The Strong Franchetta Conjecture in arbitrary characteristics

Habilitationsschrift

Vorgelegt von
Stefan Schröer

Introduction

Let M_g be the coarse moduli space of smooth curves of genus $g \geq 3$ over an arbitrary ground field k. Deligne and Mumford [9] showed that the normal quasiprojective scheme M_g is irreducible. Let $\eta \in M_g$ be its generic point and $C = M_{g,1} \to M_g$ the tautological curve. The generic fiber C_η is a smooth curve of genus g over the function field $\kappa(\eta)$ of the moduli space M_g. We call it the *generic curve*.

Franchetta [12] conjectured $\mathrm{Pic}(C_\eta) = \mathbb{Z}K_{C_\eta}$. Arbarello and Cornalba [2] proved this over the complex numbers using Harer's calculation [28] of the second homology for the mapping class group of Riemann surfaces. The latter is a purely topological result. Later, Arbarello and Cornalba [3] gave an algebro-geometric proof over the complex numbers. Mestrano [40] and Kouvidakis [36] deduced the *Strong Franchetta Conjecture* over $\mathbb{C}$, which states that the rational points in the Picard scheme $\mathrm{Pic}_{C_\eta/\eta}$ are precisely the multiples of the canonical class.

The first goal of this paper is to give an algebraic proof for the Strong Franchetta Conjecture in all characteristics $p \geq 0$. The idea is to construct special stable curves showing that any divisor class violating Franchetta's Conjecture must be nontorsion. Having this, we use Moriwaki's recent calculation [42] of $\mathrm{Pic}(\overline{M}_{g,n+1}) \otimes \mathbb{Q}$ in characteristic $p > 0$ to infer the Strong Franchetta Conjecture. I also prove the Franchetta Conjecture for generic *pointed* curves: Their Picard groups are freely generated by the canonical class and the marked points. Actually, I use the pointed case as an essential step in the proof for the unpointed case.

The second goal of this paper is to show that there are many other non-closed points $x \in M_{g,n}$ such that the marked points and the canonical class generates $\mathrm{Pic}(C_x)$, at least up to torsion. This seems to be new even in characteristic zero. We shall see that over uncountable ground fields, there is an uncountable dense set of such points with $\dim \overline{\{x\}} \leq 2$.

This relies on *Hilbert's Irreducibility Theorem* for function fields. The idea is to view C_η as the generic fiber of some fibered surface Y, extend this to a family of fibered surfaces $\mathfrak{Y} \to S$ over some parameter space S, and apply Hilbert's Irreducibility Theorem and the *Tate–Shioda Formula* to the resulting family of Néron–Severi scheme $s \mapsto \mathrm{NS}_{\mathfrak{Y}_s/s}$. Such specialization arguments are problematic in characteristic $p > 0$, because ungeometric properties such as regularity behave badly in families. However, we overcome these difficulties by using the theory of *geometric unibranch singularities*.

Here is a plan for the paper. The first chapter contains basic notation and fundamental facts. In Section 1.1 we recall the definition and properties of the Picard scheme. In Section 1.2 we define the Néron–Severi scheme as the quotient of the Picard scheme by the connected component of the origin. We discuss its relation to the Néron–Severi group and its behaviour under universal homeomorphisms. In Section 1.3 we review the definition of stable curves and some properties of the moduli space. Section 1.4 deals with curves of compact type.

Chapter 2 contains the proof of the Strong Franchetta Conjecture. In Section 2.1, we first prove that the Mordell–Weil group of the generic Jacobian is torsion free. This result is further improved in Section 2.2. Building on this, we prove the Strong Franchetta Conjecture in Section 2.3. As an application I deduce in Section 2.4 that the generic curve in characteristic $p = 2$ does not admit a tamely ramified morphism to the projective line. In Section 2.5 we construct an explicit stable curve X of genus g over the rational function field F so that the $\mathrm{Pic}^0_{X/F}$-torsor $\mathrm{Pic}^1_{X/F}$ has order $2g - 2$ in the Weil–Châtelet group.

In the last Chapter, we extend Franchetta's Conjecture to other nonclosed points $x \in M_{g,n}$, at least up to torsion. Section 3.1 contains some facts about geometrically unibranch schemes in characteristic $p > 0$, which we shall use as substitutes for normal schemes. In Section 3.2 we use Hilbert's Irreducibility Theorem, which comes from Galois theory, to study Picard numbers in families of proper schemes. We shall apply these results in Section 3.3: There we first construct fibered surfaces with small Picard number over transcendental extension fields. Specializing them, we show that there are many nonclosed points $x \in M_{g,n}$ such that $\mathrm{Pic}(C_x)$ has the same rank as $\mathrm{Pic}(C_\eta)$. Section 3.4 contains a list of some open problems.

