THE HILBERT SCHEME OF POINTS FOR SUPERSINGULAR ABELIAN SURFACES

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Abstract. We study the geometry of Hilbert schemes of points on abelian surfaces and Beauville's generalized Kummer varieties in positive characteristics. The main result is that, in characteristic two, the addition map from the Hilbert scheme of two points to the abelian surface is a quasifibration, such that all fibers are nonsmooth. In particular, the corresponding generalized Kummer surface is nonsmooth, and minimally elliptic singularities occur in the supersingular case. We unravel the structure of the singularities in dependence of p-rank and a-number of the abelian surface. To do so, we establish a McKay Correspondence for Artin's wild involutions on surfaces. Along the line, we find examples of canonical singularities that are not rational singularities.

Introduction

This paper circles, in positive characteristics, about the following subjects: The *McKay Correspondence*, *Artin's Wild Involutions*, and the *Hilbert–Chow morphism.* My point of departure is Beauville's generalized Kummer construction, which works as follows:

Fix a complex abelian surface A and let $\mathrm{Hilb}^n(A)$ be its Hilbert scheme of subschemes of length n. One knows that $\mathrm{Hilb}^n(A)$ is smooth with trivial dualizing sheaf. It is a crepant resolution of the symmetric product, given by the Hilbert–Chow morphism $\gamma : \mathrm{Hilb}^n(A) \to \mathrm{Sym}^n(A)$. From this one gets an addition map $\mathrm{Hilb}^n(A) \to A$, and Beauville [8] introduced the *generalized Kummer variety* $\mathrm{Km}^n(A)$ as the fiber of the addition map over the origin. It turns out that $\mathrm{Km}^n(A)$ is smooth, and its dualizing sheaf is trivial as well. In fact, generalized Kummer varieties are one of the few examples of *hyperkähler manifolds.*

The same construction works over ground fields of characteristic $p > 0$. The Hilbert scheme $\mathrm{Hilb}^n(A)$ is still smooth with trivial dualizing sheaf. For the generalized Kummer varieties, however, entirely new geometric phenomena arise: As we shall see, $\mathrm{Km}^n(A)$ is not necessarily smooth, and may even be nonnormal. The goal of this paper is to study this in the simplest accessible case, namely in characteristic $p = 2$ for $n = 2$ points. The first main result is the following:

Theorem. *In characteristic two, all fibers of the addition map* $\mathrm{Hilb}^2(A) \to A$ *are nonsmooth. They are always geometrically reduced, and geometrically normal if and only if the abelian variety* A *is not superspecial.*

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Whence $\mathrm{Hilb}^2(A) \to A$ is an example of a *quasifibration*, that is, its schematic generic fiber is regular but not geometrically regular. Such a violation of Sard's Lemma is only possible in positive characteristics. So far, there is little systematic study of quasifibrations, except for the special case of quasielliptic surfaces, which play a crucial role in the extension of the Enriques classification to positive characteristics [9].

The generalized Kummer surface $\mathrm{Km}^2(A)$ is related to the classical Kummer surface, which is the quotient $A/\{\pm 1\}$ of the abelian surface by the sign involution. For such quotients Artin's classification [3] of involutions on surfaces in characteristic two applies. Shioda [42] and Katsura [30] proved that the singularities on the normal surface $A/\{\pm 1\}$ are certain rational or elliptic double points. This is in startling contrast to the complex situation, where we always have sixteen ordinary double points. The second main result of this paper is a description of the singularities on $\mathrm{Km}^2(X)$ in relation to the singularities on $A/\{\pm 1\}$:

Theorem. *Suppose the ground field k is of characteristic $p = 2$. Then Beauville's Kummer surface $\mathrm{Km}^2(A)$ is crepant partial desingularization of the classical Kummer surface $A/\{\pm 1\}$ obtained by blowing-up the the schematic image of the fixed scheme on A.*

The precise structure of the singularities on $\mathrm{Km}^2(A)$ will be determined in Sections 5, 6, and 7. The following table gives a rough idea of the situation:

p-rank or a-number of A	$\sigma = 2$	$\sigma = 1$	$\sigma = 0, a = 1$	$a = 2$
Singularities on $A/\{\pm 1\}$	$4D_4^1$	$2D_8^2$	elliptic double point	elliptic double point
Singularities on $\mathrm{Km}^2(A)$	$12A_1$	$2A_3 + 2D_4^0$	elliptic triple point	nonnormal

Here the first row contains the two basic numerical invariants of abelian varieties in positive characteristic, namely the p-rank σ and the a-number a. The upper indices in D_n^r determines the isomorphism class of rational double points of type D_n in characteristic two, according to Artin's classification [5]. The supersingular case is most challenging: Here our analysis depends on Laufer's theory of *minimally elliptic singularities* [35].

The existence of a crepant partial resolution holds true in general for quotients of surfaces by involutions in characteristic two, and is closely related to G-Hilbert schemes. Recall the complex *McKay Correspondence* in dimension two was established in various degrees of generality: Ito and Nakamura [28] showed that the minimal resolution of singularities for rational double points is isomorphic to a suitable G-Hilbert scheme. This was extended by Kidoh [32] to cyclic quotients singularities, and by Ishii [26] to arbitrary quotient singularities. Ito and Nakajima [27] generalized this to dimension three for abelian groups.

The situation appears to be rather involved in positive characteristics. Suppose that S is a quasiprojective smooth surface in characteristic $p = 2$, endowed with an action of the group of order two $G = \{\pm 1\}$ having a single fixed point $s \in S$. Let $T = S/G$ be the quotient surface. Then the image $t = q(s)$ of the fixed point under the quotient map $q : S \to T$ is an isolated singularity. The third main result of this paper describes the McKay Correspondence in this situation:

Theorem. *The blowing-up $g : T' \to T$ of the image of the fixed scheme $q(S^G) \subset T$ is a crepant partial resolution with $R^1 g_*(\mathcal{O}_{T'}) = 0$. The scheme T' is isomorphic to the reduced G-Hilbert scheme $\mathrm{Hilb}^G_{\mathrm{red}}(S)$.*

The scheme T' is usually nonnormal, and the G-Hilbert scheme $\mathrm{Hilb}^G(S)$ usually contains embedded components. Note that the two descriptions $T' = \mathrm{Hilb}^G_{\mathrm{red}}(S)$ are entirely different: The first is suitable for explicit computations, the second is useful for theoretical considerations.

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1. Artin's wild involutions

Fix a ground field k of characteristic $p = 2$, and let S be a quasiprojective smooth surface, endowed with an involution $\iota : S \to S$. In other words, the cyclic group $G = \{\pm 1\}$ of order two acts on S. Then the quotient $T = S/G$ is a quasiprojective normal surface. Let $q : S \to T$ be the quotient morphism. To simplify, we assume that G acts freely except for a single rational fixed point $s \in S$. Let $t \in T$ be the image of this fixed point, such that $\mathrm{Sing}(T) = \{t\}$.

The goal of this section is to study the singularity $t \in T$ in terms of the blowing-up $T' \to T$ with center the image of the fixed scheme $S^G \subset S$. We shall see that this blowing-up behaves cohomologically like the resolution of singularities for rational double points. It should be seen as a partial resolution of singularities. The scheme T', however, is usually not normal. In the next section, we shall identify our partial resolution with the underlying reduced subscheme of the G-Hilbert scheme of S.

To start with, I recall Artin's work on involutions on surfaces in characteristic two. In contrast to the case of complex numbers or odd characteristics, the involution ι acting on the complete local ring $\mathcal{O}^\wedge_{S,s}$ is in general neither linearizable nor splits into a product action, such that no obvious description of the quotient springs to mind. However, Artin [3] obtained the following structure result:

Proposition 1.1. *There is a parameter system $x, y \in \mathcal{O}^\wedge_{S,s}$, a regular system of parameters $u, v \in \mathcal{O}^\wedge_{S,s}$, and a parameter system $a, b \in k[[x, y]]$ so that*

$$u^2 + au + x = 0 \quad \textit{and} \quad v^2 + bv + y = 0, \tag{1}$$

and we have $\mathcal{O}^\wedge_{T,t} = k[[x, y, z]]/(z^2 + abz + xb^2 + ya^2)$.

Note that the involution $\iota : \mathcal{O}^\wedge_{S,s} \to \mathcal{O}^\wedge_{S,s}$ must interchange the roots of the two quadratic equations in (1), whence is given by

$$u \longmapsto u + a \quad \text{and} \quad v \longmapsto v + b. \tag{2}$$

Consequently, we have

$$x = u^2 + au = u\bar{u} \quad \text{and} \quad y = v^2 + bv = v\bar{v} \quad \text{and} \quad z = u\bar{v} + \bar{u}v = ub + va, \tag{3}$$

where $\bar{v} = \iota(v)$ etc. denotes the action of the involution.

I find it difficult to make general statements about the structure of the singularity $\mathcal{O}_{T,t}$. Throughout, $\tilde{T} \to T$ denotes the minimal resolution of singularities. The exceptional divisor $E \subset \tilde{T}$ has the following property:

Proposition 1.2. *The Picard scheme $\mathrm{Pic}^0_{E/k}$ is unipotent.*

Proof. We shall use the *local fundamental group* $\pi_1^{\mathrm{loc}}(\mathcal{O}_{T,t})$, which is by definition the fundamental group of $\mathrm{Spec}(\mathcal{O}_{T,t}) \smallsetminus \{t\}$. For the problem at hand, we may assume that the ground field k is separably closed, and replace T by the formal completion at $t \in T$. Then $\pi_1^{\mathrm{loc}}(\mathcal{O}_{T,t})$ is cyclic of order two. Seeking a contradiction, we now assume that $\mathrm{Pic}^0_{E/k}$ is not unipotent. Then $\mathrm{Pic}(E)$ contains nonzero elements of finite order prime to $p=2$, say of order three. Such an elements element extends to a an element of order three in $\varprojlim \mathrm{Pic}(nE)$, which follows from the exact sequences

$$H^1(E, \mathcal{O}_E(-nE)) \longrightarrow \mathrm{Pic}((n+1)E) \longrightarrow \mathrm{Pic}(nE) \longrightarrow H^2(E, \mathcal{O}_E(-nE)).$$

By Grothendieck's Existence Theorem, this corresponds to an element in $\mathrm{Pic}(\tilde{T})$ of order three. In turn, we obtain an invertible sheaf $\mathcal{L}$ on $U = T \smallsetminus \{t\}$ of order three. Choosing a trivialization $\mathcal{L}^{\otimes 3}$, we endow $\mathcal{A} = \mathcal{O}_U \oplus \mathcal{L} \oplus \mathcal{L}^{\otimes 2}$ with an algebra structure, and in turn have a connected finite étale covering of degree three, contradiction. □

It follows that the integral components $E_i \subset E$ have genus zero, for otherwise the Picard scheme would contain abelian varieties. Moreover, the intersection graph for the E_i must be a tree, because otherwise the Picard scheme would contain copies of $\mathbb{G}_m$. Compare the discussion in [11], Chapter 9.

It is easy to determine the schematic fiber $q^{-1}(t) \subset S$ of the singular point $t \in T$:

Lemma 1.3. *The ideal of the schematic fiber $q^{-1}(t) \subset S$ is generated by the elements $u^2, v^2 \in \mathcal{O}^\wedge_{S,s}$.*

Proof. We have to check the equality of ideals $(x,y,z)\mathcal{O}^\wedge_{S,s} = (u^2, v^2)$. The inclusion "$\supset$" follows directly from (1). To check the reverse inclusion, we use (1) to compute

$$x \equiv u^2 + a_x u^3 + a_y uv^2 \quad \text{and} \quad y \equiv v^2 + b_x u^2 v + b_y v^3 \quad \text{modulo } (u,v)^4,$$

where $a_x, a_y, b_x, b_y \in k$ are the coefficients of the linear monomials in the Taylor expansion $a = a_x x + a_y y + O(2)$ and $b = b_x x + b_y y + O(2)$. From this we deduce

$$x, y \in (u^2, v^2). \tag{4}$$

Using $z = ub + va$ from (3), we also have $z \in (u^2, v^2)$. □

Remark 1.4. We may describe the ideal of the fiber without passing to the formal completion as follows: If $u', v' \in \mathcal{O}_{S,s}$ is any regular parameter system, then we have $(u^2, v^2) = (u'^2, v'^2)$ inside the formal completion. Hence $(u'^2, v'^2) \subset \mathcal{O}_{S,s}$ is the ideal of the fiber $q^{-1}(t) \subset S$. This leads to a coordinate free description of the ideal for $q^{-1}(t) \subset S$ as the *bracket ideal* $\mathfrak{m}_s^{[2]} = (f^2 \mid f \in \mathfrak{m}_s)$.

Let $S^G \subset S$ be the *fixed scheme* of the G-action. This is the largest closed subscheme on which the G-action is trivial. In light of (2), its ideal is generated by the parameter system $a, b \in \mathcal{O}^\wedge_{S,s}$. Note that the Artin scheme S^G is never reduced.

Lemma 1.5. *The schematic image $q(S^G) \subset T$ of the fixed scheme $S^G \subset S$ under the quotient map $q: S \to T$ is defined by the parameter ideal $(a,b,z) \subset \mathcal{O}^\wedge_{T,t}$.*

Proof. Let $\mathfrak{a} = \mathcal{O}^\wedge_{T,t} \cap (a,b)\mathcal{O}^\wedge_{S,s}$ be the ideal of the schematic image $q(S^G) \subset T$. We have $z = ub + va$ by (3), hence $z \in \mathfrak{a}$, and therefore $(a,b,z) \subset \mathfrak{a}$. To check the reverse inclusion, let $f(x,y) + zg(x,y) = r(u,v)a + s(u,v)b$ be an element from the ideal $\mathfrak{a}$. Since we already know $z \in \mathfrak{a}$, we may as well assume $g = 0$. It remains to check that f vanishes in $k[[x,y]]/(a,b)$. But this is true, because $k[[x,y]] \subset k[[u,v]]$ is faithfully flat and f vanishes in $k[[u,v]]/(a,b)k[[u,v]]$. □

It follows that the ideal $(a,b,z) \subset \mathcal{O}^\wedge_{T,t}$ has a coordinate-free description as the ideal of the schematic image of the fixed scheme. Let $g: T' \to T$ be the blowing-up of this ideal, or equivalent the blowing-up with center $q(S^G) \subset T$. The following result asserts that this morphism behaves cohomologically like the resolution of rational double points. One should keep in mind, however, that the scheme T' is usually not normal, as we shall see in due course.

