) = 0$)
- If $p \in P$, then $p$ is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$.
- If $n > 0$, $p \in I_n$ then, for any choice of $a \in U$ or $b \in V$ there is a $q \in I_{f(n)}$ extending $p$ and with $a \in \text{dom}(q)$, resp. $b \in \text{Im}(q)$.
- $\psi^U$, $(\neg \psi)^V$ hold — here we use the relativization property $\text{Rel}(\mathcal{L})$ to build the formulas $\psi^U$, resp $(\neg \psi)^V$ saying that $\psi$, resp $\neg \psi$ hold on the sub-$S$-structures given by $U$, resp $V$. To form $\neg \psi$ we use that $\mathcal{L}$ contains Boolean connectives.

These are finitely many sentences (concrete fully formal sentences can be found in Ebbinghaus/Flum/Thomas) so we can form their conjunction, using $\text{Bool}(\mathcal{L})$, and call the result $\gamma$. 
Prop.: For any finite $S_0 \subseteq S$ there are $S$-structures $A$, $B$ with $A \models L \psi$, $B \models L \neg \psi$ and $A|_{S_0} \cong_p B|_{S_0}$.

Proof: Consider $\Gamma := \{\gamma\} \cup \{\text{"W has at least } m \text{ elements"} \mid m \in \mathbb{N}\}$. Since we have the $S^+$-structures $\mathcal{K}_m$, all finite subsets are satisfiable. By Comp($L$) the whole set is satisfiable, i.e. there exists an $S^+$-structure $M$ with $M \models \Gamma$.

This $M$ has $W \subseteq M$ an infinite totaly ordered set with maximal element, and has sub-$S$-structures $A$, $B$ with $A \models \psi$, $B \models \neg \psi$.

Define $I := \{p \in P \mid p \in I_{f(n)}(c) \text{ for some } n \in \mathbb{N}\}$ (where $f^{(n)}$ means the $n$-fold application of the predecessor function). Every $p \in I$ is infinitely extendable (since $W$ is infinite), so the the set $I$ is a $p$-isomorphism $I : A \cong_p B$. □
We want to improve the result of Step 2 to saying the following:

For any finite $S_0 \subseteq S$ there are \textit{countable} $p$-isomorphic $S$-structures $\mathcal{A}$, $\mathcal{B}$ with $\mathcal{A} \models L\psi$, $\mathcal{B} \models L\neg\psi$, and $\mathcal{A}|_{S_0} \cong_p \mathcal{B}|_{S_0}$. Being $p$-isomorphic and countable they will then be isomorphic.

For this it is of no use to apply $\text{L"osko}(L)$ to the structures $\mathcal{A}$, $\mathcal{B}$ directly: $\text{L"osko}(L)$ merely allows us to replace $\mathcal{A}$, $\mathcal{B}$ with countable, elementarily equivalent structures, but nothing guarantees that the two outcomes are $p$-isomorphic again.
Instead we take the structure $M$ from the proof of Step 2 and apply LöSko($\mathcal{L}$) to $M$ to get a countable model of our sentence $\gamma$ from before. The two substructures $A$, $B$ will then also be countable.

The last thing that we have to take care of is that the ordered set that is the interpretation of $\mathcal{W}$ is still infinite (then we can, as in the proof of Step 2, get a $p$-isomorphism). To this end we enhance the signature $S^+$ by one more unary predicate $Q$. In the $S^+$-structure $M$ of Step 2 we interpret this as the set of predecessors of the maximal element $c$ of $\mathcal{W}$. Then $M$ is a model of the sentence $\theta := Q(c) \land \forall x (Q(x) \rightarrow ((f(x) < x) \land Q(f(x))))$ (which says that the set of predecessors of $c$ is infinite).

Now, using LöSko($\mathcal{L}$), we pass to a countable model of $\gamma \land \theta$. The same moves as in Step 2 which defined a $p$-isomorphism, together with the countability, prove then the following proposition:

**Prop. (Step 3):** For any finite $S_0 \subseteq S$ there are $S$-structures $A$, $B$ with $A \models L \psi$, $B \models L \neg \psi$ and $A|_{S_0} \cong B|_{S_0}$.
This is not yet a contradiction, since we had to pass to a finite subsignature $S_0 \subseteq S$. We now show that this is enough, since the validity of $\psi$ itself in an $S$-structure $\mathcal{A}$ depends only on $\mathcal{A}|_{S_0}$ for a finite subsignature $S_0$.

