On non-projective normal surfaces

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Abstract

In this note we construct examples of *non-projective* normal proper algebraic surfaces and discuss the somewhat pathological behaviour of their Neron-Severi group. Our surfaces are birational to the product of a projective line and a curve of higher genus.

1. Introduction

The aim of this note is to construct some simple examples of *non-projective* normal surfaces, and discuss the degeneration of the Neron-Severi group and its intersection form. Here the word *surface* refers to a 2-dimensional proper algebraic scheme.

The criterion of Zariski [3, Cor. 4, p. 328] tells us that a normal surface Z is projective if and only if the set of points $z \in Z$ whose local ring $\mathcal{O}_{Z,z}$ is not $\mathbb{Q}$-factorial allows an *affine* open neighborhood. In particular, every resolution of singularities $X \to Z$ is projective. In order to construct Z, we therefore have to start with a regular surface X and contract at least *two* suitable connected curves $R_i \subset X$.

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Our surfaces will be modifications of $Y = P^1 \times C$, where C is a smooth curve of genus $g > 0$; the modifications will replace some fibres $F_i \subset Y$ over P^1 with *rational* curves, thereby introducing non-rational singularities and turning lots of Cartier divisors into Weil divisors.

The Neron-Severi group $\mathrm{NS}(Z) = \mathrm{Pic}(Z)/\mathrm{Pic}^\circ(Z)$ of a non-projective surface might become rather small, and its intersection form might degenerate. Our first example has $\mathrm{NS}(Z) = \mathbb{Z}$ and trivial intersection form. Our second example even has $\mathrm{Pic}(Z) = 0$. Our third example allows a birational morphism $Z \to S$ to a projective surface.

These examples provide answers for two questions concerning surfaces posed by Kleiman [2, XII, rem. 7.2, p. 664]. He has asked whether or not the intersection form on the group $\mathrm{N}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^\tau(X)$ of numerical classes is always non-degenerate, and the first example shows that the answer is negative. Here $\mathrm{Pic}^\tau(X)$ is the subgroup of all invertible sheaves $\mathcal{L}$ with $\deg(\mathcal{L}_A) = 0$ for all curves $A \subset X$. He also has asked whether or not a normal surface with an invertible sheaf $\mathcal{L}$ satisfying $c_1^2(\mathcal{L}) > 0$ is necessarily projective, and the third example gives a negative answer. This should be compared with a result on smooth complex *analytic* surfaces [1, IV, 5.2, p. 126], which says that such a surface allowing an invertible sheaf with $c_1^2(\mathcal{L}) > 0$ is necessarily a projective *scheme*.

In the following we will work over an arbitrary ground field k with *uncountably* many elements. It is not difficult to see that a normal surface over a finite ground field is always projective. It would be interesting to extend our constructions to countable fields.

2. A surface without ample divisors

In this section we will construct a normal surface Z which is not embeddable into any projective space. The idea is to choose a suitable smooth curve C of genus $g > 0$ and perform certain modifications on $Y = P^1 \times C$ called *mutations*, thereby destroying many Cartier divisors.

(2.1) We start by choosing a smooth curve C such that $\mathrm{Pic}(C) \otimes \mathbb{Q}$ contains uncountably many different classes of rational points $c \in C$. For example, let C be an elliptic curve with at least two rational points. We obtain a Galois covering $C \to P^1$ of degree 2 such that the corresponding involution

$i : C \to C$ interchanges the two rational points. Considering its graph we conclude that i has at most finitely many fixed points; since there are uncountably many rational points on the projective line, the set $C(k)$ of rational points is also uncountable.

Since the group scheme of n-torsion points in the Picard scheme $\mathrm{Pic}_{C/k}$ is finite, the torsion subgroup of $\mathrm{Pic}(C)$ must be countable. Since C is a curve of genus $g > 0$, any two different rational points $c_1, c_2 \in C$ are not linearly equivalent, otherwise there would be a morphism $C \to P^1$ of degree 1. We conclude that $\mathrm{Pic}(C) \otimes \mathbb{Q}$ contains uncountably many classes of rational points.