Theorem 1.6. *The scheme T' is locally of complete intersection. The fiber $g^{-1}(t)$ is isomorphic to the infinitesimal extension of $\mathbb{P}^1$ by $\mathcal{O}_{\mathbb{P}^1}(-1)$. Furthermore, we have $R^1g_*(\mathcal{O}_{T'}) = 0$, and the relative dualizing sheaf $\omega_{T'/T}$ is trivial.*

Proof. To verify this we may assume that our scheme T actually equals the spectrum of the ring $k[x,y,z]/(z^2+abz+xb^2+ya^2)$ and forget about formal completions, which simplifies notation a little bit. The blowing-up $g : T' \to T$ of the ideal $(a,b,z) \subset \mathcal{O}_{T,t}$ is covered by three affine charts: The a-chart, the b-chart, and the z-chart.

The z-chart is generated by variables $x, y, z, a/z, b/z$, subject to the relations $1 + (a/z)(b/z)z + x(b/z)^2 + y(a/z)^2 = 0$ and $a/z \cdot z = a$ and $b/z \cdot z = b$. The exceptional divisor is given by the additional relation $z = 0$, which is easily seen to be empty. We may therefore concentrate on the a-chart, the situation for the b-chart being symmetric.

The a-chart is generated by four variables $x, y, b/a, z/a$ modulo two relations

$$(z/a)^2 + a\cdot b/a\cdot z/a + x(b/a)^2 + y = 0 \quad \text{and} \quad b/a\cdot a = b. \tag{5}$$

These equations clearly correspond to a regular sequence. It follows that T' is locally of complete intersection.

We next examine the fiber $F = g^{-1}(t)$, which is a Weil divisor. Obviously, it is covered by two affine charts: The a-chart has generators $b/a, z/a$, with only relation $(z/a)^2 = 0$. The b-chart has generators $a/b, z/b$ with relation $(z/b)^2 = 0$. The infinitesimal generators are related by $(z/a) = (b/a)(z/b)$ on the overlap. We infer that $F = g^{-1}(t)$ is an infinitesimal extension of the projective line $\mathbb{P}^1$ by the invertible sheaf $\mathcal{O}_{\mathbb{P}^1}(-1)$. It follows from [6], Theorem 1.2 that such extensions are unique up to isomorphism. We note in passing that F must be isomorphic to a nonreduced quadric in $\mathbb{P}^2$.

To proceed, consider the Cartier divisor $C \subset T'$ whose ideal is the tautological sheaf $\mathcal{O}_{T'}(1) \subset \mathcal{O}_{T'}$ attached to the blowing-up. Note that C is an infinitesimal extension of the fiber $F = g^{-1}(t)$. The a-chart for C has generators $x, y, b/a, z/a$, modulo the relations a, b, and $(z/a)^2 + x(b/a)^2 + y$. From this we infer that $\mathcal{O}_C$ has a decomposition series

$$0 = \mathcal{I}_0 \subset \mathcal{I}_1 \subset \ldots \subset \mathcal{I}_l = \mathcal{O}_C, \tag{6}$$

whose factors are isomorphic to an extension of $\mathcal{O}_{\mathbb{P}^1}$ by $\mathcal{O}_{\mathbb{P}^1}(-1)$, in other words, $\mathcal{I}_i/\mathcal{I}_{i-1} \simeq \mathcal{O}_F$. The length l of the composition series is also the length of the Artin ring $\mathcal{O}_{T,t}/(a,b,z) \simeq k[[x,y]]/(a,b)$, which defines the center for our blowing-up $T' \to T$. Now let $B_i \subset C$ be the closed subscheme defined by $\mathcal{I}_i \subset \mathcal{O}_C$. We infer that an invertible $\mathcal{O}_C$-module $\mathcal{L}$ with $\mathcal{L}\cdot F \geq 0$ has $H^1(C,\mathcal{L}) = 0$; this follows inductively from the exact sequences

$$H^1(F,\mathcal{L}_F) \longrightarrow H^1(B_{i+1},\mathcal{L}_{B_{i+1}}) \longrightarrow H^1(B_i,\mathcal{L}_{B_i}).$$

In particular, we have $H^1(C,\mathcal{O}_C) = 0$.

Next, let C_n be the n-th infinitesimal neighborhood of the Cartier divisor $C = C_0$ and F_n be the n-th infinitesimal neighborhood of the fiber $F = g^{-1}(t)$. Given $n \geq 0$, there is an $m \geq n$ with $C_n \subset F_m$ and $F_n \subset C_m$, and the restriction maps

$$H^1(F_m, \mathcal{O}_{F_m}) \longrightarrow H^1(C_n, \mathcal{O}_{C_n}) \quad \text{and} \quad H^1(C_m, \mathcal{O}_{C_m}) \longrightarrow H^1(F_n, \mathcal{O}_{F_n})$$

are surjective. This implies that the two inverse systems of groups $H^1(F_m, \mathcal{O}_{F_m})$ and $H^1(C_m, \mathcal{O}_{C_m})$ have isomorphic inverse limits. Using the Theorem on Formal Functions, we infer that the canonical map

$$R^1 g_*(\mathcal{O}_{T'}) \longrightarrow \varprojlim H^1(C_n, \mathcal{O}_{C_n})$$

is bijective. On the other hand, the short exact sequence of coherent sheaves supported by the exceptional locus $0 \to \mathcal{I}^n/\mathcal{I}^{n+1} \to \mathcal{O}_{C_{n+1}} \to \mathcal{O}_{C_n} \to 0$ yields an exact sequence

$$H^1(C, \mathcal{I}^n/\mathcal{I}^{n+1}) \longrightarrow H^1(C_{n+1}, \mathcal{O}_{C_{n+1}}) \longrightarrow H^1(C_n, \mathcal{O}_{C_n}) \longrightarrow 0,$$

where $\mathcal{I} = \mathcal{O}_{T'}(1)$. The sheaf $\mathcal{I}^n/\mathcal{I}^{n+1} \simeq \mathcal{O}_C(n)$ is invertible and ample, and we saw in the preceding paragraph that this implies $H^1(C, \mathcal{I}^n/\mathcal{I}^{n+1}) = 0$. Using induction on n, we infer that $H^1(C_n, \mathcal{O}_{C_n})$ vanishes. Whence $R^1 g_*(\mathcal{O}_{T'}) = 0$.

It remains to check that the relative dualizing sheaf is trivial. Since it is trivial outside the fiber $F = g^{-1}(t)$, there is a Cartier divisor $D \in \operatorname{Div}(T')$ supported by F with $\omega_{T'/T} = \mathcal{O}_{T'}(D)$. Our task is to show $D = 0$. Consider the Cartier divisor $C \subset T'$. Its dualizing sheaf is $\omega_C = \mathcal{O}_C(C+D)$, by relative duality for the inclusion $C \subset T'$. Serre duality gives $\deg(\omega_C) = -2\chi(\mathcal{O}_C)$. Using the decomposition series (6), we infer that $\chi(\mathcal{O}_C) = -l$. Hence $\deg(\omega_C) = -2l$. On the other hand, the invertible $\mathcal{O}_F$-module $\mathcal{O}_F(-C)$ is generated on the a-chart by a, and on the b-chart by b, with $a = a/b \cdot b$ on the overlap. It follows that $\deg \mathcal{O}_F(-C) = 2$, and whence $\deg \mathcal{O}_C(C) = -2l$. Consequently, $\mathcal{O}_C(D)$ had degree zero. The curve C is irreducible, and its Picard scheme has tangent space $H^1(C, \mathcal{O}_C) = 0$. Whence we actually have $\mathcal{O}_C(D) \simeq \mathcal{O}_C$, and similarly $\mathcal{O}_{C_n}(D) \simeq \mathcal{O}_{C_n}$ on all infinitesimal neighborhoods. Applying the Theorem of Formal Functions once more, we see that $f_*\mathcal{O}_{T'}(D)$ is an invertible sheaf, which is obviously trivial outside $t \in T$. Now [23], Theorem 1.12 tells us that $f_*\mathcal{O}_{T'}(D)$ is trivial. This implies $\mathcal{O}_{T'}(D) \simeq \mathcal{O}_{T'}$. □

A partial resolution whose relative dualizing sheaf is trivial is called *crepant*. Although this terminology is usually applied to normal partial resolutions, we shall also say that our blowing-up $T' \to T$ is crepant.

In characteristic zero, any quotient singularity is rational, according to Hochster and Eagon [25]. This does not hold true in positive characteristics. In case our quotient singularity happens to be rational, this has the following consequence:

Corollary 1.7. *Suppose the $\mathcal{O}_{T,t}$ is a rational singularity. Then $\mathcal{O}_{T,t}$ is a rational double point, the surface T' is normal with only rational double points, and the minimal resolution $\tilde{T} \to T$ factors over our partial resolution $T' \to T$.*

Proof. Recall that rational double points are precisely the rational Gorenstein singularities. Our singularity $\mathcal{O}_{T,t}$ is a complete intersection, and rational by assumption, whence a rational double point. Let $\nu : T^\flat \to T'$ be the normalization, and $\tilde{T}^\flat \to T^\flat$ be the minimal resolution of singularities. Then we have a commutative diagram

$$\begin{array}{ccc} \tilde{T}^\flat & \longrightarrow & T^\flat \\ \downarrow & & \downarrow \\ \tilde{T} & \longrightarrow & T. \end{array}$$

Suppose that the normalization $\nu : T^\flat \to T'$ is not an isomorphism. Then it is not an isomorphism over the generic point of $C \subset T'$, because T' satisfies Serre's Condition (S_2). The relative dualizing sheaf $\omega_{T^\flat/T'}$ is given by the conductor ideal for the inclusion $\mathcal{O}_{T'} \subset \mathcal{O}_{T^\flat}$. Using $K_{T'/T} = 0$, we conclude that $K_{\tilde{T}^\flat/T}$ is not effective. On the other hand, we have $K_{\tilde{T}/T} = 0$ since $\mathcal{O}_{T,t}$ is a rational double point, and $K_{\tilde{T}^\flat/T'} \geq 0$ because the morphism $\tilde{T}^\flat \to T'$ decomposes into a sequence of blowing-ups of reduced points. Hence $K_{\tilde{T}^\flat/T} \geq 0$, contradiction. Therefore $T^\flat = T'$, such that T' is normal.

Finally, suppose that $\tilde{T}^\flat \to \tilde{T}$ is not an isomorphism. Then $K_{\tilde{T}^\flat/\tilde{T}} > 0$. On the other hand, we have $K_{T'/T} = 0$ by Theorem 1.6, and $K_{\tilde{T}^\flat/T^\flat} \leq 0$ because the resolution $\tilde{T}^\flat \to T^\flat$ is minimal. This gives again a contraction. □

We will examine the case of rational double points at length in Section 4. As we shall see later, neither normality nor factorization holds true with minimally elliptic singularities instead of rational singularities.

Let me now discuss the question what Cartier divisors $D \subset T'$ are supported on the Weil divisor $F = g^{-1}(t)$. Given such a Cartier divisor, we have an equality $D = mF_{\text{red}}$ of Weil divisors for some integer $m \geq 0$. The multiplicity m is related to the length $l \geq 1$ of the Artin algebra $\mathcal{O}^\wedge_{T,t}/(a,b,z)$, which defines the center for the blowing-up $T' \to T$, as follows:

Corollary 1.8. *The multiplicity m of any Cartier divisor $D \subset T'$ supported by $F = g^{-1}(t)$ is a multiple of $2l$.*

Proof. Let $C \subset T$ be the Cartier divisor corresponding to the invertible sheaf $\mathcal{O}_{T'}(1)$ attached to the blowing-up $T' \to T$. If follows from the computations in the proof of Theorem 1.6 that $F = 2F_{\text{red}}$, and $C = lF$, and $F_{\text{red}} \cdot \mathcal{O}_{T'}(1) = -1$. We infer that $-m = mF_{\text{red}} \cdot C = D \cdot C = D \cdot 2lF_{\text{red}}$. The assertion follows, because the number $D \cdot F_{\text{red}}$ is an integer. □

As a consequence, there is no simple relationship between our blowing-up of $q(S^G) \subset T$ and the blowing-up of the reduced singular point $t \in T$:

Corollary 1.9. *Our partial resolution $g : T' \to T$ factors over the blowing-up $T'' \to T$ of the singularity $t \in T$ if and only if $(a,b,z) = (x,y,z)$. In this case, the two blowing-ups coincide.*

Proof. The condition is obviously sufficient. Conversely, suppose there exist a factorization $T' \to T''$. The universal property of the blowing-up $T'' \to T$ implies that the Weil divisor $F = g^{-1}(t) \subset T$ is Cartier. We have $F = 2F_{\text{red}}$, and the preceding corollary tell us that $l = 1$, whence $(a,b,z) = (x,y,z)$. □

The following observation will be important in Section 6: Suppose our partial resolution T' is normal, and let $r : \tilde{T} \to T'$ be the minimal resolution of singularities. For any Weil divisor D on T', we then have the pullback $r^*(D) \in \operatorname{Div}(\tilde{T}) \otimes \mathbb{Q}$ in the sense of Mumford [36]. We call D *numerically Cartier* if the $\mathbb{Q}$-divisor $r^*(D)$ has integral coefficients. Then for any Weil divisor D' on Y', the intersection number $D \cdot D' \in \mathbb{Q}$ is an integer as well. In the special case that $D = mF_{\text{red}}$ is supported on the exceptional locus, the same proof as for Corollary 1.8 gives:

Corollary 1.10. *Suppose that the scheme T' is normal, and that the Weil divisor mF_{red} is numerically Cartier. Then the integer m is a multiple of $2l$.*

To finish this section, I want to clarify the dependence of the singular locus $\mathrm{Sing}(T') \subset T'$ on the parameters $a, b \in k[[x,y]]$. Let $a_x, a_y, b_x, b_y \in k$ be the scalars describing the linear part

$$a \equiv a_x x + a_y y \quad \text{and} \quad b \equiv b_x x + b_y y \quad \text{modulo } (x,y)^2$$

of our parameters. Let $c \in C$ be a rational point on the exceptional locus for the blowing-up $q : T' \to T$. We assume that c lies on the a-chart, the situation for the b-chart being symmetric. Using that $C_{\mathrm{red}} = \mathbb{P}^1$, the ideal of $c \in T'$ is generated by $x, y, z/a, b/a - \lambda$ for some scalar $\lambda \in k$.