Contents

1 Preliminaries

In this chapter I will introduce basic notation and collect several well-known results.

1.1 The Picard scheme

Throughout this paper, we shall work over a fixed ground field k of arbitrary characteristic $p \geq 0$. Usually, we shall deal with finitely generated field extensions $k \subset F$ and proper F-schemes X. We write $F \subset \bar{F}$ for an algebraic closure and $\bar{X} = X \otimes \bar{F}$.

Grothendieck [21] introduced the *relative Picard functor* $\mathrm{Pic}_{X/F}$ as the sheafification of the presheaf $S \mapsto H^0(S, R^1 \mathrm{pr}_*(\mathcal{O}_{X \times S}^\times))$, where S ranges over the F-schemes, and $\mathrm{pr} : X \times S \to S$ is the projection. Sheafification is with respect to the fpqc-topology, whose refinements are the quasicompact flat surjections. According to [27], Exposé XII, Corollary 1.5, the relative Picard functor is representable by a group scheme. Moreover, the connected component $\mathrm{Pic}_{X/F}^0$ of the origin is a group scheme of finite type ([27], Exposé XIII, Theorem 4.7)

We denote by $\mathrm{Pic}(X/F) = \mathrm{Pic}_{X/F}(F)$ the group of rational points. If X contains a rational point, this is nothing but the Picard group of X. In general we have an exact sequence

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X/F) \longrightarrow \mathrm{Br}(F) \longrightarrow \mathrm{Br}(X),$$

where $\mathrm{Br}(F)$ is the Brauer group ([24], Corollary 5.3). Tsen's Theorem implies the following (compare [24], Theorem 1.1):

Lemma 1.1 *Suppose that k is algebraically closed and that F is the function field of a smooth algebraic curve over k. Then $\mathrm{Pic}(X) = \mathrm{Pic}(X/F)$.*

If $\mathrm{Pic}_{X/F}^0$ is an abelian variety, the group of rational points $\mathrm{Pic}^0(X/F)$ is also called the *Mordell–Weil group*. Note that we do not lose rational points by passing to field extensions:

Lemma 1.2 *Let $F \subset F'$ be a field extension, and $X' = X \otimes_F F'$ the induced curve. Then the canonical map $\mathrm{Pic}(X/F) \to \mathrm{Pic}(X'/F')$ is injective.*

Proof. We have $\mathrm{Pic}_{X'/F'} = \mathrm{Pic}_{X/F} \otimes F'$, hence the fiber over the rational point $0 \in \mathrm{Pic}_{X/F}$ is nothing but the rational point $0 \in \mathrm{Pic}_{X'/F'}$. $\square$

Let $\operatorname{Pic}^0_{X/F} \subset \operatorname{Pic}_{X/F}$ be the connected component of the origin. We sometimes write $J = \operatorname{Pic}^0_{X/F}$ and call it the *Jacobian*. Suppose $T \subset \operatorname{Pic}_{X/F}$ is another connected component, or more generally a nonempty union of connected components. Then J acts freely on T, and T defines a class in the *Weil–Châtelet group* $H^1(T/J, \operatorname{Pic}^0_{X/F})$. Here cohomology is with respect to the fppf-topology, whose refinements are the flat surjections of finite presentation. Note that if the Jacobian is smooth, then we may compute the Weil–Châtelet group also in the étale cohomology, whose refinements are the quasifinite étale surjections ([24], Theorem 11.7). In characteristic $p = 0$, the Jacobian is smooth, as any group scheme. In positive characteristics, the condition $H^2(X, \mathcal{O}_X) = 0$ implies smoothness of the Jacobian by [44], Corollary on page 198.

An example for such a T is the subgroup scheme $\operatorname{Pic}^\tau_{X/F} \subset \operatorname{Pic}_{X/F}$ given by invertible sheaves having a nonzero multiple inside $\operatorname{Pic}^0_{X/F}$. Then $\operatorname{Pic}^\tau_{X/F}$ is both open and closed by [22], Theorem 1.1. Actually, it is a group scheme of finite type by [27], Exposé XIII, Theorem 4.7.

Another example for T is the following. Suppose X is a curve, and decompose $\bar{X} = \bar{X}_1 \cup \ldots \bar{X}_n$ into irreducible components. For each $d \in \mathbb{Z}^n$, let $\operatorname{Pic}^d_{X/F} \subset \operatorname{Pic}_{X/F}$ be the subset given by those invertible sheaves having degree d_i on $\bar{X}_i$. Then $\operatorname{Pic}^d_{X/F}$ is a J-torsor, provided it is nonempty. Moreover, $\operatorname{Pic}^d_{X/F}$ is nonempty for all $d \in \mathbb{Z}^n$, at least if X is geometrically reduced. The J-torsor $\operatorname{Pic}^d_{X/F}$ is trivial if and only if it contains a rational point. Note also that the Jacobian is smooth and $\operatorname{Pic}^0_{X/F} = \operatorname{Pic}^\tau_{X/F}$ holds.