**Lemma 1:** Let $\Phi \subseteq L(S)$, $\varphi \in L(S)$, $\Phi \models \mathcal{L} \varphi$. Then there exists a finite $\Phi_0 \subseteq \Phi$ such that $\Phi_0 \models \mathcal{L} \varphi$.

**Proof:** Choose $\neg \varphi$ using $\text{Bool}(\mathcal{L})$. Then $\Phi \cup \{ \neg \varphi \}$ is not satisfiable. Hence there is some finite $\Phi_0 \subseteq \Phi$ s.t. $\Phi_0 \cup \{ \neg \varphi \}$ is not satisfiable. Hence $\Phi_0 \models \mathcal{L} \varphi$. $\square$
Lemma 2: Let $\psi \in L(S)$. Then there is a finite subset $S_0 \subseteq S$ such that for all $S$-structures $A$, $B$:
If $A|_{S_0} \cong B|_{S_0}$, then $(A \models_L \psi \iff B \models_L \psi)$.

Proof: We consider a new signature $(S \cup \{U, V, f\})$ intended to talk about homomorphisms of $S$-structures: Given a homomorphism of $S$-structures $A \rightarrow B$ we can make an $(S \cup \{U, V, f\})$-structure $M$ with underlying set $A \sqcup B$ s.t. $M|_A = A$, $M|_B = B$, $U$ is a unary relation symbol interpreted as the subset $A$, $V$ is a unary relation symbol interpreted as the subset $B$, and $f$ is a binary relation symbol encoding the homomorphism between the two.

There is a set of sentences $\Phi$ of first order logic saying that $f$ is an isomorphism between the $S$-structures $M^U$ and $M^V$. Clearly we have that

$$\Phi \models \psi^U \leftrightarrow \psi^V$$

(the right hand formula is built by using the Relativization and Boolean properties of $L$, the entailment comes from the isomorphism property of $L$).
(Proof continued)

By Lemma 1 there is a finite subset $\Phi_0 \subseteq \Phi$ such that $\Phi_0 \models \psi^U \leftrightarrow \psi^V$. As $\Phi$, and hence $\Phi_0$, consist of first order sentences, there is a finite subsignature $S_0 \subseteq S$ such that $\Phi_0 \subseteq L(S_0)$.

This subsignature $S_0$ has the desired property: If $A|_{S_0} \cong B|_{S_0}$, then we have an $(S \cup \{U, V, f\})$-structure which is a model of $\Phi_0$. Because of $\Phi_0 \models \psi^U \leftrightarrow \psi^V$ we have $A \models \psi$ iff $B \models \psi$. \qed
Proof of Lindström's theorem: Let $\mathcal{L}_{\omega\omega} < \mathcal{L}$, and assume that $\mathcal{L}$ is regular and satisfies $\text{Comp}(\mathcal{L})$ and $\text{LöSk}(\mathcal{L})$. From Steps 1 – 3 we get a signature $S$, a $\psi \in L(S)$ and for all finite subsignatures $S_0 \subseteq S$ we get $S$-structures $A, B$ such that

$$A \models_\mathcal{L} \psi, \quad B \models_\mathcal{L} \neg \psi, \quad A|_{S_0} \cong B|_{S_0} \quad (*)$$

In particular this holds for the finite subsignature $S_0 \subseteq S$ of Lemma 2. But by Lemma 2 for this signature $(*)$ is a contradiction. $\square$
III. Other variants

A. More about $L_{\omega\omega}$
Sharpness of the result:

<table>
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<tr>
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<th>Comp</th>
<th>LöSko</th>
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<tr>
<td>$\mathcal{L}_{\omega\omega}$</td>
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<td>✓</td>
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<tr>
<td>$\mathcal{L}^{2nd}$</td>
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<td>✗</td>
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<tr>
<td>$\mathcal{L}_{\kappa\lambda}$ in general</td>
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<td>✗</td>
</tr>
<tr>
<td>$\mathcal{L}_{\omega_1\omega}$</td>
<td>✗</td>
<td>✓</td>
</tr>
<tr>
<td>$\mathcal{L}_{\omega\omega}(Q_1)$</td>
<td>✓</td>
<td>✗</td>
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<tr>
<td>$\mathcal{L}_{\omega\omega}(Q^R)$</td>
<td>✗</td>
<td>✗(?)</td>
</tr>
<tr>
<td>$\mathcal{L}_{\omega 2nd}$</td>
<td>✗</td>
<td>✓</td>
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</tbody>
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Thus we can not drop the condition Comp or LöSko.
Sharpness of the result (cont.):

Define a logic $\mathcal{L}$ by
\[
\mathcal{L}(S) := \{ \text{2nd order sentences of the form } \exists X_1, \ldots, X_n \psi \text{ where } \psi \text{ contains no 2nd order quantifier} \}
\]
Satisfaction relation $\models_{\mathcal{L}}$ is that of $\mathcal{L}^{2nd}$.