Proposition 1.11. *The local ring $\mathcal{O}_{T',c}$ is regular if and only if the scalar $\lambda \in k$ satisfies $b_x + \lambda a_x + b_y \lambda^2 + a_y \lambda^3 \neq 0$. In any case,* $\mathrm{edim}(\mathcal{O}_{T',c}) \leq 3$.

Proof. It follows from (5) that the embedding dimension of $\mathcal{O}_{T',c}$ is at most three, and that the k-vector space $\mathfrak{m}_c/\mathfrak{m}_c^2$ is generated by the classes of the generators $x, y, b/a - \lambda, z/a$ with relations $\lambda^2 x + y = 0$ and $\lambda a + b = 0$. The latter relation equals $\lambda(a_x x + a_y y) + (b_x x + b_y y) = 0$ modulo $\mathfrak{m}_c^2$. Substituting the former relation gives the assertion. $\square$

Corollary 1.12. *The 2-dimensional scheme T' is nonnormal if and only if both parameters $a, b \in k[[x,y]]$ have no linear part.*

Proof. We may assume that the ground field k is separably closed, such that there are infinitely many rational points on $C \simeq \mathbb{P}^1$. Then the scheme is T' is nonnormal if and only if there are infinitely many $c \in C$ so that $\mathcal{O}_{T',c}$ has embedding dimension three. According to Proposition 1.11, this holds if and only if $b_x + \lambda a_x + b_y \lambda^2 + a_y \lambda^3$ is the zero polynomial. $\square$

2. Maps to the Hilbert scheme

We keep the notation from the preceding section, such that $q : S \to T$ is the quotient morphism for an involution having an isolated rational fixed point $s \in S$, with image $t \in T$. Over the complement of the singularity $t \in T$, the quotient map $q : S \to T$ is a G-torsor, and in particular flat of degree two. This gives an embedding $T \smallsetminus \{t\} \to \mathrm{Hilb}^2(S)$ into the Hilbert scheme that parameterizes subschemes of length two. We may view this as a rational map $T \dashrightarrow \mathrm{Hilb}^2(S)$. Such rational maps extend to morphisms on suitable blowing-ups of T. It turns out that our blowing-up $g : T' \to T$ defined in the preceding section already does the job. To see this we have to come up with a family of length two subschemes over T', which we do as follows:

Recall that $g : T' \to T$ is the blowing-up of the parameter ideal (a, b, z) inside the local ring

$$\mathcal{O}^\wedge_{T,t} = k[[x,y,z]]/(z^2 + abz + xb^2 + ya^2).$$

Now let $S' \to S$ be the blowing-up of the induced ideal $(a,b,z)\mathcal{O}^\wedge_{S,s} = (a,b)\mathcal{O}^\wedge_{S,s}$ inside the local ring $\mathcal{O}^\wedge_{S,s} = k[[u,v]]$. The universal property of blowing-ups gives a morphism $h : S' \to T'$, such that the diagram

$$\begin{array}{ccc} S' & \longrightarrow & S \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ T' & \xrightarrow[g]{} & T. \end{array}$$

is commutative.

Proposition 2.1. *The induced morphism $S' \to T' \times S$ is a closed embedding, and the projection $h : S' \to T'$ is flat of degree two.*

Proof. Let us first check flatness: The scheme T' is integral outside the exceptional divisor for $T' \to T$, and clearly has no embedded components on the exceptional divisor $C \subset T'$. Whence T' is integral. The morphism $S' \to T'$ is flat of degree two over the complement of the exceptional divisor. According to [22], Theorem 9.9, it is enough to prove that $h^{-1}(C_{\text{red}}) \to C_{\text{red}}$ is flat of degree two. As in the proof for Theorem 1.6, we may assume that T is the spectrum of the ring $k[x,y,z]/(z^2 + abz + xb^2 + ya^2)$, which simplifies notation a bit.

It follows from (5) that the a-chart for the the exceptional divisor C equals the spectrum of

$$k[x,y,b/a,z/a]/(a,b,(z/a)^2 + (b/a)^2x + y).$$

Whence the reduction C_{red} has relations $x,y,z/a$, because $a,b \in k[[x,y]]$ is a parameter system. Consequently, the schematic preimage $h^{-1}(C_{\text{red}})$ is isomorphic to $\operatorname{Spec} k[u,v,b/a]/(u^2,v^2,z/a)$. Using that $z/a = u \cdot b/a + v$, we see that the projection $h^{-1}(C_{\text{red}}) \to C_{\text{red}}$ is indeed flat of degree two.

Now let us check that $S' \to T' \times S$ is a closed embedding. This map is clearly proper, whence $\mathcal{O}_{S'}$ might be viewed as a coherent sheaf on $T' \times S$. By the Nakayama Lemma, it suffices to show that $S'_c \to S_c$ is a closed embedding for all points $c \in T'$. Making a field extension, we reduce to the case that $c \in T'$ is rational. There is nothing to prove if c lies outside the exceptional divisor $C \subset T'$, so let us assume $c \in C$. By symmetry, may also assume that c lies in the a-chart. Then there is a scalar $\lambda \in k$ so that the ideal defining $c \in C_{\text{red}}$ is $(x,y,b/a-\lambda,z/a) \subset k[x,y,b/a,z/a]$. The fiber $h^{-1}(c) \subset S'$ then is defined by

$$(u^2, v^2, u \cdot b/a + v, b/a - \lambda) \subset k[u,v,b/a],$$

which clearly defines a closed subscheme in S. $\square$

It follows that our rational map $T \dashrightarrow \operatorname{Hilb}^2(S)$ extends to a morphism of schemes $f : T' \to \operatorname{Hilb}^2(S)$, which is defined by the family of subschemes $h : S' \to T'$. The following fact came as a surprise to me:

Theorem 2.2. *The morphism $f : T' \to \operatorname{Hilb}^2(S)$ is a closed embedding.*

Proof. First observe that f is proper: This is clear if S is proper, because then T' is proper and the Hilbert scheme is separated. In general it follows by using a compactification $S \subset \overline{S}$.

Next, we check that the map $f : T' \to \operatorname{Hilb}^2(S)$ is injective. This is clear outside the exceptional divisor $C \subset T'$. The a-chart of the reduction C_{red} is the spectrum of $k[x,y,b/a,z/a]/(x,y,z/a) = k[b/a]$. Given a closed point $c \in C$, say given by $b/a = \lambda$, the fiber in our family $h : S' \to T'$ is $h^{-1}(c) = \operatorname{Spec}(k[u,v]/(u^2, u\lambda + v))$. Clearly, different scalars $\lambda \in k$ give different subschemes $h^{-1}(c) \subset S$, whence our map is indeed injective.

Let $h \in \operatorname{Hilb}^2(S)$ be a point with nonempty fiber Y'_h. Using the Nakayama Lemma, it suffices to check that the fiber $T'_h = f^{-1}(h)$ has length one. Making a base-change, we may assume that the ground field is algebraically closed and that $h \in \operatorname{Hilb}^2(S)$ is closed. Seeking a contradiction, we suppose that the fiber has length > 1. Then it contains a tangent vector $\operatorname{Spec}(k[\epsilon]) \subset T'_h$, where ϵ denotes an indeterminate subject to $\epsilon^2 = 0$. Let $c \in T'$ be the support of such a tangent vector. Then the tangent map $\Theta_{T'}(c) \to \Theta_{\mathrm{Hilb}^2(S)}(h)$ is not injective. We shall produce a contradiction by showing that the tangent map actually is injective. We do this by finding a basis of $\Theta_{T'}(c)$ whose image in $\Theta_{\mathrm{Hilb}^2(S)}(h)$ is linear independent.

To carry out this plan, let me recall the well-known description of the tangent space of the Hilbert scheme. For a nice discussion of these matters, see Artin's lecture notes ([4], Section I.4) or Vistoli's expository paper ([43], Section 2). Let $I \subset \mathcal{O}_S$ be the ideal of $S_c \subset S$. Suppose $J \subset \mathcal{O}_S[\epsilon]$ is a coherent ideal so that the quotient $\mathcal{O}_S[\epsilon]$ is $k[\epsilon]$-flat and $J/\epsilon J = I$. Consider the generators $f_1 = u^2$, $f_2 = \lambda u + v$ for the ideal I. Suppose $f_1', f_2' \in J$ are lifts for $f_1, f_2 \in I$. Then these lifts are necessarily generators of J. If $J' \subset \mathcal{O}_S[\epsilon]$ is another such ideal, with lifts $f_1'', f_2'' \in J'$, the differences $f_i' - f_i''$ yields an element in $\epsilon\mathcal{O}_S[\epsilon] = \epsilon\mathcal{O}_S$, and in turn a residue class in $\epsilon \cdot \mathcal{O}_S/I$. It turns out that this gives a well-defined homomorphism

$$\varphi : I/I^2 \to \epsilon \cdot \mathcal{O}_S/I, \quad f_i \longmapsto f_i' - f_i''.$$

In this way, the tangent space $\Theta_{\mathrm{Hilb}^2(S)}(h)$ becomes a torsor under the k-vector space $\mathrm{Hom}(I/I^2, \epsilon \cdot \mathcal{O}_S/I)$. Throughout, we shall identify $\mathcal{O}_S$ as a subring of $\mathcal{O}_S[\epsilon]$, and choose $f_1'' = f_1 = u^2$, $f_2' = f_2 = \lambda u + v$ as the obvious lifts. This yields an identification of vector spaces

$$\Theta_{\mathrm{Hilb}^2(S)}(h) \longrightarrow \mathrm{Hom}_k(I/I^2, \epsilon \cdot \mathcal{O}_S/I), \quad \mathcal{O}_S/J \longmapsto \varphi.$$

We are now ready for explicit computations: Clearly our point c contained in the exceptional locus $C \subset T'$ for the blowing-up $g : T' \to T$. By symmetry, we may assume that c lies in the a-chart for the blowing-up. Using the notation from (5), we have $\mathfrak{m}_c = (x, y, z/a, b/a - \lambda)$ for some scalar $\lambda \in k$. As explained in the proof for Proposition 1.11, the k-vector space $\mathfrak{m}_c/\mathfrak{m}_c^2$ is generated by the residue classes of $x, y, z/a, b/a - \lambda$, modulo the relations $x\lambda^2 = y$ and $0 = b + \lambda a$.

Consider the tangent vector $\psi : \mathfrak{m}_c/\mathfrak{m}_c^2 \to k$ that vanishes on the classes of $x, y, b/z - \lambda$ and has $\psi(z/a) = \epsilon$. Then the schematic fiber S'_ψ over $\mathrm{Spec}(k[\epsilon]) \subset T'$ is isomorphic to the spectrum of $k[u, v, b/a]/(u^2, v^2, b/a - \lambda)$, and the $k[\epsilon]$-algebra structure comes from $\epsilon \mapsto z/a$. Using $z/a = b/a \cdot u + v$ and writing

$$k[u, v, b/a]/(u^2, v^2, b/a - \lambda) = k[u, v, \epsilon]/(u^2, \epsilon + \lambda u + v),$$

we see that image of our tangent vector $f_*(\psi) \in \Theta_{\mathrm{Hilb}^2(A)}(h)$ corresponds to the homomorphism with $\varphi(u^2) = 0$, $\varphi(\lambda u + v) = \epsilon$, which is obviously nonzero. As a shorthand, we may represent the tangent vector ψ as a 1×4-matrix $(0, 0, 0, 1)$ with respect to the generating system $x, y, b/z - \lambda, z/a \in \mathfrak{m}_c/\mathfrak{m}_c^2$, and its image $f_*(\psi)$ as the 1×2-matrix $(0, \epsilon)$ with respect to the basis $u^2, \lambda u + v \in I/I^2$.

Similar computations with other tangent vectors, which we leave to the reader, yield the following data:

$\psi \in \mathrm{Hom}(\mathfrak{m}_c/\mathfrak{m}_c^2, k)$	$(0,0,0,1)$	$(0,0,1,0)$	$(1,\lambda^2,0,0)$
$f_*(\psi) \in \mathrm{Hom}(I/I^2, \epsilon \cdot \mathcal{O}_S/I)$	$(0,\epsilon)$	$(0,\epsilon u)$	$(\epsilon,0)$

Note that the last column is possible only if the embedding dimension of T' is three, as explained in Proposition 1.11. In any case, we see that the images $f_*(\psi)$ occurring in the second row are linearly independent. The upshot is that the tangent map $f_* : \Theta_{T'}(c) \to \Theta_{\mathrm{Hilb}^2(S)}(h)$ is injective. □

3. G-Hilbert schemes as partial resolutions

We keep the assumptions as in the preceding sections. Let $\mathrm{Hilb}^G(S) \subset \mathrm{Hilb}^2(S)$ be the *G-Hilbert scheme*. For me, this is the scheme that parameterizes G-invariant closed subschemes of length two on S. Note that there are various other definitions in the literature, depending on the context at hand.

The G-Hilbert scheme plays a central role in the *McKay Correspondence* for surface singularities over the complex numbers: Ito and Nakamura [28] showed that the minimal resolution of rational double points is isomorphic to a suitable G-Hilbert scheme. This was extended by Kidoh [32] to cyclic quotient singularities, and by Ishii [26] to arbitrary quotient singularities. It turns out that the situation differs drastically in positive characteristics. The goal of this section is to show that in our situation $\mathrm{Hilb}^G(S)$ usually contains embedded components, and that the underlying reduced subscheme is isomorphic to the blowing-up $T' \to T$ constructed in the preceding section.

Recall that the projection $S \to T$ is a G-torsor over the complement of the singularity $t \in T$. It follows that the embedding $T \smallsetminus \{t\}$ factors over the G-Hilbert scheme. Since $\mathrm{Hilb}^G(S) \subset \mathrm{Hilb}^2(S)$ is a closed embedding, the closed embedding $T' \subset \mathrm{Hilb}^2(S)$ factors over the G-Hilbert scheme as well.