To illustrate this, consider the 2-dimensional Fermat quadric $X \subset \mathbb{P}^2_\mathbb{R}$ given by $T_0^2 + T_1^2 + T_2^2 = 0$. This is a smooth curve of genus $g = 0$, hence $J = 0$. The sheaf $\omega_X = \mathcal{O}_X(1)$ defines a rational point in $\operatorname{Pic}^2_{X/\mathbb{R}}$, hence $\operatorname{Pic}^d_{X/\mathbb{R}} = \operatorname{Spec}(\mathbb{R})$ for all $d \in \mathbb{Z}$. The rational point in $\operatorname{Pic}^1_{X/\mathbb{R}}$, however, does not come from an invertible sheaf, because X has no rational point.

Perhaps a more interesting example is the 3-dimensional Fermat quadric $X \subset \mathbb{P}^3_\mathbb{R}$ given by $T_0^2 + \ldots + T_3^2 = 0$. Here we have $\bar{X} = \mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}$ via the Segre embedding, and again $J = 0$. Fix a rational point $c \in \mathbb{P}^3_\mathbb{R} - X$. Projecting away from c we obtain a flat double covering $X \to \mathbb{P}^2_\mathbb{R}$. This covering is injective on closed points, because $X \subset \mathbb{P}^3_\mathbb{R}$ has degree two and $X(\mathbb{R}) = \emptyset$. It follows that any two curves on X have nonempty intersection, and we conclude that none of the two geometric fibrations is defined over $\mathbb{R}$. The structure of the Picard scheme is therefore as follows: Let $\operatorname{Pic}^{(a,b)}_{X/\mathbb{R}} \subset \operatorname{Pic}_{X/\mathbb{R}}$ be the connected component corresponding to invertible sheaves of bidegree

(a, b). Using that $\mathcal{O}_X(1)$ has bidegree $(1, 1)$, we infer

$$\operatorname{Pic}_{X/\mathbb{R}}^{(a,b)} = \begin{cases} \operatorname{Spec}(\mathbb{R}) & \text{if } a = b, \\ \operatorname{Spec}(\mathbb{C}) & \text{otherwise.} \end{cases}$$

1.2 The Néron–Severi scheme

Suppose that X is a proper F-scheme. Recall that the *Néron–Severi group scheme* is the quotient $\operatorname{NS}_{X/F} = \operatorname{Pic}_{X/F} / \operatorname{Pic}_{X/F}^0$. This is an étale group scheme over F, whose points are given by finite separable field extensions ([26], Exposé VI$_A$, 5.5). Its group of rational points is denoted by $\operatorname{NS}(X/F)$, which is a finitely generated abelian group ([27], Exposé XIII, Theorem 5.1). The *Néron–Severi* group is the subgroup $\operatorname{NS}(X) \subset \operatorname{NS}(X/F)$ generated by $\operatorname{Pic}(X)$.

Lemma 1.3 *The inclusion $\operatorname{NS}(X) \subset \operatorname{NS}(X/F)$ has finite index.*

Proof. Fix a rational point $z \in \operatorname{NS}(X/F)$. The corresponding connected component $T \subset \operatorname{Pic}_{X/F}$ is a torsor under $J = \operatorname{Pic}_{X/F}^0$, and it suffices to show that its order in $H^1(F, J)$ is finite. Indeed, then T^n contains a rational point x for some $n > 0$, and mx comes from an invertible sheaf for some $m > 0$ because $\operatorname{Br}(F)$ is torsion.

Let $F \subset F'$ be some finite field extension such that $T' = T \otimes F'$ acquires a rational point y. Passing to some multiple, we may also assume that y comes from some invertible sheaf $\mathcal{M}$ on $X' = X \otimes F'$.

First, consider the special case that F' is separable. Then we may assume that F' is Galois, say with Galois group G. Write $n = |G|$. Then $\bigotimes_{g \in G} g^*(\mathcal{M})$ is a G-invariant invertible sheaf whose rational point lies on T'^n, hence descends to the desired rational point $x \in T^n$.