Then:

(i) $\mathcal{L}$ is an abstract logic
(ii) $\mathcal{L}_{\omega \omega} < \mathcal{L}$
(iii) LöSko($\mathcal{L}$), Comp($\mathcal{L}$), Repl($\mathcal{L}$), Rel($\mathcal{L}$) hold
(iv) Bool($\mathcal{L}$) does not hold
**Definition:** $\mathcal{L}$ satisfies *countable compactness* if, given a *countable* $\Phi \subseteq L(S)$ then, if all finite subsets are satisfiable, it follows that $\Phi$ is satisfiable.

**Theorem:** $L_{\omega \omega}$ is the most expressive regular abstract logic satisfying *countable compactness* and Löwenheim-Skolem.

**Theorem:** $L_{\omega \omega}$ is the most expressive regular abstract logic satisfying countable compactness and the following implication (Karp property): If $\mathcal{A} \equiv_p \mathcal{B}$ then $\mathcal{A} \equiv_L \mathcal{B}$ (i.e. the $S$-structures $\mathcal{A}, \mathcal{B}$ satisfy the same $L(S)$-sentences).

For this and the following see: Lindström, On characterizing elementary logic, in: *Logical Theory and Semantic Analysis*, Synthese Library Volume 63, 1974, pp 129–146

See also Flum, Characterizing logics, Chapter III of Barwise/Feferman, Model-theoretic logics, Springer 1985
**Definition:** $\mathcal{L}$ satisfies the *Tarski Union property* if, given a chain $M_0 \leq_{\mathcal{L}} M_1 \leq_{\mathcal{L}} M_2 \leq_{\mathcal{L}} \ldots$ of $\mathcal{L}$-elementary extensions, the inclusion $M_n \leq_{\mathcal{L}} \bigcup_i M_i$ is an $\mathcal{L}$-elementary extension for all $n$.

**Theorem (Lindström):** $L_{\omega\omega}$ is the most expressive regular abstract logic satisfying compactness and the Tarski Union property.
There is a property \( (+) \), roughly saying that one can take a sentence \( \varphi \in L(S) \), replace all \( n \)-ary relation symbols in there by \( (n + 1) \)-ary relation symbols, and then make it into a sentence again by binding the newly gained variable with a \( \forall x \).

**Theorem (Lindström):** Suppose \( L_{\omega\omega} \leq L \), \( L \) satisfies property \( (+) \) and for all \( S \)-structures one has that \( \mathcal{A} \equiv \mathcal{B} \) (elementary equivalence in \( L_{\omega\omega} \)) implies \( \mathcal{A} \equiv_L \mathcal{B} \) (elementary equivalence in \( L \)). Then \( L_{\omega\omega} \sim L \).
**Definition:** From a relational signature $S$ create a new signature $S^+$ by replacing each $n$-ary $P \in S$ with an $(n+1)$-ary $P^+$. From an $S^+$-structure $\mathcal{A}$ and an $a \in A$ we get an $S$-structure $\mathcal{A}^{(a)}$ with the same underlying set $A$ by setting $P^{\mathcal{A}^{(a)}} := \{ (a, a_1, \ldots, a_n) | \mathcal{A} \models P^+(a, a_1, \ldots, a_n) \}$. Then $\mathcal{L}$ satisfies the property (+), if for every $\varphi \in L(S)$ there is a $\varphi^+ \in L(S^+)$ such that for every $S^+$-structure $\mathcal{A}$ one has: $\mathcal{A} \models \varphi^+$ iff $\mathcal{A}^{(a)} \models \varphi$ for all $a \in A$.

**Remark:** In the usual logics this is the following: From $\varphi$ one obtains $\varphi'(x)$ by replacing $P(x_1, \ldots, x_n)$ with $P^+(x, x_1, \ldots, x_n)$ everywhere. Then $\varphi^+ = \forall x \varphi'(x)$. Indeed, property (+) follows from some extra functoriality on signatures which allows to replace relation symbols in a formula with symbols of higher arity together with the existence of quantifiers.