Proposition 3.1. *The closed embedding $T' \subset \mathrm{Hilb}^G(S)$ is bijective.*

Proof. The subscheme $T \smallsetminus \{t\} \subset \mathrm{Hilb}^G(S)$ parameterizes G-orbits disjoint from the fixed point $s \in S$, and its complement parameterizes G-invariant tangent vectors supported by $s \in S$. As explained in [14], Lemma 7.2.8, the scheme of tangent vectors supported by s is the projectivized cotangent space $\mathbb{P}^1 = \mathrm{Proj}(\mathrm{Sym}(\mathfrak{m}_s/\mathfrak{m}_s^2))$. Consequently, the closed embedding $T' \subset \mathrm{Hilb}^2(S)$ necessarily induces a bijection $T' \subset \mathrm{Hilb}^G(S)$. $\square$

Consequently, the reduction of the G-Hilbert scheme is our crepant partial resolution of singularities $T' = \mathrm{Hilb}^G_{\mathrm{red}}(S)$, which is possibly nonnormal, of the quotient surface $T = S/G$. At this point I would like to point out that there is an a priori argument that the normalization $\mathrm{Hilb}^G_{\mathrm{nor}}(S)$ is a partial resolution of singularities for the quotient surface T: We may view the embedding $T \smallsetminus \{t\} \subset \mathrm{Hilb}^G(S)$ as a rational map $T \dashrightarrow \mathrm{Hilb}^G_{\mathrm{nor}}(S)$. Suppose the inverse is undefined at some point of $\mathrm{Hilb}^G_{\mathrm{nor}}(S)$. Then [7], Lemma II.10 tells us that the rational map $T \dashrightarrow \mathrm{Hilb}^G_{\mathrm{nor}}(S)$ contracts a curve. But we know that this rational map is an open embedding outside the closed point $t \in T$, contradiction.

According to [14], Theorem 7.4.1 the Hilbert scheme $\mathrm{Hilb}^2(S)$ is smooth and has dimension four. It follows that the embedding dimensions of the G-Hilbert scheme are at most four.

Theorem 3.2. *Suppose that both parameters $a, b \in k[[x,y]]$ have no linear terms. Then $\mathrm{Hilb}^G(S)$ has an embedded component along the projective line $\mathbb{P}^1 \subset \mathrm{Hilb}^G(S)$ that parameterizes tangent vectors supported by the fixed point $s \in S$.*

Proof. Let $h \in \mathrm{Hilb}^G(S)$ be a point corresponding to a tangent vector supported by $s \in S$. We already know that $\mathrm{Hilb}^G_{\mathrm{red}}(S) = T'$ has embedding dimension three at h. The idea now is to check that the local ring $\mathcal{O}_{\mathrm{Hilb}^G(S),h}$ has embedding dimension four. Recall that the fixed scheme $S^G \subset S$ for the group action is defined by the parameter ideal $(a,b) \subset \mathcal{O}^\wedge_{S,s}$. Obviously, we have closed embeddings of Hilbert schemes $\mathrm{Hilb}^G(S) \supset \mathrm{Hilb}^2(S^G) \subset \mathrm{Hilb}^2(S)$, and it follows from Proposition 3.1 that $h \in \mathrm{Hilb}^2(S^G)$. In turn, we have inclusions between tangent spaces

$$\Theta_{\mathrm{Hilb}^G(S)}(h) \supset \Theta_{\mathrm{Hilb}^2(S^G)}(h) \subset \Theta_{\mathrm{Hilb}^2(S)}(h).$$

It suffices to prove that the inclusion on the right is bijective. To this end, let $I \subset \mathcal{O}_{S,s}^\wedge$ be the ideal of the tangent vector corresponding to $h \in \mathrm{Hilb}^2(S)$, and $\mathfrak{a} = (a,b)$ the ideal of the fixed scheme $S^G \subset S$. Let $J = I \cdot \mathcal{O}_{S,s}^\wedge/\mathfrak{a}$ be the induced ideal of the tangent vector viewed as a subscheme of S^G. Then

$$J = I/(I \cap \mathfrak{a}) = (I + \mathfrak{a})/\mathfrak{a}$$

by the Isomorphism Theorems. Similarly, we obtain identifications $J^2 = (I^2 + \mathfrak{a})/\mathfrak{a}$ and $J/J^2 = (I + \mathfrak{a})/(I^2 + \mathfrak{a})$. It follows that

$$\Theta_{\mathrm{Hilb}^2(S)}(h) = \mathrm{Hom}(I/I^2, \mathcal{O}_S/I),$$
$$\Theta_{\mathrm{Hilb}^2(S^G)}(h) = \mathrm{Hom}((I + \mathfrak{a})/(I^2 + \mathfrak{a}), \mathcal{O}_S/I).$$

Hence it suffices to check that the canonical surjection $I/I^2 \to (I + \mathfrak{a})/(I^2 + \mathfrak{a})$ is bijective. The kernel is $(I^2 + \mathfrak{a})/I^2$, whence it remains to verify $\mathfrak{a} \subset I^2$. We have inclusions

$$(x,y) \subset (u^2,v^2) \subset (u,v)^2 \subset I, \tag{7}$$

the first inclusion coming from (4), the last from [14], Lemma 7.2.6. By assumption $a, b \in k[[x,y]]$ have no linear parts, hence $(a,b) \subset (x,y)^2$. Using (7), we obtain the desired inclusion $(a,b) \subset I^2$. □

4. Example: Rational double points

Keeping the same general assumptions as in the preceding sections, we now consider the special case that the 2-dimensional singularity $\mathcal{O}_{T,t}$ is a rational. For simplicity we also assume that our ground field k is algebraically closed. By Corollary 1.7, the singularity is a rational double point, and our partial crepant resolution $T' = \mathrm{Hilb}^G_{\mathrm{red}}(S)$ is normal.

Recall that in characteristic zero, and for all primes ≥ 7 as well, rational double points are *taut* in the sense of Laufer (compare [33] and [34]). This means that two rational double points are isomorphic if and only if the minimal resolution of singularities have the same intersection graph, which in turn correspond to the Dynkin diagrams. As Artin computed in [5], this does not hold true in characteristics two, three, and five. For example, in characteristic two there are exactly $\lfloor m/2 \rfloor$ different rational double points of Dynkin type D_m, which are denoted by D_m^r with $0 \leq r \leq \lfloor m/2 \rfloor - 1$, and five different rational double points of Dynkin type E_8, which are denoted by $E_8^0, \ldots, E_8^4$. Among other things, the isomorphism classes within a given Dynkin type differ by the *Tjurina number*, which is length of the scheme of nonsmoothness. Recall that if $f(x,y,z) = 0$ defines an isolated singularity, its Tjurina number is the length of the *Tjurina algebra*

$$T = k[[x,y,z]]/(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}).$$

Our first observation is that the amazing diversity of rational double points in characteristic two does not occur among invariant rings for involutions:

Proposition 4.1. *In our situation, the rational double point $\mathcal{O}_{T,t}$ is of type D_{4r}^r for some integer $r \geq 1$, or of type E_8^2. The ring $\mathcal{O}_{T,t}^\wedge$ is isomorphic to the quotient of the power series ring $k[[x,y,z]]$ by the polynomial $z^2 + xy^r z + xy^{2r} + x^2 y$ or by the polynomial $z^2 + yx^2 z + x^5 + y^3$, respectively.*

Proof. In our situation, the local fundamental group $\pi_1^{\mathrm{loc}}(\mathcal{O}_{T,t})$ is cyclic of order two, and the universal covering is given by the regular local ring $\mathcal{O}_{S,s}$. Artin computed in [5], Section 4 the local fundamental groups of all rational double points and gathered information about some unramified coverings as well. We exploit his results as follows: First of all, the singularities of type A_m have no 2-torsion at all in their local fundamental group, which rules them out. Among the rational double points of Dynkin type E_m, $m = 6, 7, 8$, only E_8^2 has local fundamental group of order two. This singularity is given, according to classification, by the polynomial $z^2 + yx^2 z + x^5 + y^3$. As we saw in Proposition 1.1, this equation indeed describes the invariant ring of an involution on $\mathcal{O}_{S,s} = k[[u,v]]$.

It remains to treat the case of rational double points of Dynkin type D_m. If $m = 2n+1$ is odd, the singularity D_m^r has local fundamental group of order two if and only if $n = 2r$. However, the universal covering is then given by a rational double point of type A_1. We conclude that $m = 2n$ must be even. In case $r < m/4$, the singularity D_m^r is simply connected. In case $m/4 < r$, the singularity D_m^r admits a finite unramified covering by an $A_{8r-2m-1}$ singularity, which is never regular. So the only remaining case is $m = 4r$. By Artin's classification, rational double points of type D_{4r}^r are given by the equation $z^2 + xy^r z + xy^{2r} + x^s y = 0$. Indeed, their local fundamental group is cyclic of order two. By Proposition 1.1, this equation actually describes the invariant ring of an involution on $\mathcal{O}_{S,s} = k[[u,v]]$. $\square$

Now let $T' = \mathrm{Hilb}^G_{\mathrm{red}}(S)$ be our crepant partial resolution of T constructed in Section 1, which is the blowing-up whose center is the image of the fixed scheme $S^G \subset S$. According to Corollary 1.7, the scheme $T' = \mathrm{Hilb}^G_{\mathrm{red}}(S)$ is normal, and the minimal resolution $\tilde{T} \to T$ factors over T'. Whence T' is obtained from $\tilde{T}$ by contracting all but one exceptional divisor. We have to determine which exceptional divisor are contracted, and the isomorphism class of singularities created on T'. First, we treat the case that our rational double point $\mathcal{O}_{T,t}$ is of type E_8.

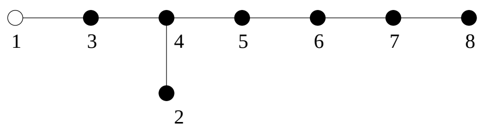

Figure 1: The Dynkin diagram E_8.

The enumeration of vertices in the Dynkin diagram is always as in the Bourbaki tables [13]. These vertices corresponding to the exceptional curves $C_1, \ldots, C_8 \subset \tilde{T}$.

Proposition 4.2. *Suppose $\mathcal{O}_{T,t}$ is an E_8^2-singularity. Then the partial resolution $T' \to T$ is obtained from the minimal resolution $\tilde{T} \to T$ by contracting all exceptional divisors except $C_1 \subset \tilde{T}$. The singular locus of T' consists of a rational double point of type D_7^0. The situation is depicted in Figure 1.*

Proof. We saw that $\mathcal{O}_{T,t}$ is given by the polynomial $z^2 + yx^2z + x^5 + y^3$, and $T' \to T$ is the blowing-up of the ideal (y, x^2, z). We may decompose this blowing-ups into a blowing up of (x, y, z), followed by a blowing-up of $(y/x, x, z/x)$ on the x-chart. Now recall the following simple but useful fact, which I learned from Torsten Ekedahl: For any rational double point, the blowing-up of the reduced singular point introduces a single exceptional curve, and this curve corresponds to the vertex in the Dynkin diagram adjacent to the *longest root* of the root system in question. It follows that in the blowing-up of an E_8-singularity the exceptional divisor corresponds to C_8, and that the exceptional divisor for an iterated blowing-up corresponds to C_1, confer the Bourbaki tables [13]. We infer that $T' \to T$ is obtained from the minimal resolution by contracting all exceptional divisors except for C_1, and the singular locus of T' consists of a rational double point of Dynkin type D_7.

To determine which D_7^r-singularity actually occurs, we use Tjurina numbers. We saw in the proof of Proposition 1.6 that the a-chart of T' is generated by $x, y, b/a, z/a$, modulo the two relations in (5). The scheme of nonsmoothness is thus defined by the 2×2-minors of the matrix of partial derivatives of the two relations. A straightforward computations reveals that this determinantal scheme has length 12. According to [5], page 15, the D_7^r-singularities have Tjurina number $12 - 2r$, so our singularity must be of type D_7^0. $\square$

Next, we consider the rational double points of type D_{4r}.

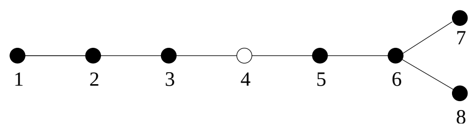

Figure 2: The Dynkin diagram D_8.

Proposition 4.3. *Suppose $\mathcal{O}_{T,t}$ is a D_{4r}^r-singularity. Then the partial resolution $T' \to T$ is obtained from the minimal resolution $\tilde{T} \to T$ by contracting all exceptional divisors $C_i \subset \tilde{T}$, except the component C_{2r}. The singular locus of T' consists of two rational double points, one of type A_{2r-1}, the other of type D_{2r}^0. The special case $r = 2$ is depicted if Figure 2. In the boundary case $r = 1$, we have to interpret D_2^0 as a pair of A_1-singularities.*

Proof. The singularity $\mathcal{O}_{T,t}$ is given by the equation $z^2 + xy^rz + x^{2r} + yx^2$, and $T' \to T$ is the blowing-up of (z, x, y^r). We may decompose this into a sequence of r blowing-ups or reduced points, given on the $(i+1)$-th step by blowing-up the ideal $(z/y^i, x/y^i, y)$. As in the preceding proof we infer that the last exceptional divisor corresponds to the curve $C_{2r} \subset \tilde{T}$ on the minimal resolution. Whence there are precisely two singularities on T', one of type A_{2r-1}, and one of type D_{2r}. The Tjurina number on the a-chart of T' turns out to be $2r$, which corresponds to the A_{2r-1}-singularity. The Tjurina number on the b-chart is $4r$, which implies that our singularity is of type D_{2r}^0. $\square$

5. Sign involution on abelian surfaces

In this section we apply our general results to the following special case, which was my initial motivation to study Artin's wild involutions: Let A be an abelian surface over an algebraically closed field k of characteristic two, and $\iota : A \to A$ be the sign involution. This gives an action of the group $G = \{\pm 1\}$ on the abelian surface A. The fixed points are precisely the 2-torsion points. The kernel $A[2] \subset A$ of the multiplication-by-2 morphism $2 : A \to A$ is a group scheme of length $2^{2g} = 16$, and necessarily of the form

$$A[2] = (\mathbb{Z}/2\mathbb{Z} \oplus \mu_2)^\sigma \oplus N$$

for some integer $\sigma = \sigma(A)$ subject to $0 \leq \sigma \leq 2$, and some local-local group scheme N of length $4^{2-\sigma}$. The integer σ is called the *p-rank* of the abelian surface, and A is called *ordinary* if $\sigma = 2$. Another important invariant is the embedding dimension of N , which is called the *a-number* $a = a(A)$. The a-number is also the dimension of $\mathrm{Hom}(\alpha_p, A)$ viewed as a vector space over $k = \mathrm{End}(\alpha_p)$, where $\alpha_p \subset \mathbb{G}_a$ is the local additive group scheme of order p. We have $0 \leq a \leq 2$. Abelian surfaces are called *supersingular* if $a \geq 1$, and *superspecial* if $a = 2$.