Next, consider the special case that we are in characteristic $p > 0$ and F' is purely inseparable. Then the projection $X' = X \otimes F' \to X$ is a finite universal homeomorphism of F-schemes. Similarly, the n-fold relative Frobenius morphism $X' \to X'^{(p^n)}$ is a finite universal homeomorphism. Recall that $X'^{(p^n)}$ is defined by the cartesian diagram

$$\begin{array}{ccc} X'^{(p^n)} & \longrightarrow & X' \\ \downarrow & & \downarrow \\ \operatorname{Spec}(F) & \xrightarrow{\operatorname{Fr}^n} & \operatorname{Spec}(F). \end{array}$$

According to [35], Proposition 6.6, we have a factorization $X' \to X \to X'^{(p^n)}$ for some $n \geq 0$. It follows that $\mathcal{M}^{\otimes p^n}$ comes from the invertible $\mathcal{O}_X$-module $\mathcal{L} = \mathcal{M}^{(p^n)}|X$.

The general case follows from the special cases, because any finite field extension is given by a purely inseparable extension followed by a separable extension. $\qquad\square$

The Néron–Severi scheme behaves well under universal homeomorphism in characteristic $p > 0$:

Proposition 1.4 *Let* $f : X \to Y$ *be a universal homeomorphism of proper F-schemes in characteristic* $p > 0$. *Then kernel and cokernel of the induced map* $\mathrm{NS}(Y/F) \to \mathrm{NS}(X/F)$ *are finite p-groups. In particular, we have* $\rho(Y) = \rho(X)$.

Proof. By [35], Proposition 6.6, we have a factorization $X \to Y \to X^{(p^n)}$ for some $n \geq 0$. Observe that the cartesian diagram

$$
\begin{array}{ccc}
X^{(p^n)} & \longrightarrow & X \\
\downarrow & & \downarrow \\
\mathrm{Spec}(F) & \xrightarrow{\ \ \mathrm{Fr}^n\ \ } & \mathrm{Spec}(F).
\end{array}
$$

induces a bijection $\mathrm{NS}(X/F) \to \mathrm{NS}(X^{(p^n)}/F)$. This is because Fr^n is a universal homeomorphism, and sections for the étale scheme $\mathrm{NS}_{X/F}$ correspond to points $l \in \mathrm{NS}_{X/F}$ such that $\mathrm{Spec}\,\kappa(l) \to \mathrm{Spec}(F)$ is a universal homeomorphism ([20], Corollary 17.9.3). It follows that kernel and cokernel of $\mathrm{NS}(X^{(p^n)}/F) \to \mathrm{NS}(X/F)$ are p-groups. Hence the cokernel of $\mathrm{NS}(Y/F) \to \mathrm{NS}(X/F)$ is a p-group.

Finally, suppose $L \in \mathrm{NS}(Y/F)$ vanishes on X. As above, there is a factorization $Y \to X^{(p^n)} \to Y^{(p^m)}$ for some $m \geq 0$. Then $p^m L$ comes from a point $M \in \mathrm{NS}(X^{(p^n)}/F)$, and $p^n M$ vanishes on X. It follows that some p-power annihilates L. $\qquad\square$

Remark 1.5 The preceding result does not hold in characteristic zero. To see an example, set $X = \mathbb{P}^2_{\mathbb{C}}$, such that $\mathrm{Pic}(X) = \mathbb{Z}$. According to [29], Exercise 5.9 on page 232, there is a first order infinitesimal extension $X \subset Y$ with ideal $\mathcal{O}_X(-3) = \omega_X$ such that $\mathrm{Pic}(Y) = 0$.

The intersection pairing defines a symmetric integral valued multilinear form

$$\mathcal{L}_1, \mathcal{L}_2 \ldots, \mathcal{L}_d \longmapsto (\mathcal{L}_1 \cdot \mathcal{L}_2 \cdot \ldots \cdot \mathcal{L}_d)$$

on $\mathrm{NS}(X/F)$, where $d = \dim(X)$. If X admits an ample invertible sheaf $\mathcal{H}$, this form is nondegenerate in the following sense: Given an invertible sheaf $\mathcal{L}_1$ whose class in $\mathrm{NS}(X)$ is nontorsion, there is another invertible sheaf $\mathcal{L}_2$ with $(\mathcal{L}_1 \cdot \mathcal{L}_2 \cdot \mathcal{H} \cdot \ldots \cdot \mathcal{H}) \neq 0$. Note, however, that the intersection pairing might be trivial on nonprojective proper schemes, see [51].

1.3 The moduli space of stable curves

We now discuss the coarse moduli space of stable curves. For us, a *curve* over k is a 1-dimensional proper k-scheme X with $k = H^0(X, \mathcal{O}_X)$. The latter condition implies that X is geometrically connected.

A curve X is called *stable* if the induced curve $\bar{X}$ over the algebraic closure $\bar{k}$ is reduced, has only ordinary double points, and satisfies the stability condition: Any smooth rational component contains at least three double points, any singular rational component contains at least two double points, and any elliptic component contains at least one double point. Equivalently, the group scheme $\mathrm{Aut}_{X/k}$ is finite, or the invertible sheaf $\omega_{X/k}$ is ample.