**Theorem (Lindström):** Suppose $\mathcal{L}_{\omega\omega} \leq \mathcal{L}$, $\mathcal{L}$ satisfies (+) and for all $S$-structures one has that $\mathcal{A} \equiv \mathcal{B}$ (elementary equivalence in $\mathcal{L}_{\omega\omega}$) implies $\mathcal{A} \equiv_{\mathcal{L}} \mathcal{B}$ (elementary equivalence in $\mathcal{L}$). Then $\mathcal{L}_{\omega\omega} \sim \mathcal{L}$.
Definition: $\mathcal{L}$ satisfies the *upward Löwenheim-Skolem property* if every $\varphi \in \mathcal{L}(S)$ that has an infinite model, has an uncountable model.

**Theorem (Lindström):** Among the regular abstract logics with the property (+), $\mathcal{L}_{\omega\omega}$ is the most expressive satisfying the upward and the downward Löwenheim-Skolem properties.
Effective versions: Suppose now that $L(S)$ is made of strings of symbols from some finite alphabet.

Definition: (a) $L$ satisfies completeness if the set of valid sentences is recursively enumerable (i.e. there is some complete proof procedure).
(b) $L$ has effective negation and conjunction if the negations and disjunction in the previous sense can be computed effectively.
(c) $L \leq_{\text{eff}} L'$ means that there is an effective procedure associating to each $\varphi \in L$ a $\varphi' \in L'(S)$ which has the same models (i.e. is logically equivalent).

Theorem (Lindström): Suppose $L_{\omega\omega} \leq_{\text{eff}} L$, $L$ has the downward Löwenheim property, is complete and has effective negation and conjunction. Then $L_{\omega\omega} \sim_{\text{eff}} L$.

One can define what it means to have an effective tableau method for determining the valid sentences.

Theorem (Lindström): Suppose $L_{\omega\omega} \leq_{\text{eff}} L$, $L$ has an effective tableau method and has effective negation and conjunction. Then $L_{\omega\omega} \sim_{\text{eff}} L$. 
Remark: $L_{\omega\omega}$ is the most expressive regular abstract logic satisfying a version of the Omitting types theorem.

(See: Flum, Characterizing logics, Thm 2.2.2. Originally: Lindström, Omitting uncountable types and extensions of elementary logic. Theoria 44 (1978), no. 3, 152–156 but only considering extensions of $L_{\omega\omega}$ by quantifiers)

Open question: Is $L_{\omega\omega}$ the most expressive regular abstract logic satisfying compactness and Craig interpolation?

III. Other variants

B. Other logics
**Definition:** A logic $\mathcal{L}$ is *bounded*, if for any $S$ containing a binary relation $<$ and any $\varphi \in L(S)$ having only models with $<$ a well-ordering, there is an ordinal $\alpha$, such that the order type of $<$ in any model is always smaller than $\alpha$.

**Theorem:** $\mathcal{L}_\infty \omega$ is the most expressive regular logic that is bounded and has the Karp property (i.e. if $\mathcal{A} \cong_p \mathcal{B}$ then $\mathcal{A} \equiv_\mathcal{L} \mathcal{B}$).

See Flum, Characterizing Logics, Thm. 3.1. The Karp property is a substitute for downward Löwenheim-Skolem, the boundedness is a substitute for compactness.
Definition: (a) A logic $\mathcal{L}$ has occurrence number $\alpha$, if $\alpha$ is the smallest cardinal such that for all $S$ one has $L(S) = \{L(T) \mid T \subseteq S, |T| < \alpha\}$. 
Notation: $oc(\mathcal{L})$
(b) Consider all $\varphi \in L(S)$ having only models with $<$ a well-ordering, and for which there is an ordinal $\alpha$, such that the order type of $<$ in every model is smaller than $\alpha$. The supremum of all $\alpha$ occurring thus is called the well-ordering number, $wo(\mathcal{L})$.

Theorem: $\mathcal{L}_{\kappa\omega}$ is the most expressive regular logic that is bounded, has the Karp property and has $oc(\mathcal{L}) \leq \kappa$ and $wo(\mathcal{L}) \leq \kappa$.

See Flum, Characterizing Logics, Thm. 3.2
**Definition:** For a signature $S$ denote by $S – Str$ the class of $S$-structures. The *elementary topology* is the topology on $S – Str$ with the elementary classes $\text{Mod}(\varphi) = \{M | M \models \varphi\}$ as open basis.

Because we have negation, it is also a closed basis (i.e. a clopen basis). It follows that the topology is *regular*, i.e. a closed set and a point outside of it can be separated by disjoint open sets.