(2.2) We will examine the product ruled surface $Y = P^1 \times C$, and the corresponding projections $\mathrm{pr}_1 : Y \to P^1$ and $\mathrm{pr}_2 : Y \to C$. Let $y \in Y$ be a rational point, $f : X \to Y$ the blow-up of this point, $E \subset X$ the exceptional divisor, and $R \subset X$ the strict transform of $F = \mathrm{pr}_1^{-1}(\mathrm{pr}_1(y))$. Then we can view f as the contraction of the curve $E \subset X$, and I claim that there is also a contraction of the curve $R \subset X$. Let $D \subset X$ be the strict transform of $\mathrm{pr}_2^{-1}(\mathrm{pr}_2(y))$ and $\mathcal{L} = \mathcal{O}_X(D)$ the corresponding invertible sheaf. Obviously, the restriction $\mathcal{L} \mid D$ is relatively ample with respect to the projection $\mathrm{pr}_1 \circ f : X \to P^1$; according to [4] some $\mathcal{L}^{\otimes n}$ with $n > 0$ is relatively base point free, hence the homogeneous spectrum of $(\mathrm{pr}_1 \circ f)_*(\mathrm{Sym}\,\mathcal{L})$ is a normal projective surface Z, and the canonical morphism $g : X \to Z$ is the contraction of R, which is the only relative curve disjoint to D. We call Z the *mutation* of Y with respect to the center $y \in Y$.

(2.3) We observe that the existence of the contraction $g : X \to Z$ is *local* over P^1; hence we can do the same thing simultaneously for finitely many rational points $y_1, \ldots, y_n$ in pairwise different closed fibres $F_i = \mathrm{pr}_1^{-1}(\mathrm{pr}_1(y_i))$. If $f : X \to Y$ is the blow-up of the points y_i, and $E_i \subset X, R_i \subset X$ are the corresponding exceptional curves and strict transforms respectively, we can construct a normal proper surface Z and a contraction $g : X \to Z$ of the union $R = R_1 \cup \ldots \cup R_n$ by patching together quasi-affine pieces over P^1. Since Z is obtained by patching, there is no reason that the resulting proper surface should be projective. We also will call Z the *mutation* of Y with respect to the centers $y_1, \ldots, y_n$.

(2.4) Let us determine the effect of mutations on the Picard group. One easily sees that the maps

$$H^1(C, \mathcal{O}_C) \longrightarrow H^1(Y, \mathcal{O}_Y) \longrightarrow H^1(X, \mathcal{O}_X)$$

are bijective. Let $\mathfrak{X}$ be the formal completion of X along $R = \cup R_i$; since the composition

$$H^1(C, \mathcal{O}_C) \longrightarrow H^1(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \longrightarrow H^1(R, \mathcal{O}_R)$$

is injective, the same holds for the map on the left. Hence the right-hand map in the exact sequence

$$0 \longrightarrow H^1(Z, \mathcal{O}_Z) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$$

in injective, and $H^1(Z, \mathcal{O}_Z)$ must vanish. We deduce that the group scheme $\mathrm{Pic}^\circ_{Z/k}$, the connected component of the Picard scheme, is zero. Since the Neron-Severi group of Y is torsion free, the same holds true for Z, and we conclude $\mathrm{Pic}^\tau(Z) = 0$.

(2.5) Now let $F_1, F_2 \subset Y$ be two different closed fibres over rational points of P^1 and $y_1 \in F_1$ a rational point. The idea is to choose a second rational point $y_2 \in F_2$ in a *generic* fashion in order to eliminate all ample divisors on the resulting mutation. Let Z' be the mutation with respect to y_1. By finiteness of the base number, $\mathrm{Pic}(Z')$ is a countable group, in fact isomorphic to $\mathbb{Z}^2$. On the other hand, $\mathrm{Pic}(F_2)$ is uncountable, and there is a rational point $y_2 \in F_2$ such that the classes of the divisors ny_2 in $\mathrm{Pic}(F_2)$ for $n \neq 0$ are not contained in the image of $\mathrm{Pic}(Z')$. Let Z be the mutation of Y with respect to the centers y_1, y_2.

I claim that there is no ample Cartier divisor on Z. Assuming the contrary, we find an ample effective divisor $D \subset Z$ disjoint to the two singular points $z_1 = g(R_1)$ and $z_2 = g(R_2)$ of the surface. Hence the strict transform $D' \subset Z'$ is a divisor with

$$D' \cap F_2 = \{y_2\},$$

contrary to the choice of $y_2 \in F_2$. We conclude that Z is a non-projective normal surface. More precisely, there is no divisor $D \in \mathrm{Div}(Z)$ with $D \cdot F > 0$, where $F \subset Z$ is a fibre over P^1, since otherwise $D + nF$ would be ample for n sufficiently large. Hence the canonical map $\mathrm{Pic}(P^1) \to \mathrm{Pic}(Z)$ is bijective, $\mathrm{Pic}(Z)/\mathrm{Pic}^\tau(Z) = \mathbb{Z}$ holds, and the intersection form on $N(Z)$ is zero.