Suppose X is a smooth curve of genus $g = h^1(\mathcal{O}_X)$. It then follows that $\omega_{X/k}^{\otimes 3}$ is very ample, and there is a closed embedding $X \subset \mathbb{P}_k^{2gn-2n-g}$. Using these 3-canonical embeddings, Mumford [45] constructed the coarse moduli space M_g of smooth curves of genus g as a quasiprojective normal scheme having only quotient singularities. Its closed points correspond to isomorphism classes of smooth curves over k, provided that k is algebraically closed. Later, Mumford [46] extended these ideas and constructed the coarse moduli space $\overline{M}_g$ of stable curves of genus g as a projective scheme. It comes along with an open dense inclusion $M_g \subset \overline{M}_g$. Another proof for the existence of an ample sheaf on $\overline{M}_g$ in characteristic $p = 0$ appeared in [33].

Deligne and Mumford [9] used the compactification $M_g \subset \overline{M}_g$ to prove that M_g is irreducible. We shall usually denote its generic point by $\eta \in M_g$. Note that M_g is not a fine moduli space, so there is no universal curve over the moduli space. However, for $g \geq 3$ there is a dense open subset $\overline{M}_g^0 \subset \overline{M}_g$ such that for each geometric point $\mathrm{Spec}(F) \to \overline{M}_g^0$, the corresponding curve X over F has no automorphism. This is a classical result about Riemann surfaces [4], see also [48] for an algebraic proof. It then follows that there is

10

a universal family $C^0 \to \overline{M}_g^0$. We call its generic fiber C_η the *generic curve of genus* $g \geq 3$. This is a smooth curve over the function field $k(\eta)$ of the moduli space M_g.

It will be important to consider *stable pointed curves* of genus $g \geq 2$ as well. An n-pointed curve $(X, x_1, \ldots, x_n)$ is a curve endowed with n distinct rational points $x_i \in X$ lying in the smooth locus. These points are also called the marked points. The stability condition for pointed curves is the following: Each smooth rational component in $\bar{X}$ contains either three double points, or two double points and a marked point, or three marked points; each singular rational component contains either an additional double point or a marked point; each elliptic component contains either a double point or a marked point.

As in the unpointed situation, there is a quasiprojective coarse moduli space $M_{g,n}$ of smooth pointed curves, and a projective coarse moduli space $\overline{M}_{g,n}$ of stable pointed curves. Moreover, these moduli spaces are irreducible and $M_{g,n} \subset \overline{M}_{g,n}$ is a dense open subset. We shall write $\eta_n \in M_{g,n}$ for the generic point.

Recall that $\overline{M}_g$ does not admit a universal family. However, it admits a tautological family $C = \overline{M}_{g,1} \to \overline{M}_g$. On geometric points, the projection sends the isomorphism class of a pointed curve (X, x_1) to the isomorphism class of the underlying unpointed curve X. This family is indeed universal over the open subset $\overline{M}_{g,n}^0$, but acquires nonreduced fibers over points corresponding to curves with automorphisms.

More generally, the moduli spaces $\overline{M}_{g,n}$ with g fixed are related by several tautological maps. We have a projection morphism $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ sending the isomorphism class of $(X, x_1, \ldots, x_n, x_{n+1})$ to the isomorphism class of $(X', x_1', \ldots, x_n')$. Here X' is obtained from X by forgetting the last marked point, and contracting the possibly occurring rational component violating stability. The x_i' are the images of the x_i.

Let $\eta_n \in \overline{M}_{g,n}$ be the generic point and $C_n = \overline{M}_{g,n+1}$ the tautological family of curves. The generic fiber C_{η_n} is a smooth curve of genus g, which is endowed with marked rational points $c_1, \ldots, c_n \in C_{\eta_n}$. We call C_{η_n} the *generic n-pointed curve of genus g.*

1.4 Curves of compact type

A stable curve X over F is called of *compact type* if $\mathrm{Pic}^0_{X/F}$ is an abelian variety. Equivalently, the map $\mathrm{Pic}^0_{X/F} \to \mathrm{Pic}^0_{\tilde{X}/F}$ induced by the normalization $\tilde{X} \to X$ is injective.

We may calculate the kernel of $\mathrm{Pic}(X) \to \mathrm{Pic}(\tilde{X})$ as follows: Let $D \subset X$ and $C \subset \tilde{X}$ be the closed subscheme defined by the conductor ideal for the inclusion $\mathcal{O}_X \subset \mathcal{O}_{\tilde{X}}$. Then the diagram

$$
\begin{array}{ccc}
C & \longrightarrow & \tilde{X} \\
\downarrow & & \downarrow \\
D & \longrightarrow & X
\end{array}
$$

is both cartesian and cocartesian. The 0-dimensional schemes C and D are smooth because $\tilde{X}$ has only ordinary double points.