**Facts:**
- The open sets are closed under isomorphism of $S$-structures (=:the topology is *invariant*).
- The reduction map $S_1 – Str \to S_0 – Str$ coming from an inclusion of signatures $S_0 \subseteq S_1$ is continuous (also “renamings”).
Some reformulations:

Compactness theorem ⇔ $S - Str$ is compact ⇔ every ultrafilter has a limit ⇔ Łoś’s theorem

Downward Löwenheim-Skolem ⇔ the countable $S$-structures are dense

**Topological Lindström theorem (Caicedo):** For each $S$ let $\Gamma_S$ be a regular, invariant topology on $S - Str$ such that the countable structures are dense, reduct and renaming maps are continuous, the $\Gamma_S$ are compact and at least as fine as the elementary topology. Then the $\Gamma_S$ are the elementary topologies.

**Definition:** \( \mathcal{L} \) is a weak extension of \( \mathcal{L} (\mathcal{L} \leq_w \mathcal{L}') \) :iff each sentence of \( \mathcal{L}(S) \) is equivalent to a theory of \( \mathcal{L}'(S) \). Write \( \mathcal{L} \sim_w \mathcal{L}' \) for \( \mathcal{L} \leq_w \mathcal{L}' \) and \( \mathcal{L} \geq_w \mathcal{L}' \).

Consider logics without negation. Denote by \( \text{LöSko}_2 \) the following version of downward Löwenheim-Skolem: A sentence that is true in all countable models of a theory is true in all models of that theory. We define a new topology on \( S\text{-Str} \) by declaring the classes \( \text{Mod}(\varphi) \) to be a sub-basis of *closed* classes.

**Theorem (Caicedo):** Any regular (now meaning: induces a regular topology on each class \( S\text{-Str} \)), compact logic with \( \mathcal{L}_{\omega\omega} \leq_w \mathcal{L} \), having disjunctions and satisfying \( \text{LöSko}_2 \) satisfies \( \mathcal{L} \sim_w \mathcal{L}_{\omega\omega} \).
Lindström theorems for modal logics

New issues arise for modal logics:
1. They are fragments of first order logic – we cannot import 1st order formulas in the proofs, as before.
2. They are interpreted in other structures, coming with their own notions of (partial) isomorphisms, back and forth etc.

**Definition:**

(a) A *Kripke model* is an $S$-structure $\mathcal{M}$ for the signature

\[ S := \{ A, R_1, R_2, R_3, \ldots \} \]

($M$ the set of worlds, $A^\mathcal{M}$ the accessibility relation, $R_i^\mathcal{M}$ encodes a valuation for the variable $x_i$ at each world)

(b) A *pointed Kripke model* is a pair $(\mathcal{M}, w \in M)$

(c) An *abstract modal logic* is a pair $\mathcal{L} = (Fm_\mathcal{L}, \models_\mathcal{L})$ where $Fm_\mathcal{L}$ is a set ("$\mathcal{L}$-formulas") and $\models_\mathcal{L}$ is a relation between pointed Kripke models and $\mathcal{L}$-formulas.

Standing assumption: $\mathcal{L}$-formulas are invariant under isomorphism, $\mathcal{L}$ has Boolean operations, we have renaming and relativization.
Definition: (a) A *bisimulation* between Kripke models $\mathcal{M}$, $\mathcal{N}$ is a binary relation $Z$ between $M$ and $N$, such that:

(i) If $wZv$ then $R^m_i(w) \iff R^n_i(v)$

(ii) If $wZv$ and $wA^m w'$ there is a $v'$ s.t. $vA^n v'$ and $w'Zv'$

(iii) If $wZv$ and $vA^n v'$ there is a $w'$ s.t. $wA^m w'$ and $w'Zv'$

(b) Two pointed Kripke models $(\mathcal{M}, w)$, $(\mathcal{N}, v)$ are *bisimilar* if there exists a bisimulation $Z$ with $wZv$.

(c) A formula $\varphi$ is *bisimulation invariant* if, given bisimilar $(\mathcal{M}, w)$, $(\mathcal{N}, v)$ one has $(\mathcal{M}, w) \models \varphi \iff (\mathcal{N}, v) \models \varphi$

(d) A logic is *bisimulation invariant* if all its formulas are.
Theorem (van Benthem, 2007): An abstract modal logic extending basic modal logic and satisfying compactness and bisimulation invariance is equally expressive as the basic modal logic $K$.

See ten Cate/Väänänen/van Benthem: Lindström theorems for fragments of first order logic, Logical Methods in Computer Science 5(3): 3 (2009)

Further results: Lindström theorem by S. Enqvist for Kripke frames axiomatizable by “strict first order Horn clauses” (2013), Lindström theorems for coalgebra semantics (Kurz/Venema, Enqvist), other results by de Rijke, Otto/Piro, Vuković, ...