Shioda [42] and Katsura [30] studied the singularities on the classical Kummer surface $A/\{\pm 1\}$, in dependence on p-rank and a-number. The goal of this and the next section is to complete the results of Shioda and Katsura and determine the isomorphism class and equations in normal form for these singularities. Furthermore, we will determine the structure of the crepant partial resolution furnished by the G-Hilbert scheme.

Suppose first that A is not supersingular. Then there are either two or four 2-torsion points on A, and we have to cope with with the following slight complication: The G-Hilbert scheme $\mathrm{Hilb}^G(A)$ is no longer connected, because pairs of 2-torsion point make up entire connected components. However, the reduced connected component $\mathrm{Hilb}^{G,\circ}_{\mathrm{red}}(A)$ that is 2-dimensional indeed yields a crepant partial resolution $T' = \mathrm{Hilb}^{G,\circ}_{\mathrm{red}}(A)$ of the quotient surface $T = A/\{\pm 1\}$.

Proposition 5.1. *Suppose A is ordinary, that is, has p-rank $\sigma = 2$. Then the singular locus of $A/\{\pm 1\}$ comprises four rational double points of type D_4^1. Over each such singularity, the crepant partial resolution $\mathrm{Hilb}^{G,\circ}_{\mathrm{red}}(A)$ contains precisely three singularties, which are rational double points of type A_1.*

Proof. The singular points on $A/\{\pm 1\}$ are the images of the 2-torsion points on A, which are four in number. According to Shioda [42] and Katsura [30], Proposition 3, each singularity is a rational double point of type D_4. These must be singularities of type D_4^1 by Proposition 4.1. The statement about the G-Hilbert scheme follows from Proposition 4.3. □

The same arguments settle the following case as well:

Proposition 5.2. *Suppose the abelian surface A has p-rank $\sigma = 1$. Then the singular locus of $A/\{\pm 1\}$ comprises two rational double points of type D_8^2. Over each such singularity, the crepant partial resolution $\mathrm{Hilb}^{G,\circ}_{\mathrm{red}}(A)$ contains precisely two singularties, which are rational double points of type A_3 and D_4^0.*

The main task now is to understand the supersingular case, which is far more challenging. Suppose A is supersingular. Then only the origin of A is 2-torsion, such that the normal surface $A/\{\pm 1\}$ contains precisely one singularity. Katsura showed in [30], Lemma 12 that the complete local ring at the singularity on $A/\{\pm 1\}$ is isomorphic to $k[[x,y,z]]$ modulo the polynomial

$$f = q^4z^4 + (1 + (q^4 - q)q^2x^3 + q^2x^2y^2)z^2 +$$
$$((q^4 - q)x^3 + q^2x^4y + x^2y^2)z + (q^4 - q)^2x^3 + q^4x^5y^2 + x^4y + xy^4,$$

for some parameter $q \in k$. This is, however, not in Artin's normal form. Our first task is to put Katsura's polynomial into Artin's normal form:

Proposition 5.3. *Suppose A is supersingular. Then the singularity on $A/\{\pm 1\}$ is formally isomorphic to the spectrum of $k[[x,y,z]]/(z^2 + x^2bz + xb^2 + yx^4)$, where we have $b = (q^4 - q)x + y^2$ for some parameter $q \in k$.*

Proof. We simply check that Katsura's polynomial f is *right equivalent* to our polynomial

$$g = z^2 + x^2((q^4 - q)x + y^2)z + x^4y + x((q^4 - q)x + y^2)^2.$$

This simply means there is an automorphism of $k[[x,y,z]]$ sending f to g. Using the substitution $\tilde{z} = z + q^2z^2$, we may rewrite Katsura's polynomial as

$$f = \tilde{z}^2 + x^2((q^4 - q)x + y^2)\tilde{z} + x^4y + x((q^4 - q)x + y^2)^2 + x^4y(q^2z + q^4xy).$$

Whence the inverse of the automorphism $z \mapsto z + q^2z^2$ maps Katsura's polynomial f to a power series of the form $g + x^4y\epsilon$ for some power series $\epsilon \in \mathfrak{m}$. So it remains to check that our polynomial g and power series of the form $g + x^4y\epsilon$ are right equivalent. One achieves this by inductively using substitutions of the form $y \mapsto y + y\epsilon$. □

The parameter $q \in k$ has the following geometric meaning: Oort showed that any supersingular abelian surface A is of the form $(E\times E)/\alpha_2$, where E is a supersingular elliptic curve, and $\alpha_2 \subset E \times E$ is an embedding of group schemes ([37], Corollary 7). Such embeddings depend on a single parameter $q \in \mathbb{P}^1(k)$. If necessary, we may interchange the factors in $E \times E$, and assume that $q \neq \infty$. The resulting scalar $q \in k$ is precisely the parameter in our polynomial defining the singularity.

Let me now recall the following three facts: First, any product of $n \geq 2$ supersingular elliptic curves yields isomorphic abelian varieties. In other words, there is only one superspecial abelian variety in a given dimension $n \geq 2$. Second, in characteristic two there is only one supersingular elliptic curve, which is given by the Weierstrass equation $y^2 = x^3 + x$. Third, $(E \times E)/\alpha_2$ is superspecial if and only if $q \in \mathbb{F}_4$.

Having the equation for the singularity $T = A/\{\pm 1\}$, we now may easily infer the following facts:

Corollary 5.4. *Suppose the abelian surface A is supersingular. Then the singularity on the classical Kummer surface $A/\{\pm 1\}$ is not rational. The crepant partial resolution $T' = \operatorname{Hilb}^G_{\mathrm{red}}(A)$ is normal if and only if A is not superspecial.*

Proof. The first statement is due to Katsura [30], who determined the resolution graph for the singularities. We shall come back to this in the next section.

Now suppose A is superspecial. Then the parameter $q \in k$ satisfies $q^4 - q = 0$, and our equation defining the singularity reduces to $z^2 + x^2y^2z + xy^4 + yx^4$. According to Corollary 1.12, the crepant partial resolution T' is nonnormal. □

To understand the situation, we have to enter the theory of elliptic singularities, which we do in the next section.

6. Minimally elliptic singularities

Let me make a short digression and recall some facts on elliptic singularities. Suppose $T = \operatorname{Spec}(\mathcal{O}_t)$ is a normal 2-dimensional local scheme, with minimal resolution of singularities $f : \tilde{T} \to T$, and reduced exceptional divisors $E_1 \cup \ldots \cup E_n \subset \tilde{T}$. The *fundamental cycle* $Z = \sum n_i E_i$ is the smallest nonzero cycle supported on the exceptional locus with integer coefficient so that $Z \cdot E_i \leq 0$. The *relative canonical cycle* $K = \sum d_i E_i$ is defined as the $\mathbb{Q}$-valued cycle supported on the exceptional locus satisfying the system of linear equations $Z \cdot E_i + E_i^2 = -2\chi(\mathcal{O}_{E_i})$. Its coefficients $d_i \in \mathbb{Q}$ are called the *discrepancies.* The next result due to Laufer ([35], Theorem 3.4 and Theorem 3.10) is fundamental in the theory of surface singularities:

Theorem 6.1. *The following are equivalent:*

(i) *$K = -Z$ as cycles holds.*
(ii) *The scheme T is Gorenstein and the sheaf $R^1 f_* \mathcal{O}_{\tilde{T}}$ has length one.*
(iii) *We have $\chi(\mathcal{O}_Z) = 0$, and $\chi(\mathcal{O}_{Z'}) = -2$ for any subcycle $Z' \subsetneqq Z$.*

Singularities satisfying these equivalent condition are called *minimally elliptic.* They constitute a very interesting class of singularities, in importance second only to rational double points. They are special cases of *elliptic singularities*, which are defined by the somewhat weaker condition $p_a = 1$. Note, however, that Reid [38] uses slightly different terminology.

Now back to our main interest: Throughout this section we assume that A is a supersingular abelian surface.

Proposition 6.2. *The singularity on the classical Kummer surface $A/\{\pm 1\}$ is minimally elliptic. If A is not superspecial, then the singularity on the crepant partial resolution $T' = \operatorname{Hilb}^G_{\mathrm{red}}(A)$ is minimally elliptic as well.*

Proof. Let $f : \tilde{T} \to T$ be the minimal resolution of singularities of $T = A/\{\pm 1\}$. By minimality, the relative canonical cycle satisfies $K_{\tilde{T}/T} \cdot E_i \geq 0$ for alle exceptional divisors E_i. The singularity on T is not rational, whence we actually have $K_{\tilde{T}/T} \cdot E_i > 0$ for some exceptional divisor. We conclude that $K_{\tilde{T}/T} < 0$. Clearly $K_T = 0$, whence $K_{\tilde{T}} < 0$, and therefore $H^2(\tilde{T}, \mathcal{O}_{\tilde{T}}) = 0$. This implies that the Picard scheme $\operatorname{Pic}_{\tilde{T}/k}$ is smooth, of expected dimension $h^1(\mathcal{O}_{\tilde{T}})$. Let $\tilde{T} \to P$ be the Albanese morphism into the abelian variety P dual to $\operatorname{Pic}^0_{\tilde{T}/k}$. Then the induced map $H^1(P, \mathcal{O}_P) \to H^1(\tilde{T}, \mathcal{O}_{\tilde{T}})$ is bijective. In light of Proposition 1.2, all exceptional curves $E_i \subset \tilde{T}$ are mapped to points on P. This means that the boundary map $H^1(\tilde{T}, \mathcal{O}_{\tilde{T}}) \to H^0(T, R^1 f_* \mathcal{O}_{\tilde{T}})$ is zero. The Leray–Serre spectral sequence for $f : \tilde{T} \to T$ yields an exact sequence

$$H^1(\tilde{T}, \mathcal{O}_{\tilde{T}}) \longrightarrow H^0(T, R^1 f_* \mathcal{O}_{\tilde{T}}) \longrightarrow H^2(T, \mathcal{O}_T).$$

The term on the right is Serre dual to $H^0(T, \omega_T)$. The latter is 1-dimensional, because ω_A and hence ω_T are trivial. We conclude that $R^1 f_*(\mathcal{O}_{\tilde{T}})$ has length at most one. It must have length at least one, because the singularity on T is nonrational. The assertion on $T' = \operatorname{Hilb}^G_{\mathrm{red}}(A)$ follows in a similar way, because $\omega_{T'/T}$ is trivial. $\square$

Now suppose that A is not superspecial. According to [30], the minimal resolution $\tilde{T} \to T$ for the singularity on $T = A/\{\pm 1\}$ has the following intersection graph:

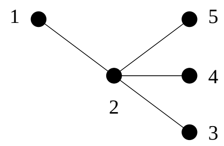

Figure 3: Resolution graph for a minimally elliptic double point.

All components are isomorphic to $\mathbb{P}^1$, and the selfintersection numbers are

$$C_1^2 = -3 \quad \text{and} \quad C_2^2 = \ldots = C_5^2 = -2.$$

From this one directly computes the fundamental cycle

$$Z = -K = C_1 + 2C_2 + C_3 + C_4 + C_5. \tag{8}$$

The singularity appears in Laufer's Table 1 ([35], page 1288) under the name $A_{1,****}$. It also appears in Wagreich's list of elliptic double points ([44], Theorem 3.8) under the symbol $④^1_{0,1}$. The singularity has indeed multiplicity two, according to [35], Theorem 3.13, because $Z^2 = -1$.

By Proposition 1.11, our crepant partial resolution $T' = \operatorname{Hilb}^G_{\mathrm{red}}(A)$ contains precisely one singularity $t' \in T'$. To describe it, we define an iterated blowing-up of reduced points on $\tilde{T}$ as follows: First, blow-up a point on $C_2 \subset \tilde{T}$ not contained in any other curves C_i. Second, blow-up a point on the resulting exceptional divisor not contained in strict transforms of any C_i. Call this two-fold blowing-up $\hat{T} \to \tilde{T}$. Let $\hat{C}_1, \ldots, \hat{C}_5 \subset \hat{T}$ be the strict transforms of the $C_i \subset \tilde{T}$, and let $\hat{C}_6, \hat{C}_7 \subset \hat{T}$ be the two new exceptional curves. They form the following intersection graph:

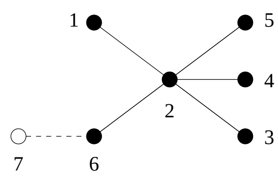

Figure 4: Resolution graph for a minimally elliptic triple point.

The resulting selfintersection numbers are

$$\hat{C}_1^2 = \hat{C}_2^2 = -3, \quad \hat{C}_3^2 = \ldots = \hat{C}_6^{\,2} = -2, \quad \text{and} \quad \hat{C}_7^2 = -1.$$

The configuration of curve $\hat{C}_1 \cup \ldots \cup \hat{C}_6 \subset \hat{T}$ is negative definite and contractible. (Let me remark in passing that its contractibility would be a problem in characteristic zero, compare [39].) The resulting normal singularity is minimally elliptic, with fundamental cycle $\hat{Z} = \hat{C}_1 + 2\hat{C}_2 + \hat{C}_3 + \hat{C}_4 + \hat{C}_5$, which has $\hat{Z}^2 = -3$. Whence this minimally elliptic singularity has embedding dimension three and multiplicity three, by Laufer's result [35], Theorem 3.13. It appears in loc. cit. Table 3, page 1293 under the name $A_{*,o} + A_{*,o} + A_{*,o} + A_{*,o} + A_{*,o}$.

