BAER'S RESULT: THE INFINITE PRODUCT OF THE INTEGERS HAS NO BASIS

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1. INTRODUCTION

In a course on linear algebra one learns that a vector space V over a field k contains a basis $e_\alpha \in V$, $\alpha \in I$. This means that any vector $v \in V$ can be written in a unique way as a linear combination

$$v = \sum_{\alpha \in I} \lambda_\alpha e_\alpha.$$

A little care has to be taken if the index set I is infinite: Then one demands that all but finitely many coefficients $\lambda_\alpha \in k$ vanish. The upshot is that the linear map

$$\bigoplus_{\alpha \in I} k \longrightarrow V, \quad (\lambda_\alpha) \longmapsto \sum_{\alpha \in I} \lambda_\alpha e_\alpha$$

is bijective. This sounds easy enough. One should bear in mind, however, that nobody is able to write down explicitly a basis for such innocent $\mathbb{Q}$-vector spaces as the real numbers $V = \mathbb{R}$ or the infinite product $V = \prod_{m=1}^\infty \mathbb{Q}$. Indeed, to establish the existence of a basis in infinite-dimensional vector spaces one has to use the axiom of choice, or equivalently Zorn's Lemma, and thereby abandons all explicitness. To emphasize this aspect, especially in analysis, such a basis is sometimes referred to as a *Hamel basis*.

The existence of a basis does not carry over from vector spaces to modules. Indeed, a module M over a ring R is called *free* if it contains a basis. Throughout, we are interested in modules over the ring of integers $\mathbb{Z}$, that is, abelian groups. Some abelian groups are free, others are not. For example, the abelian group of complex n-th roots of unity

$$\mu_n = \left\{1, e^{2\pi i/n}, e^{2\pi i 2/n}, \ldots, e^{2\pi i(n-1)/n}\right\} \subset \mathbb{C}^\times$$

does not contain a basis for $n > 1$. Obvious reason: This group is nonzero and finite; if were free, the group would be infinite.

Sometimes, it is a bit more tricky to decide whether or not a given abelian group contains a basis. Consider, for example, the rational numbers $\mathbb{Q}$. To see that $\mathbb{Q}$, viewed as an abelian group, is not free, one may proceed as follows: Suppose there were a basis $e_\alpha \in \mathbb{Q}$, $\alpha \in I$. Write the rational number $1/1 = n_1 e_{\alpha_1} + \ldots + n_r e_{\alpha_r}$ as a linear combination this basis. After reordering the summands, we may assume $n_1 \neq 0$. To proceed, choose an integer n not dividing n_1, and express the rational number $1/n = m_1 e_{\beta_1} + \ldots + m_s e_{\beta_s}$ as a linear combination of our basis. Equating the two expressions for the rational number $n \cdot 1/n = 1/1$, we obtain

$$nm_1 e_{\beta_1} + \ldots + nm_s e_{\beta_s} = n_1 e_{\alpha_1} + \ldots + n_r e_{\alpha_r}.$$

The uniqueness of such linear combinations implies that the common factor n in the coefficients of the left hand side divides the coefficient n_1 on the right hand side, a contradiction.

The following is my favorite example of an abelian group without a basis: The infinite product of copies of the integers

$$\prod_{m=1}^{\infty} \mathbb{Z} = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \ldots .$$

Its elements are unrestricted infinite tuples $(n_1, n_2, n_3, \ldots)$ of integers. This group is sometimes called the *Baer-Specker group*. At first glance, one might suspect that this group is free. However, it follows from results of Baer dating back to 1937 that the group is not free [1]:

Theorem. *The abelian group $\prod_{m=1}^{\infty} \mathbb{Z}$ has no basis.*

I am very fond of surprising facts like this. Nevertheless, I always used to forget the reasoning. Consequently, from time to time I found myself racking my brains to recall the proof. In the next section I give a direct, down-to-earth proof that I find easy to remember.

The infinite direct product $\prod_{m=1}^{\infty} \mathbb{Z}$ contains an obvious free subgroup, namely the infinite direct sum $\bigoplus_{m=1}^{\infty} \mathbb{Z}$. In some sense, however, this subgroup is rather small, because the infinite direct sum has the cardinality of the set of natural numbers, whereas the infinite direct product has the cardinality of the set of real numbers. Actually, Specker [4] proved that every countable subgroup of the Baer-Specker group is free.