Following Deligne and Rapoport [10], Section 3.5, we define a bicolored graph $\Gamma(X)$ as follows: The white vertices are the connected components of $\tilde{X}$. The black vertices are the connected components of D. The edges are the connected components of C. An edge $c \in C$ connects a white vertex $\tilde{X}' \subset \tilde{X}$ with a black vertex $d \in D$ if $c \in \tilde{X}'$ and $c \mapsto d$.

The graph $\Gamma(X)$ is labeled with fields in a canonical way: The label at a white vertex is $\Gamma(\tilde{X}', \mathcal{O}_{\tilde{X}'})$, the label at a black vertex is $\kappa(d)$, and the label along the edges are $\kappa(c)$. We may view this as an abelian sheaf $\mathcal{F}$ on $\Gamma(X)$ and compute its cohomology groups $H^1(\Gamma(X), \mathcal{F})$ and $H^1(\Gamma(X), \mathcal{F}^\times)$. The exact sequence

$$
1 \longrightarrow \mathcal{O}_X^\times \longrightarrow \mathcal{O}_D^\times \oplus \mathcal{O}_{\tilde{X}}^\times \longrightarrow \mathcal{O}_C^\times \longrightarrow 1
$$

implies that we have an exact sequence

$$
0 \longrightarrow H^1(\Gamma(X), \mathcal{F}^\times) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(\tilde{X}). \tag{1}
$$

Proposition 1.6 *Let X be a stable curve. If X is of compact type, then $H^1(\Gamma(X), \mathcal{F}^\times) = 0$. Conversely, if F is algebraically closed and $H^1(\Gamma(X), \mathbb{Z})$ vanishes, then X is of compact type.*

Proof. The first part follows from the preceding exact sequence (1). Now suppose that F is algebraically closed and $H^1(\Gamma(X), \mathbb{Z}) = 0$. First note that $H^1(\Gamma(X), F^\times)$ vanishes as well, because the canonical map

$$
H^1(\Gamma(X), \mathbb{Z}) \otimes F^\times \longrightarrow H^1(\Gamma(X), F^\times)
$$

is bijective by the Künneth formula [56], Chapter 5, Theorem 10. The abelian sheaf $\mathcal{F}$ on the graph $\Gamma(X)$ is nothing but the constant sheaf F, because F is algebraically closed. The statement now follows from the exact sequence (1). $\qquad \square$

Remark 1.7 There are many funny stable curves of compact type over non-closed fields. Consider, for example, a complex elliptic curve E viewed as a real scheme. Let $e \in E$ be a closed point, and consider the curve X over $\mathbb{R}$ obtained by replacing the residue field $\kappa(e) = \mathbb{C}$ by $\mathbb{R}$. Then X is a stable curve of compact type. Its normalization is $\tilde{X} = E$, and $\bar{X} = X \otimes \mathbb{C}$ is the union of two copies of E glued together at a single common point.

Stable curves of compact type have nice properties:

Lemma 1.8 *Let* $F \subset F'$ *be a purely transcendental field extension,* X *a stable curve over* F, *and* $X' = X \otimes F'$ *the induced curve. If* X *is of compact type, then the preimage map* $\mathrm{Pic}(X) \to \mathrm{Pic}(X')$ *is bijective.*

Proof. By the usual limit argument ([20], Theorem 8.5.2), we may assume that F' is finitely generated. Applying induction, we reduce to the case $\mathrm{trdeg}(F') = 1$. Now we may view F' as the function field of $\mathbb{P}^1_F$.

The task is to check that $\mathrm{Pic}(X) \to \mathrm{Pic}(X')$ is surjective. First we do the special case that X is smooth. Then $X \times \mathbb{P}^1_F$ is factorial, so the restriction map $\mathrm{Pic}(X \times \mathbb{P}^1_F) \to \mathrm{Pic}(X')$ is surjective. Using $\mathrm{Pic}(X \times \mathbb{P}^1_F) = \mathrm{Pic}(X) \oplus \mathbb{Z}$ we deduce that $\mathrm{Pic}(X) \to \mathrm{Pic}(X')$ is surjective as well.