Theorem 6.3. *Suppose A is supersingular, but not superspecial. Then the singularity on the partial crepant resolution $T' = \operatorname{Hilb}^G_{\mathrm{red}}(A)$ is obtained as described* *above: Make a two-fold blowing-up $\hat{T} \to \tilde{T}$ and contract $\hat{C}_1 \cup \ldots \cup \hat{C}_6 \subset \hat{T}$. The exceptional curve for $T' \to A/\{\pm 1\}$ is the image of the (-1)-curve $\hat{C}_7 \subset \hat{T}$.*

Proof. Let $\hat{T} \to T'$ be the minimal resolution of singularities, such that we have a commutative diagram

$$\begin{tikzcd} \hat{T} \arrow[r] \arrow[d] & T' \arrow[d] \\ \tilde{T} \arrow[r] & T. \end{tikzcd}$$

Recall that $K_{T'/T} = 0$ by Theorem 1.6. On the other hand, $K_{\tilde{T}} = K_{\tilde{T}/T}$ is the inverse of the fundamental cycle for $\tilde{T} \to T$, according to Proposition 6.2. We infer that $\hat{T} \to \tilde{T}$ is not an isomorphism. Instead, it factors into a sequence

$$\hat{T} = \hat{T}_{n+5} \longrightarrow \hat{T}_{n+4} \longrightarrow \ldots \longrightarrow \hat{T}_6 \longrightarrow \hat{T}_5 = \tilde{T}$$

of $n \geq 1$ blowing-ups with reduced points $t_{i+1} \in \hat{T}_i$ as centers. We now define curves $\hat{C}_i \subset \hat{T}$ for $1 \leq i \leq n+5$ as follows: For $1 \leq i \leq 5$, let $\hat{C}_i$ be the strict transform of the exceptional curve $C_i \subset \tilde{T}$. For $6 \leq i \leq n+5$, let $\hat{C}_i$ be the strict transform of the exceptional curve for the blowing-up $\hat{T}_i \to \hat{T}_{i-1}$. For convenience, we also denote by $\hat{C}_i \subset \hat{T}_j$ the images of $\hat{C}_i \subset \hat{T}$; this ambiguity should not cause any confusion.

We have $t_{i+1} \in \hat{C}_i \subset \hat{T}_i$ by minimality of $\hat{T} \to T'$. Since the exceptional curve for $T' \to T$ is irreducible, the morphism $\tilde{T} \to T'$ contracts $\hat{C}_1 \cup \ldots \cup \hat{C}_{n+4} \subset \hat{T}$ but not $\hat{C}_{n+5}$. We now use the following observation: Since the singularity $t' \in T'$ is minimally elliptic and $K_{T'/T} = 0$, the multiplicities of $K_{\hat{T}_i}$ at the exceptional divisors $\hat{C}_1, \ldots, \hat{C}_{n+4}$ are negative, and is zero at $\hat{C}_{n+5}$.

Seeking a contradiction, we now suppose that the first center $t_6 \in \tilde{T}$ is contained in $C_1 \smallsetminus C_2$. Then $K_{\hat{T}_6}$ has multiplicity zero along $\hat{C}_6$, which means $n = 1$ and $\hat{T} = \hat{T}_6$. We obtain the desired contradiction as follows: By Corollary 1.10, the Weil divisor $2g^{-1}(t)_{\text{red}} \subset T'$, which is the image of $2\hat{C}_6$, is not numerically Cartier. On the other hand, the vector

$$\begin{pmatrix} -4 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 1 & 1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & -2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = - \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

in integer-valued. This implies that $2g^{-1}(t)_{\text{red}} \subset T'$ is numerically Cartier, contradiction. The matrix whose inverse appears above on the left is the intersection matrix $(\hat{C}_i \cdot \hat{C}_j)_{1 \leq i,j \leq 5}$, and the vector on the left comprises the intersection numbers $(\hat{C}_i \cdot 2\hat{C}_6)_{1 \leq i \leq 5}$. In a similar way one excludes the cases $t_6 \in C_i \smallsetminus C_2$ for $i \geq 3$. The case $t_6 \in C_i \cap C_2$, $i \neq 2$ is impossible as well, because the fiber $g^{-1}(t) \subset T'$ has multiplicity two by Theorem 1.6.

Hence we have $t_6 \in C_2$, and t_5 is not contained in any C_i, $i \neq 2$. Again using that $g^{-1}(t) \subset T'$ has multiplicity two, we infer that $t_7 \in \hat{C}_6$ is not contained in any other strict transform. The canonical class $K_{\hat{T}_7}$ has multiplicity zero in $\hat{C}_7$. As discussed above, this implies $\hat{T} = \hat{T}_7$, and the assertion follows. $\square$

7. The superspecial case

We now assume that our abelian surface A is superspecial, that is, isomorphic to $E \times E$, where E is the supersingular elliptic curve, which has Weierstrass equation $y^2 = x^3 + x$. The minimally elliptic singularity $t \in T = A/\{\pm 1\}$ is formally given by the equation $z^2 + x^2y^2z + xy^4 + yx^4 = 0$, according to Proposition 5.3. Katsura showed in [30] that the minimal resolution of singularities has the following intersection graph:

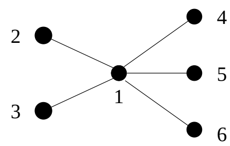

Figure 5: Resolution graph for a minimally elliptic double point.

The intersection numbers are

$$E_1^2 = -3 \quad \text{and} \quad E_2^2 = \ldots = E_5^2 = -2.$$

The fundamental cycle is $Z = 2E_1 + E_2 + \ldots + E_5$, which has $Z^2 = -2$. Whence this minimally elliptic singularity has multiplicity two. It appears in Laufer's Table 2 ([35], page 1290) under the name $A_{*,o} + A_{*,o} + A_{*,o} + A_{*,o} + A_{*,o}$, and in Wagreich's list ([44], Theorem 3.8) under the symbol 19$_0$.

Recall that our crepant partial resolution $T' \to T$ is given by the blowing-up of the parameter ideal (x^2, y^2, z). We now compare it with the blowing-up $T'' \to T$ of the maximal ideal (x, y, z).

Proposition 7.1. *The schemes T', T'' have trivial dualizing sheaf. Both are non-normal, and their normalizations are isomorphic. The common normalization S is obtained from the minimal resolution of singularities $\tilde{T}$ by contracting all exceptional curves except $C_1 \subset \tilde{T}$, compare Figure 5.*

Proof. First, consider the blowing-up $T' \to T$ of the parameter ideal (x^2, y^2, z). Over the x^2-chart, the affine ring of T' is given by four generators $x, y, y^2/x^2, z/x^2$ modulo two relations $(z/x^2)^2 + x^2(y^2/x^2)(z/x^2) + x(y^2/x^2)^2 + y$ and $(y^2/x^2)x^2 = y^2$. The rational function y/x is clearly integral, and the normalization is given by three generators $x, y/x, z/x^2$ modulo the relation $(z/x^2)^2 + x^2(y/x)^2(z/x^2) + x(y/x)^4 + x(y/x)$.

Now consider $T'' \to T$ be the blowing-up of the maximal ideal (x, y, z). Over the x-chart, the affine ring of T'' is given by three generators $x, y/x, z/x$ modulo the relation $(z/x)^2 + x^3(y/x)^2(z/x) + x^3(y/x)^4 + x^3(y/x)$. The rational function z/x^2 is clearly integral, and the normalization is given by generators $x, y/x, z/x^2$ modulo the relation $(z/x^2)^2 + x^2(y/x)^2(z/x^2) + x(y/x)^4 + x(y/x)$. We conclude that the schemes T', T'' have isomorphic normalizations S. It is easy to check that S contains five rational double points of type A_1, whose resolution yield $\tilde{T}$.

It remains to verify the triviality of $\omega_{T''/T}$. This can be done as in the proof for Theorem 1.6. $\square$

Next, we compute fibers of the singularity $t \in T$ on the normalized blowing-up S. It is easy to see that S_t is the trivial infinitesimal extension of the projective line in the coordinate y/x by the invertible sheaf $\mathcal{O}_{\mathbb{P}^1}(-2)$. For simplicity, we write $S_t = \mathbb{P}^1 \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$.

Next, recall that finite birational maps line $S \to T'$ are determined by the conductor ideal $\mathfrak{c} \subset \mathcal{O}_S$, which is the largest $\mathcal{O}_{T'}$-ideal that is at the same time an $\mathcal{O}_S$-ideal.

Proposition 7.2. *The conductor ideals inside $\mathcal{O}_S$ for the normalizations $S \to T'$ and $S \to T''$ coincide, and this conductor ideal is given by $\mathfrak{c} = \mathcal{O}_S(-S_t) \simeq \omega_S$.*

Proof. First note $\omega_{T'}$ and $\omega_{T''}$ are trivial, whence $\omega_{S/T'} = \omega_S = \omega_{S/T''}$. By duality theory, the conductor ideals for $S \to T'$ and $S \to T'$ coincide with the respective relative dualizing sheaves. We conclude that the two conductor ideals coincide. To finish the proof, recall that the fundamental cycle of the minimally elliptic singularity $t \in T$ is $Z = 2C_1 + C_2 + \ldots + C_6$, which implies $\omega_S = \mathcal{O}_S(-S_t)$. □

To proceed, we have to compute the schematic image on T' and T'' of the conductor scheme $S_t = \mathrm{Spec}(\mathcal{O}_S/\mathfrak{c}) \subset S$. We leave the following easy verification to the reader:

Proposition 7.3. *The schematic image of $S_t = \mathbb{P}^1 \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ on T' is the fiber $T'_t = \mathbb{P}^1 \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$; the induced morphism between reduced subschemes is the Frobenius. On the other hand, the schematic image of S_t on T'' is the reduced fiber $(T''_t)_{\mathrm{red}} = \mathbb{P}^1$; the induced morphism between reduced subschemes is the identity.*

Summing up, the two finite birational maps $S \to T'$ and $S \to T''$ are given by the two cartesian and cocartesian squares

$$\begin{array}{ccc} \mathbb{P}^1 \oplus \mathcal{O}_{\mathbb{P}^1}(-2) & \longrightarrow & S \\ {\scriptstyle g'}\downarrow & & \downarrow \\ \mathbb{P}^1 \oplus \mathcal{O}_{\mathbb{P}^1}(-1) & \longrightarrow & T' \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{P}^1 \oplus \mathcal{O}_{\mathbb{P}^1}(-2) & \longrightarrow & S \\ {\scriptstyle g''}\downarrow & & \downarrow \\ \mathbb{P}^1 & \longrightarrow & T''. \end{array}$$

Here the glueing map $g'' : \mathbb{P}^1 \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathbb{P}^1$ is just the identity on the underlying reduced subschemes. In contrast, the glueing map $g' : \mathbb{P}^1 \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathbb{P}^1 \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ is given by the relative Frobenius morphism $\mathrm{Fr} : \mathbb{P}^1 \to \mathbb{P}^1$ on the underlying reduced subschemes, together with $\mathrm{Fr}^* \mathcal{O}_{\mathbb{P}^1}(1) \simeq \mathcal{O}_{\mathbb{P}^1}(2)$. The passage from one denormalization $S \to T''$ to another denormalization $S \to T'$ might be called an *infinitesimal flip*.

We thus have completely unraveled the structure of $A/\{\pm 1\}$ and its nonnormal crepant partial resolution $T' = \mathrm{Hilb}^G_{\mathrm{red}}(A)$.

8. Serre conditions for symmetric products

In the forthcoming sections, we shall apply our results on Kummer surfaces to the geometry of Hilbert scheme of points on abelian surface. To do this, it is first necessary to collect some facts on symmetric products and their Cohen–Macaulay properties. Fix a ground field k, for the moment of arbitrary characteristic $p \geq 0$, and let X be quasiprojective connected smooth scheme, say of dimension $g = \dim(X)$. Throughout, we fix an integer $n \geq 0$. Recall that n-fold *symmetric product* $\mathrm{Sym}^n(X)$ is the quotient of the n-fold product $X^n = X \times \ldots \times X$ by the action of the symmetric group S_n that permutes the factors. Such a quotient exists as a scheme, and we have

$$\mathrm{Sym}^n(X) = \bigcup \mathrm{Spec}(\mathrm{TS}^n(R)).$$

Here the union runs over all affine open subsets $\mathrm{Spec}(R) \subset X$, and $\mathrm{TS}^n(R) \subset T^n(R)$ is the subalgebra of symmetric tensors, which are by definition the S_n-invariant tensors. For an account of symmetric tensors we refer to Bourbaki [12], Chapter IV, §5. The geometric points on $\mathrm{Sym}^n(X)$ correspond to formal linear combinations $\sum n_i x_i$ of pairwise different geometric points with $\sum n_i = n$, $n_i > 0$. As explained in Brion and Kumar's nice account ([14], Section 7.1), we have the following basic properties:

Proposition 8.1. *The symmetric product* $\mathrm{Sym}^n(X)$ *is connected, quasiprojective, normal, $\mathbb{Q}$-factorial, of dimension ng, and its dualizing sheaf is invertible.*

Throughout the paper, we follow the convention adopted by most researchers in the field and call an algebraic scheme *Gorenstein* if it is Cohen–Macaulay and its dualizing sheaf is invertible. Note that our quasiprojective scheme $S = \mathrm{Sym}^n(X)$ is in general *not* Cohen–Macaulay.

Proposition 8.2. *The symmetric product* $\mathrm{Sym}^n(X)$ *is Gorenstein if and only if it is Cohen–Macaulay. These equivalent conditions hold provided $p = 0$ or $p > n$.*

Proof. The first statement follows from the fact that the dualizing sheaf is invertible, by Proposition 8.1. As to the second statement, suppose that $p = 0$ or $p > n$. Then the order $n!$ of the symmetric group S_n is invertible in the ground field k. According to Hochster and Eagon [25], Proposition 13, the quotient of the Cohen–Macaulay scheme X^n by the S_n-action must again be Cohen–Macaulay. $\square$

Note that symmetric products are usually not Cohen–Macaulay in positive characteristics. For example, Aramova [1], Proposition 2.8 computed explicitely that $\mathrm{Sym}^n(X)$ for $g \geq 3$, $n \geq 2$ is not Cohen–Macaulay in characteristic two. We now have a closed look into this matter.

Recall that a locally noetherian scheme S satisfies *Serre's condition* (S_k) if for each point $s \in S$, the complete local ring $\mathcal{O}^\wedge_{S,s}$ is either Cohen–Macaulay, or contains a regular sequence of length at least k, that is, $\mathrm{depth}(\mathcal{O}^\wedge_{S,s}) \geq k$. The next result ensures that our symmetric products satisfy sufficiently many Serre Conditions:

Theorem 8.3. *The symmetric product* $\mathrm{Sym}^n(X)$ *satisfies Serre's Condition* (S_{g+2}).