3. A surface without invertible sheaves

In this section we will construct a normal surface S with $\mathrm{Pic}(S) = 0$. We start with $Y = P^1 \times C$, pass to a suitable mutation Z, and obtain the desired surface as a contraction of Z.

(3.1) Let $y_1, y_2 \in Y$ be two closed points in two different closed fibres $F_1, F_2 \subset Y$ as in (2.5) such that the mutation with respect to the centers y_1, y_2 is non-projective. Let $y_0 \in Y$ be another rational point in $\mathrm{pr}_2^{-1}(\mathrm{pr}_2(y_1))$, and consider the mutation

$$Y \xleftarrow{f} X \xrightarrow{g} Z$$

with respect to the centers y_0, y_1, y_2. We obtain the following configuration of curves on X:

Please insert figure 1 here.

Here A is the strict transform of $\mathrm{pr}_2^{-1}(\mathrm{pr}_2(y_1))$ and B is the strict transform of $\mathrm{pr}_2^{-1}(\mathrm{pr}_2(y_2))$. Consider the effective divisor $D = 3B + 2R_0 + 2R_1$; one easily calculates

$$D \cdot B = 1, \quad D \cdot R_0 = 1, \quad \text{and} \quad D \cdot R_1 = 1,$$

hence the associated invertible sheaf $\mathcal{L} = \mathcal{O}_X(D)$ is ample on $D \subset X$. According to [4], the homogeneous spectrum of $\Gamma(X, \mathrm{Sym}\,\mathcal{L})$ yields a normal projective surface and a contraction of $A \cup R_2$. On the other hand, the curves R_0 and R_1 are also contractible. Since the curves R_0, R_1 and $A \cup R_2$ are disjoint, we obtain a normal surface S and a contraction $h : Z \to S$ of A by patching.

(3.2) Let $\mathcal{L}$ be an invertible $\mathcal{O}_S$-module; then $\mathcal{M} = h^*(\mathcal{L})$ is an invertible $\mathcal{O}_Z$-module which is trivial in a neighborhood of $A \subset Z$. Since the maps in

$$\mathrm{Pic}(P^1) \longrightarrow \mathrm{Pic}(Z) \longrightarrow \mathrm{Pic}(A)$$

are injective, we conclude that $\mathcal{M}$ is trivial. Hence S is a normal surface such that $\mathrm{Pic}(S) = 0$ holds.

4. A counterexample to a question of Kleiman

In this section we construct a non-projective normal surface Z containing an integral Cartier divisor $D \subset Z$ with $D^2 > 0$. We obtain such a surface by constructing a non-projective normal surface Z which allows a birational morphism $h : Z \to S$ to a projective surface S; then we can find an integral ample divisor $D \subset S$ disjoint to the image of the exceptional curves $E \subset Z$.

(4.1) Again we start with $Y = P^1 \times C$ and choose two closed points $y_1, y_2 \in Y$ as in (2.5) such that the resulting mutation is non-projective. Let y_2' be the intersection of $F_2 = \mathrm{pr}_1^{-1}(\mathrm{pr}_1(y_2))$ with $\mathrm{pr}_2^{-1}(\mathrm{pr}_2(y_1))$, and $f : X \to Y$ the blow up of y_1, y_2 and y_2'. We obtain a configuration of curves

Please insert figure 2 here.

on X. Here A is the strict transform of $\mathrm{pr}_2^{-1}(\mathrm{pr}_2(y_2))$, and A' is the strict transform of $\mathrm{pr}_2^{-1}(\mathrm{pr}_2(y_2'))$. One easily sees that there is a contraction $X \to S$ of the curve $R_1 \cup R_2 \cup E_2$ and another contraction $X \to Z$ of the curve $R_1 \cup R_2$. The divisor A' is relatively ample on S and shows that this surface is projective. On the other hand, I claim that there is no ample divisor on Z. Assuming the contrary, we can pick an integral divisor $E \subset Z$ disjoint to the singularities; its strict transform $D \subset Y$ satisfies

$$D \cap F_1 = \{y_1\} \quad \text{and} \quad D \cap F_2 = \{y_2, y_2'\},$$

where F_i are the fibres containing y_i. Since $A' \cap F_2 = \{y_2'\}$ holds, the class of some multiple $ny_2 \in \mathrm{Div}(F_2)$ is the restriction of an invertible $\mathcal{O}_Y$-module, contrary to the choice of y_2. Hence the surface Z and the morphism $Z \to S$ are non-projective.

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figure 1:

A

E_0 E_1 R_2

R_0 R_1 E_2

B

figure 2:

A'

E'_2

E_1

R_2

R_1

E_2

A