Now suppose X is arbitrary, and let $\tilde{X} \to X$ be the normalization. The maps $\mathrm{Pic}(X) \to \mathrm{Pic}(\tilde{X})$ and $\mathrm{Pic}(X') \to \mathrm{Pic}(\tilde{X}')$ are bijective, because X is of compact type. Applying the preceding special case to $\tilde{X}$, we infer that $\mathrm{Pic}(X) \to \mathrm{Pic}(X')$ is surjective. $\qquad \square$

Recall that, in characteristic $p > 0$, an abelian variety A over F has a *p-rank* f defined by $\mathrm{Hom}(\mu_p, A \otimes \overline{F}) = (\mathbb{Z}/p\mathbb{Z})^f$. The abelian variety is called *ordinary* if $f = \dim(A)$. This condition means that the group of geometric p-torsion points in $A(\overline{F})$ is as large as possible, namely $(\mathbb{Z}/p\mathbb{Z})^{\dim(A)}$. In characteristic zero, abelian varieties are ordinary by definition. The following specialization result will play a role in the sequel:

Lemma 1.9 *Suppose* B *is a discrete valuation ring with residue field* F *and field of fractions* Q. *Let* $Z \to \mathrm{Spec}(B)$ *be a relative stable curve such that*

Z_0 is of compact type and $\text{Pic}^0_{Z_0/F}$ is ordinary. If $\text{Pic}^0(Z_0/F)$ is torsion free, then $\text{Pic}^0(Z_\eta/Q)$ is torsion free as well.

Proof. Consider the relative generalized Jacobian $J = \text{Pic}^0_{Z/B}$. This is a separated group scheme of finite type over B by [11], Proposition 3.4. It is an abelian scheme because Z_0 is of compact type. Let $J[m]$ be the relative kernel of the multiplication map $[m] : J \to J$. Then $J[m]$ is a finite flat group scheme over B whose fibers have length m^{2g}, where g is the genus of Z_0.

We claim that $J[m](Q) = 0$. Indeed, suppose we have a rational point $x \in J[m](Q)$. We now assume that the characteristic $p > 0$ is positive, the case $p = 0$ being similar. Decompose $m = lq$, where l is prime to p, and q is a power of p. By construction, the geometric closed fiber of $J[m]$ is isomorphic to $(\mathbb{Z}/l\mathbb{Z})^{2g} \oplus (\mathbb{Z}/q\mathbb{Z})^g \oplus \mu_q^g$, where μ_q is the local group scheme of q-th roots of unity (here we use that J_0 is ordinary). The same holds for all geometric fibers, because the number of geometric connected components is lower semicontinuous in flat proper families ([19], Proposition 15.5.9). By [20], Proposition 15.5.1, each of the $l^{2g}q^g$ geometric points in the closed fiber extend to disjoint sections over the strict henselization $B \subset B^{\text{sh}}$, which defines $l^{2g}q^g$ different geometric points over the generic fiber. It now follows from [25], Exposé VIII, Theorem 4.1, that the closure $\overline{\{x\}} \subset J[m]$ is disjoint from the zero section. In other words, the specialization map $J[m](Q) \to J[m](F)$ is injective. Since $J(F)$ is torsion free we have $J[m](Q) = 0$. $\qquad\square$

2 Franchetta's Conjecture

In this Chapter we shall prove the Strong Franchetta Conjecture.

2.1 Torsion points

Fix an integer $n \geq 0$ and a genus $g \geq 3$. Let C_{η_n} be the generic n-pointed curve of genus g. In this section we take care of torsion points:

Proposition 2.1 *The group* $\mathrm{Pic}^0(C_{\eta_n}/\eta_n)$ *is torsion free.*

The proof requires some preparation. First note that by Lemma 1.2, we may replace the ground field k by any extension field. For the rest of the section, we assume that k is algebraically closed, and write $F = k(T)$ for the rational function field in one indeterminate.

Proposition 2.2 *There is an ordinary elliptic curve E over the rational function field F satisfying* $\mathrm{Pic}^0(E) = 0$.

Proof. The idea is to use special Halphen pencils. Fix an ordinary elliptic curve E_0 over k. Such a curve exists because there are only finitely many supersingular elliptic curves ([29], Chapter IV, Corollary 4.23). Let $x \in E_0$ be the origin and consider the closed embedding $E_0 \subset \mathbb{P}_k^2$ defined via the very ample sheaf $\mathcal{O}_{E_0}(3x) \simeq \mathcal{O}_{E_0}(1)$.

Lines and quadrics in $\mathbb{P}_k^2$ are uniquely determined by their intersection with the cubic E_0, because the restriction maps

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n)) \longrightarrow H^0(E_0, \mathcal{O}_{E_0}(3nx))$$

are bijective for $n = 1, 2$. Let $L \subset \mathbb{P}_k^2$ be the unique line with $L \cap E_0 = 3x$. Setting $E_\infty = 3L$, we have $E_0 \cap E_\infty = 9x$. Consider the pencil of cubics E_t, $t \in \mathbb{P}_k^1$ generated by E_0 and E_∞. Then E_t is integral for $t \neq \infty$. Indeed, any quadric $Q \subset E_t$ satisfies $Q \cap E_0 = 6x$, hence $Q = 2L$. Moreover, any line $H \subset E_t$ has $H \cap E_0 = 3x$, hence $H = L$.