The proof for the nonfreeness of $\prod_{m=1}^{\infty} \mathbb{Z}$ is somewhat buried under more general statements in Baer's original paper ([1], Theorem 12.4). His reasoning, which combines cardinality arguments with reduction modulo prime numbers, also appears in very readable form in Kaplansky's monograph on infinite abelian groups ([2], Theorem 21). The proof given below is related more to Kaplansky's technique of splitting-off countable summands [3].

2. THE PROOF OF BAER'S RESULT

Let $G = \prod_{m=1}^{\infty} \mathbb{Z}$ be the infinite product of copies of the integers. Our task is to show that this abelian group is not free. Seeking a contradiction, we assume that there is a basis $e_\alpha \in G$, $\alpha \in I$. Note that the index set I must be uncountable. This is because any infinite product of sets with at least two elements is uncountable, whereas any abelian group with a countable number of generators is countable.

To start with, consider the standard vectors

$$s_k = (\ldots, 0, 1, 0, \ldots) \in G,$$

whose sole nonzero entry is a 1 in the k-th position. Note that the s_k, $k \geq 1$, generate the infinite direct sum inside the infinite direct product. Write the standard vectors $s_k = \sum_{\alpha \in I} \lambda_{\alpha k} e_\alpha$ as linear combinations of our basis, where all but finitely many coefficients are zero. Let $J \subset I$ be the subset of all indices $\alpha \in I$ with the property that $\lambda_{\alpha k} \neq 0$ for at least one index $k \geq 1$. Being the countable union of finite sets, the set J is countable. The idea of the proof is to consider the subgroup $H \subset G$ generated by the basis elements e_α with $\alpha \in J$.

We shall exploit the following two properties of $H \subset G$: First, H is countable and contains the infinite direct sum $\bigoplus_{m=1}^{\infty} \mathbb{Z}$. Second, the quotient group G/H contains a basis, namely the cosets of the e_α with $\alpha \in I - J$.

To achieve the desired contradiction, consider elements $y \in G$ whose entries are strictly increasing in the multiplicative sense; this means

$$y = (n_1, n_2, n_3, \ldots) \in G$$

so that each quotient n_{i+1}/n_i is an integer $\neq \pm 1$. Any sequence $(q_1, q_2, \ldots)$ of integers $q_i \neq \pm 1$ yields such a $y = (n_1, n_2, \ldots)$ by setting $n_i = q_1 q_2 \ldots q_i$. As noted in the first paragraph of this section, there are uncountably many such $(q_1, q_2, \ldots)$. Whence there are also uncountably many $y \in G$ as above. The upshot is that we may pick one such $y \in G$ so that the coset $\bar{y} \in G/H$ is nonzero, because H is merely countable. Now comes the crucial observation: Since

$$y \equiv (\underbrace{0, \ldots, 0}_{i \text{ entries}}, n_{i+1}, n_{i+2}, \ldots) \quad \text{modulo } H,$$

the equation $\bar{y} = nx$ has a solution $x \in G/H$ for all the integers $n = n_1, n_2, \ldots$. According to the lemma below, this is impossible in free abelian groups. By construction, however, the quotient G/H contains a basis, a contradiction. $\square$

Lemma. *Let A be a free abelian group, and let $v \in A$ be a nonzero element. Then there are only finitely many integers $n \in \mathbb{Z}$ so that the equation $v = nx$ admits a solution $x \in A$.*

Proof. We already saw the argument in Section 1, in the special case $A = \mathbb{Q}$; let me repeat it for the sake of completeness. Without loss of generality we may assume $A = \bigoplus_{\alpha \in I} \mathbb{Z}$. Write $v = (n_\alpha)_{\alpha \in I}$, and choose some index $\beta \in I$ with $n_\beta \neq 0$. If the equation $v = nx$ has a solution $x \in A$, then n divides n_β. Being a nonzero integer, n_β has only finitely many divisors. $\square$

References

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