Proof. In the special cases $n \leq 1$ or $g \leq 1$, the symmetric product $\mathrm{Sym}^n(X)$ is smooth. Hence it suffices to treat the case $n, g \geq 2$. Fix a point $s \in \mathrm{Sym}^n(X)$. According to [20], Corollary 6.7.2, we are allowed to extend our ground field k, whence we may assume that s is a rational point of the form $s = \sum_{i=1}^r n_i x_i$, for certain rational points $x_i \in X$. Our task is to check $\mathrm{depth}(\mathcal{O}^\wedge_s) \geq g + 2$.

We first reduce with a standard argument to the case $r = 1$: Consider a preimage

$$x = (\underbrace{x_1, \ldots, x_1}_{n_1}, \underbrace{x_2, \ldots, x_2}_{n_2}, \ldots, \underbrace{x_r, \ldots, x_r}_{n_r}) \in X^n$$

of $s \in \mathrm{Sym}^n(X)$. As explained in [14], Lemma 7.1.3, the complete local ring $\mathcal{O}^\wedge_s$ is isomorphic to the ring of invariants for the group $G = S_{n_1} \times \ldots \times S_{n_r}$ inside $\mathcal{O}^\wedge_x$. It follows that

$$\mathcal{O}^\wedge_s \simeq \mathcal{O}^\wedge_{\mathrm{Sym}^{n_1}(X), n_1 x_1} \hat{\otimes} \ldots \hat{\otimes} \mathcal{O}^\wedge_{\mathrm{Sym}^{n_r}(X), n_r x_r}.$$

This complete local ring is formally smooth if $n_1 = \ldots = n_r = 1$. Hence it suffices to treat the case $n_1 \geq 2$. By flatness of the tensor factors, it suffices to check that the complete local ring of $n_1x_1 \in \mathrm{Sym}^{n_1}(X)$ has depth $\geq g+2$. In other words, we have reduced our problem to the case $r = 1$.

We now suppose that our point is of the form $s = nx_1$ for some rational point $x_1 \in X$. The corresponding point on X^n is $x = (x_1, \ldots, x_1)$. Its complete local ring is of the form

$$\mathcal{O}_x^\wedge = k[[\underbrace{u_{11}, \ldots, u_{1g}}_{g}, \underbrace{u_{21}, \ldots, u_{2g}}_{g}, \ldots, \underbrace{u_{n1}, \ldots, u_{ng}}_{g}]],$$

and the permutations $\sigma \in S_n$ act via $\sigma(u_{ij}) = u_{\sigma(i),j}$. This is also the completion of the symmetric algebra $\mathrm{Sym}(V)$ at the irrelevant ideal, where $V = \oplus_{j=1}^g W$ is the g-fold sum of the standard permutation representation of S_n on $W = k^{\oplus n}$. Clearly, the invariant subspace W^{S_n} is 1-dimensional. If follows that $V^{S_n} = \oplus_{j=1}^g (W^{S_n})$ has dimension g. By the work of Ellingsrud and Skjelbred ([17], Theorem 3.9), this implies that the irrelevant ideal of $\mathrm{Sym}(V)$ has depth $\geq g+2$. $\square$

In dimension two, this tells us the following:

Corollary 8.4. *Suppose X is 2-dimensional. Then $\mathrm{Sym}^2(X)$ is Cohen–Macaulay.*

Proof. The 4-dimensional scheme $\mathrm{Sym}^2(X)$ satisfies Serre's Condition (S_4) by the preceding theorem, whence is Cohen–Macaulay. $\square$

In characteristic two, we may determine precisely what Serre Conditions hold. This also shows that the preceding theorem gives the best general bound possible:

Proposition 8.5. *Suppose $p = 2$ and $g \geq 3$. Then the $2g$-dimensional scheme $\mathrm{Sym}^2(X)$ does not satisfy Serre's condition (S_{g+3}).*

Proof. We may assume that the ground field k is algebraically closed. Clearly, $\mathrm{Sym}^2(X)$ is Cohen–Macaulay outside the image of the fixed points of the S_2-action on X^2. The fixed points are the diagonal points $(x_1, x_1) \in X^2$. The complete local ring at a closed fixed point $x = (x_1, x_1)$ is of the form $\mathcal{O}_x^\wedge = k[[u_1, \ldots, u_g, v_1, \ldots, v_g]]$, and S_2 acts via the involution $u_i \mapsto v_i$, $v_i \mapsto u_i$. This complete local ring is also the completion of the symmetric algebra $\mathrm{Sym}(V)$ at the irrelevant ideal, with $V = k^{\oplus 2g}$, and group action given by the block matrix

$$\begin{pmatrix} 0 & \mathrm{id}_g \\ \mathrm{id}_g & 0 \end{pmatrix} \in \mathrm{GL}(2g, k).$$

Whence V decomposes into g irreducible 2-dimensional representations of $S_2 = \mathbb{Z}/2\mathbb{Z}$. According to Ellingsrud and Skjelbred [17], Corollary 3.2, the depth of the 2g-dimensional invariant ring $\mathrm{Sym}(V)^{\mathbb{Z}/2\mathbb{Z}}$ at the irrelevant ideal equals two plus the number of irreducible representations in V. The assertion follows. $\square$

We also have the following negative result for higher dimensional schemes, which works in all characteristics:

Proposition 8.6. *Suppose $g \geq 3$. Then the symmetric product $\mathrm{Sym}^n(X)$ is Cohen–Macaulay if and only if $n < p$.*

Proof. As we already observed in Proposition 8.1, the condition is sufficient by Hochster and Eagon [25], Proposition 13. For the converse, suppose that $n \geq p$. To proceed we may assume that the ground field is algebraically closed. Set $S =$

$\mathrm{Sym}^n(X)$, and fix a point $s \in S$ of the form $s = nx_1$ for some closed point $x_1 \in X$. The complete local ring at the point $x = (x_1, \ldots, x_1) \in X^n$ is of the form

$$\mathcal{O}_x = \hat{\bigotimes_{i=1,\ldots,n}} k[[x_1, \ldots, x_g]],$$

and the symmetric group S_n acts via permutations of the tensor factors. Let $P \subset S_n$ be a Sylow p-subgroup. According to a sly computation of Campbell, Geramita, Hughes, Shank, and Wehlau ([10], Theorem 1.2), the invariant subring $(\mathcal{O}_x^\wedge)^P \subset \mathcal{O}_x^\wedge$ cannot be Cohen–Macaulay. This implies that the full invariant subring $(\mathcal{O}_x^\wedge)^{S_n}$ is not Cohen–Macaulay either. The latter follows from the existence of a relative trace map $(\mathcal{O}_x^\wedge)^P \to (\mathcal{O}_x^\wedge)^{S_n}$ given by $f \mapsto \sum_{\sigma \in S_n/P} \sigma(f)$, which shows that the submodule $(\mathcal{O}_x^\wedge)^{S_n} \subset (\mathcal{O}_x^\wedge)^P$ is a direct summand (compare [12], Chapter IV, §5, No. 1). □

Drawing from the huge amount of literature on depths in invariant rings, we may generalize the preceding result as follows:

Proposition 8.7. *Suppose* $\max(3,p) \leq n < 2p$. *Then* $\mathrm{Sym}^n(X)$ *satisfies Serre's condition* (S_{g+2}), *but not* (S_{g+3}).

Proof. We may assume that the ground field k is algebraically closed. According to Proposition 8.3, the scheme $S = \mathrm{Sym}^n(X)$ satisfies (S_{g+2}). To proceed, fix a closed point $x_1 \in X$ and consider the closed point $s = nx_1 \in \mathrm{Sym}^n(X)$. We finish the proof by showing $\mathrm{depth}(\mathcal{O}_{S,s}) = g+2$. As in the preceding proof, the complete local ring $\mathcal{O}_{S,s}^\wedge$ is the formal completion of $\mathrm{Sym}(V)^{S_n}$, where $V = (k^{\oplus g})^{\otimes n}$, and the symmetric group S_n acts via permutations of the tensor factors. We now invoke a result of Kemper ([31], Theorem 3.3) and have to verify some hypothesis. First of all, our condition $p \leq n < 2p$ ensures that the Sylow p-subgroup $P \subset S_n$ has order p, and P equals its own normalizer. This S_n-action on V leaves the standard basis invariant, and the Sylow p-subgroup P decomposes this standard basis into g orbits. According to loc. cit., the depths of the invariant subring $\mathrm{Sym}(V)^{S_n}$ at the irrelevant ideal is $\min(g+2, ng)$. In light of $3 \leq n$, this minimum equals $g+2$. □

I close this section with an observation concerning rational singularities. Let Y be a normal irreducible scheme of finite type that admits a resolution of singularities $f : X \to Y$. Recall that Y is said to have only *rational singualrities* if $R^i f_*(\mathcal{O}_X) = 0$ and $R^i f_*(\omega_X) = 0$ for all $i > 0$. Note that the condition on the higher direct images of the dualizing sheaf are superfluous in characteristic zero, thanks to the Grauert–Riemenschneider Vanishing Theorem [18].

Now suppose that Y has only rational singularities. Then the shifted sheaf $f_*(\omega_X)[d]$, $d = \dim(Y)$ is a dualizing complex on Y, and in particular Y must be Cohen–Macaulay. As a consequence of the preceding two propositions, we obtain the following fact:

Corollary 8.8. *Suppose either* $g \geq 3$ *and* $n \geq p$, *or* $\max(3,p) \leq n < 2p$. *Then the symmetric product* $\mathrm{Sym}^n(X)$ *contains a nonrational singularity.*

9. Symmetric products of abelian varieties

Now let A be a g-dimensional abelian variety. In this section we study its symmetric products $\mathrm{Sym}^n(A)$. The first thing to say is that, since ω_A is trivial, the dualizing sheaf of $\mathrm{Sym}^n(A)$ is trivial as well, as explained in [14], Section 7.1. Another special feature of the situation is the *addition map*

$$+ : A^n \longrightarrow A, \quad (a_1, \ldots, a_n) \longmapsto a_1 + \ldots + a_n.$$

It is clearly S_n-invariant, whence descends to an addition map $\mathrm{Sym}^n(A) \to A$. Given a point $t \in A$, we denote by $\mathrm{Sym}^n_t(A)$ the fiber over $t \in A$ of the addition map. More generally, if $\phi : T \to A$ is a morphism of schemes, we define $\mathrm{Sym}^n_\phi(A)$ via the pull back

$$\begin{tikzcd} \mathrm{Sym}^n_\phi(A) \arrow[r] \arrow[d] & \mathrm{Sym}^n(A) \arrow[d, "+"] \\ T \arrow[r, "\phi"'] & A. \end{tikzcd}$$

We shall be particularly interested in the pullback under the multiplication-by-n map $n : A \to A$. Therefore, I formulate the next result in the abstract language of T-valued points:

Proposition 9.1. *Let T be a scheme, and $\phi, \psi : T \to A$ be two morphisms. Suppose that the group element $\phi - \psi \in A(T)$ is n-divisible. Then the choice of $d \in A(T)$ with $\phi - \psi = nd$ induces an isomorphism of T-schemes $\mathrm{Sym}^n_\psi(A) \to \mathrm{Sym}^n_\phi(A)$.*

Proof. The morphism $d : T \to A$ corresponds to the section $d \times \mathrm{id}$ for the projection $A \times T \to T$. It yields a translation map

$$\tau_d : A \times T \longrightarrow A \times T, \quad (a, t) \longmapsto (a + d(t), t),$$

which is a T-morphism. This induces T-morphisms $\tau_d^n = \tau_d \times \ldots \times \tau_d$ and $\mathrm{Sym}^n(\tau_d)$ on the n-fold product and symmetric product, respectively, such that the diagram

$$\begin{tikzcd} A^n \times T \arrow[r, "\tau_d^n"] \arrow[d] & A^n \times T \arrow[d] \\ \mathrm{Sym}^n(A) \times T \arrow[r, "\mathrm{Sym}^n(\tau_d)"] \arrow[d, "+"'] & \mathrm{Sym}^n(A) \times T \arrow[d, "+"] \\ A \times T \arrow[r, "\tau_{nd}"'] & A \times T \end{tikzcd}$$

is commutative. Similarly, $\phi, \psi : T \to A$ yield sections for the projection $A \times T \to T$, and we have a commutative diagram

$$\begin{tikzcd} A \times T \arrow[r, "\tau_{nd}"] & A \times T \\ T \arrow[u, "\psi\times\mathrm{id}"] \arrow[r, "\mathrm{id}_T"'] & T. \arrow[u, "\phi\times\mathrm{id}"'] \end{tikzcd}$$

Hence $\mathrm{id}_T \times \mathrm{Sym}^n(\tau_d)$ induces an isomorphism

$$(T, \psi \times \mathrm{id}) \times_{(A\times T)} (\mathrm{Sym}^n(A) \times T) \longrightarrow (T, \phi \times \mathrm{id}) \times_{(A\times T)} (\mathrm{Sym}^n(A) \times T).$$

Identifying these fiber products with $\mathrm{Sym}^n_\psi(A)$ and $\mathrm{Sym}^n_\phi(A)$, respectively, we obtain the desired result. $\square$

Applying this with the multiplication-by-n map, we reach the following:

Corollary 9.2. *Let $\psi : A \to A$ be the multiplication-by-n map. Then the pull-back* $\mathrm{Sym}^n_\psi(A) = \mathrm{Sym}^n(A) \times_A (A, n)$ *is isomorphic to the product* $\mathrm{Sym}^n_0(A) \times_k A$.

Proof. Set $T = A$. Let ϕ be the zero map $0 : A \to A$. Then $\psi - \phi = n\,\mathrm{id}_A$ is obviously n-divisible, so we may apply Proposition 9.1. □

This means that the A-scheme $\mathrm{Sym}^n(A)$ is a *twisted form* of the product A-scheme $\mathrm{Sym}^n_0(A) \times_k A$, with respect to the finite flat topology. In particular, the schemes $\mathrm{Sym}^n_\eta(A)$ over the function field $\kappa(A)$ is a twisted form of $\mathrm{Sym}^n_0(A) \otimes_k \kappa(A)$. Note that singularities may dissappear upon passing to twisted forms, as explained in [41].

Next, let $A^n_0 \subset A$ be the kernel of the addition map $A^n \to A$, which is an abelian variety of dimension $ng - 1$. Clearly, this kernel is invariant under the permutation action of S_n. So we may form the quotient scheme A^n_0/S_n, which is normal, and obtain a morphism $A^n_0/S_n \to \mathrm{Sym}^n(A)$, which factors over the closed fiber $\mathrm{Sym}^n_0(A) \subset \mathrm{Sym}^n(A)$ of the addition map.

Proposition 9.3. *Suppose $g \geq 2$. Then the canonical morphism $A^n_0/S_n \to \mathrm{Sym}^n_0(A)$ is an isomorphism.*

Proof. We may assume that k is algebraically closed. Let $U \subset A^n$ be the open subset whose closed points are the $(a_1, \ldots, a_n)$ with pairwise different entries, and let $V \subset \mathrm{Sym}^n(A)$ be its image. Then for all closed points $u \in U$, the stabilizer $G_u \subset G$ is trivial. Then the projection $U \to V$ is flat of degree $n!$, and the formation of the quotient $V = U/S_n$ commutes with arbitrary base change in V. Using that $U_0 = A^n_0 \cap U$ is the preimage of $V_0 = \mathrm{Sym}^n_0(A) \cap V$, we deduce that the morphism $U_0/S_n \to V_0$ is an isomorphism. Clearly, the complement $A^n_0 \smallsetminus U \subset A^n_0$ has codimension g, and we have $g \geq 2$ by assumption. We infer that the morphism $A^n_0/S_n \to \mathrm{Sym}^n_0(A)$ is an isomorphism in codimension ≤ 1.

Since A^n_0/S_n is normal, we see that $\mathrm{Sym}^n_0(A)$ is regular in codimension ≤ 1. Moreover, $\mathrm{Sym}^n_0(A)$ satisfies Serre's Condition (S_{g+2}) by Theorem 8.3. According to Zariski's Main Theorem, the finite morphism $A^n_0/S_n \to \mathrm{Sym}^n_0(A)$ must be an isomorphism. □

As an application, we infer that the generic fiber $\mathrm{Sym}^n_\eta(A)$ for the addition map contains no hidden singularities. Note that the corresponding statement for Hilbert schemes does not hold, as we shall see in Section 10.

Corollary 9.4. *Suppose $g \geq 2$, and let $y \in \mathrm{Sym}^n_\eta(A)$ be a point. Then the local ring $\mathcal{O}_y$ is regular if and only if it is geometrically regular as $\kappa(\eta)$-algebra.*

Proof. The condition is obviously sufficient. To check that it is also necessary, set $F = \kappa(\eta)$ and choose an algebraic closure $F \subset \bar{F}$. According to Corollary 9.2, the geometric generic fiber $\mathrm{Sym}^n_\eta(A) \otimes_F \bar{F}$ is isomorphic to $\mathrm{Sym}^n_0(A) \otimes_k \bar{F}$, and the latter is isomorphic to $A^n_0/S_n \otimes \bar{F}$ by the preceding theorem. Set $B = A^n_0$ and $G = S_n$, and let $y \in B/G$ a point whose local ring $\mathcal{O}_{B/G,y}$ is regular. Since B is Cohen–Macaulay, the projection $f : B \to B/G$ is flat near y, whence the schematic fiber $f^{-1}(y)$ has length n. Then the induced morphism $B \otimes \bar{F} \to B/G \otimes \bar{F}$ is flat at the preimages of y. Since $B \otimes \bar{F}$ is regular, the scheme $B/G \otimes \bar{F}$ must be regular at the preimages of y. □

We now turn to the special case $n = 2$. Then the antidiagonal $A \to A^2$, $a \mapsto (a, -a)$ yields an isomorphism $A \to A^2_0$, and the permutation action of the symmetric group S_2 restricts to the action of $\{\pm 1\}$ via the sign involution $a \mapsto -a$. Summing up, we have:

Corollary 9.5. *For $g \geq 2$ we have an isomorphism $A/\{\pm 1\} \to \mathrm{Sym}_0^2(A)$.*

This allows to apply our results on the sign involution from Section 5 to twofold symmetric products.

10. The Hilbert–Chow morphism and quasifibrations

Let k be a ground field of arbitrary characteristic $p \geq 0$. Recall that for any k-scheme of finite type X, the Hilbert scheme is related to symmetric products via the *Hilbert–Chow* morphism

$$\gamma : \mathrm{Hilb}^n(X) \to \mathrm{Sym}^n(X).$$

It sends a finite subscheme $A \subset X$ to the sum of points $\sum n_i x_i$, where the coefficients are the lengths of the Artin rings $\mathcal{O}_{A,x_i}$. In my opinion, the fact that such a map exists as a morphism of schemes is rather nontrivial. Iversen [29] worked this out, using his theory of linear determinants. If X is a smooth connected surface, then $\mathrm{Hilb}^n(X)$ is again smooth and connected, of dimension $2n$, and the Hilbert–Chow morphism $\gamma : \mathrm{Hilb}^n(X) \to \mathrm{Sym}^n(S)$ is a crepant resolution of singularities, as explained in [14], Section 7.4.

Now let A be an abelian surface. Then $\mathrm{Hilb}^n(A)$ is smooth and connected, and its dualizing sheaf is trivial. We may compose the Hilbert–Chow morphism with the addition map and obtain another addition map

$$\mathrm{Hilb}^n(A) \xrightarrow{\gamma} \mathrm{Sym}^n(A) \xrightarrow{+} A.$$

As with symmetric products, we denote by $\mathrm{Hilb}^n_t(A)$, $t \in A$ its fibers. More generally, if $\psi : T \to A$ is a morphism, we define $\mathrm{Hilb}^n_\psi(A)$ as the corresponding base change. The analogue of Proposition 9.1 holds true for Hilbert schemes, with the same proof:

Proposition 10.1. *Let T be a scheme, and $\phi, \psi : T \to A$ be two morphisms. Suppose that the group element $\phi - \psi \in A(T)$ is n-divisible. Then the choice of $d \in A(T)$ with $\phi - \psi = nd$ induces an isomorphism $\mathrm{Hilb}^n_\psi(A) \to \mathrm{Hilb}^n_\phi(A)$ of T-schemes.*

As a consequence:

Corollary 10.2. *The addition map $f : \mathrm{Hilb}^n(A) \to A$ is flat, and the canonical map $\mathcal{O}_A \to f_*(\mathcal{O}_{\mathrm{Hilb}^n(A)})$ is bijective.*

Proof. By Proposition 10.1, the pull-back $\mathrm{Hilb}^n(A) \times_A (A, n)$ with respect to the multiplication-by-n map $n : A \to A$ is isomorphic to the product $\mathrm{Hilb}^n_0(A) \times_k A$. This implies flatness of the addition map. To proceed, consider the commutative diagram:

$$\begin{tikzcd}
\mathrm{Hilb}^n(A) \arrow[r, "\gamma"] \arrow[dr, "f"'] & \mathrm{Sym}^n(A) \arrow[d, "g"] & A^n \arrow[l, "q"'] \arrow[dl, "h"] \\
& A &
\end{tikzcd}$$

Here the lower arrows are the addition maps. The composition of the two inclusions $\mathcal{O}_A \subset g_*(\mathcal{O}_{\mathrm{Sym}^n(A)}) \subset h_*(\mathcal{O}_{A^n})$ is bijective, whence $\mathcal{O}_A = g_*(\mathcal{O}_{\mathrm{Sym}^n(A)})$. This implies that $\mathrm{Sym}^n_0(A)$ is connected. As explained in [14], Section 5, the Hilbert–Chow morphism $\gamma : \mathrm{Hilb}^n(A) \to \mathrm{Sym}^n(A)$ is birational. Using that $\mathrm{Sym}^n(A)$ is normal, we infer with Zariski's Main Theorem that $\mathcal{O}_{\mathrm{Sym}^n(A)} = \gamma_*(\mathcal{O}_{\mathrm{Hilb}^n(A)})$. The result follows. $\square$

The splitting of $\mathrm{Hilb}^n(A) \times_A (A, n)$ is in line with Beauville' splitting result on Kähler manifolds with zero Ricci curvature ([8], Theorem 1). Let us now take a closer look at the fiber over the origin, which is Beauville's *generalized Kummer variety* $\mathrm{Km}^n(A) = \mathrm{Hilb}^n_0(A)$:

Proposition 10.3. *Beauville's generalized Kummer variety* $\mathrm{Km}^n(A)$ *is integral, locally of complete intersection, and has trivial dualizing sheaf. The induced morphism* $\gamma_0 : \mathrm{Km}^n(A) \to \mathrm{Sym}^n_0(A)$ *is birational, and a crepant partial resolution.*

Proof. The fiber $\mathrm{Hilb}^n_\eta(A)$ over the generic point $\eta \in A$ is obviously integral, regular hence locally of complete intersection, and has trivial dualizing sheaf. Moreover, the morphism $\gamma_\eta : \mathrm{Hilb}^n_\eta(A) \to \mathrm{Sym}^n_\eta(A)$ is a crepant resolution of singularities. Over the algebraic closure of the function field $\kappa(\eta)$, this schemes and morphism become isomorphic to $\mathrm{Km}^n(A)$ and $\gamma_0 : \mathrm{Km}(A) \to \mathrm{Sym}^n_0(A)$, after base-changing to $\overline{\kappa(\eta)}$. The assertions now follow from descend theory. $\square$

We already saw that the canonical map $A^n_0/S_n \to \mathrm{Sym}^n_0(A)$ is an isomorphism. In particular, $\mathrm{Sym}^n_0(A)$ is normal. In contrast, $\mathrm{Km}^n(A)$ is not necessarily normal, as we shall see. To understand this, let us consider the case $n = 2$. The group $G = \{\pm 1\}$ acts on the abelian surface A via the sign involution. Recall that $\mathrm{Hilb}^{G,\circ}_{\mathrm{red}}(A) \subset \mathrm{Hilb}^2(A)$ is the 2-dimensional integral component inside the fixed scheme $\mathrm{Hilb}^2(A)^G$.

Proposition 10.4. *We have* $\mathrm{Hilb}^{G,\circ}_{\mathrm{red}}(A) = \mathrm{Km}^2(A)$ *as subschemes of* $\mathrm{Hilb}^2(A)$.

Proof. Since both subschemes are integral, it suffices to check that both are mapped onto $\mathrm{Sym}^2_0(A)$ under the Hilbert–Chow morphism $\gamma : \mathrm{Hilb}^2(A) \to \mathrm{Sym}^2(A)$, which is obvious. $\square$

So far, the results hold true in arbitrary characteristics. We now specialize to the case of characteristic two:

Theorem 10.5. *Suppose* $p = 2$. *Then all fibers of the addition map* $\mathrm{Hilb}^2(A) \to A$ *are geometrically integral, but nonsmooth. They are geometrically normal if and only if the abelian surface* A *is not superspecial. In particular, this applies to Beauville's generalized Kummer variety* $\mathrm{Km}^2(A) = \mathrm{Hilb}^2_0(A)$.

Proof. It follows from Proposition 10.1 that for any point $t \in A$, say with residue field $F = \kappa(t)$, there exists an isomorphism $\mathrm{Hilb}^2_t(A) \otimes_F \bar{F} \simeq \mathrm{Km}^2(A) \otimes_k \bar{F}$. Hence it suffices to treat the zero fiber $\mathrm{Km}^2(A)$. By Proposition 10.4, we have an identification of $\mathrm{Km}^2(A)$ with the G-Hilbert scheme $\mathrm{Hilb}^{G,\circ}_{\mathrm{red}}$. On the other hand, by Proposition 3.1 we have $T' = \mathrm{Hilb}^{G,\circ}_{\mathrm{red}}$, where T' is the canonical blowing-up attached to the quotient $T = A/\{\pm 1\}$. We analyzed this partial crepant resolution in Sections 5, 6, and 7. In all cases, T' is integral, but not regular. It is normal if and only if A is not superspecial. $\square$

Let me call a morphism $f : X \to Y$ between smooth proper connected schemes with $\mathcal{O}_Y = f_*(\mathcal{O}_X)$ a *quasifibration* if the generic fiber X_η is not smooth. The generic fiber X_η is always a regular scheme. In characteristic zero, this implies that X_η is smooth over $\kappa(\eta)$, so there are no quasifibrations. However, there are quasifibrations in positive characteristics. The most prominent are the quasielliptic surfaces in characteristic two and three. We just saw that the addition map $\mathrm{Hilb}^2(A) \to A$ is a quasifibration in characteristic two. If A is supersingular but not superspecial, the geometric generic fiber contains a minimally elliptic singularity. Recently, Hirokado [24] studied quasifibrations with simple elliptic singuarities. It would be interesting to determine under what conditions on p, n, and A the addition map $\mathrm{Hilb}^n(A) \to A$ is a quasifibration.

Let me close this paper with an observation on canonical singularities: Suppose Y is a connected normal scheme of finite type, admitting a resolution of singularities $f : X \to Y$. One says that Y has only *canonical singularities* Y if the reflexive rank-one sheaf $\omega_Y^{[r]} = (\omega^{\otimes r})^{\vee\vee}$ is invertible for some integer $r \geq 1$, and the canonical inclusion $f_*(\omega_X^{\otimes r}) \subset \omega_Y^{[r]}$ is bijective. Elkik [16] proved that canonical singularities in characteristic zero are rational. This does not hold true in positive characteristics. Indeed, our symmetric products of abelian surfaces yield counterexamples:

Proposition 10.6. *Suppose* $\max(3,p) \leq n < 2p$. *Then the singularities of the symmetric product* $\mathrm{Sym}^n(A)$ *are canonical, but not rational.*

Proof. We saw in Proposition 8.8 that $\mathrm{Sym}^n(A)$ has not only rational singualrities. On the other hand, this scheme is normal with $\omega_{\mathrm{Sym}^n(A)} = \mathcal{O}_{\mathrm{Sym}^n(A)}$, and the Hilbert–Chow morphism $\gamma : \mathrm{Hilb}^n(A) \to \mathrm{Sym}^n(A)$ is a crepant resolution. Hence

$$\gamma_*(\omega_{\mathrm{Hilb}^n(A)}) = \gamma_*(\mathcal{O}_{\mathrm{Hilb}^n(A)}) = \mathcal{O}_{\mathrm{Sym}^n(A)} = \omega_{\mathrm{Sym}^n(A)}$$

trivially holds. □

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