Now let $g' : Y' \to \mathbb{P}_k^2$ be the blowing-up of the nonreduced center $E_0 \cap E_\infty$, and $f' : Y' \to \mathbb{P}_k^1$ the induced fibration. The Jacobian of the generic fiber Y'_η is the desired elliptic curve E. To see this, note that $R' = g'^{-1}(E_0 \cap E_\infty)$ is isomorphic to the projective line over the Artin algebra $k[v]/(v^9)$. A local computation like in [43], page 417, reveals that Y' is smooth except for a rational double point $y' \in Y'$ of type A_8 lying on R'. Let $Y \to Y'$ be the

minimal resolution of this singularity. Its exceptional locus is a string of eight smooth rational (-2)-curves. It follows that the strict transform $R \subset Y$ of R'_{red} is a smooth rational (-1)-curve.

Let $f : Y \to \mathbb{P}^1_k$ be the induced fibration and consider its generic fiber Y_η. We now check that R generates $\mathrm{Pic}(Y_\eta)$. Indeed, the restriction map $\mathrm{Pic}(Y) \to \mathrm{Pic}(Y_\eta)$ is surjective, and $\mathrm{Pic}(Y)$ is generated by the exceptional curves for the sequence of blowing ups $g : Y \to \mathbb{P}^2_k$ and $g^*(L)$. The strict transform of L is disjoint from Y_η, because the strict transform of $3L$ is disjoint from Y_η, and we conclude that the restriction of $f^*(L)$ to Y_η is a multiple of R.

Summing up, the Jacobian $E = \mathrm{Pic}^0_{Y_\eta/\eta}$ contains no rational point but the origin. Then $\mathrm{Pic}^0(E) = 0$, because E is isomorphic to its own Jacobian. To check that the elliptic curve E is ordinary, look at the relative Jacobian of $f : Y \to \mathbb{P}^1_k$ near $0 \in \mathbb{P}^1_k$. Its closed fiber E_0 is ordinary. Since this is an open condition, the generic fiber E is ordinary as well. $\qquad\square$

Remark 2.3 (i) The rational double point $y' \in Y'$ maps to $\infty \in \mathbb{P}^1_k$. To see this, write $\mathcal{O}^\wedge_{\mathbb{P}^2,x} = k[[u,v]]$ so that $E_0, L \subset \mathbb{P}^2_k$ correspond to $u = 0$ and $v^3 = u$, respectively. Then $\mathcal{O}^\wedge_{Y',y'} = k[[u,v,w]]/(uw = v^9)$. You easily check that the equation $v^3 = u$ remains indecomposable inside $k[[u,v,w]]/(uw = v^9)$. This equation defines the preimage $g'^*(L) \subset Y'$, which decomposes as a Weil divisor into two irreducible components. So the strict transform of L in not Cartier, hence passes through the singular point $y' \in Y'$.
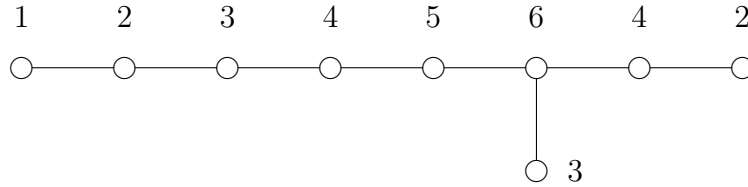

Figure 1: The Dynkin diagram $\widetilde{E}_8$

(ii) The degenerate fiber $X_\infty \subset X$ is of type II^* in Kodaira's notation, that is, it corresponds to the root lattice $\widetilde{E}_8$. Indeed, note that the string of eight (-2)-curves in X_∞ hits the strict transform of L in precisely one point. Moreover, the strict transform is not a (-1)-curve, because the intersection form on X_∞ is negative semidefinite. Glancing at Kodaira's classification ([34],

Theorem 6.2), we deduce that X_∞ must be of type II*. The corresponding Dynkin diagram appears in Figure 1, where the labels at the vertices denote the multiplicity of the corresponding integral component $C \subset X_\infty$.

Next, we construct a stable curve X of genus g over the rational function field $F = k(T)$ as follows: Let $E_1, \ldots, E_g$ be copies of the elliptic curve E from Proposition 2.2, and choose rational points $p_1, \ldots, p_g \in \mathbb{P}^1_F$. Let $X = E_1 \cup \ldots \cup E_g \cup \mathbb{P}^1_F$ be the curve obtained by identifying the rational point $p_i \in \mathbb{P}^1_F$ with the origins $0 \in E_i$ for $i = 1, \ldots, g$. This curve is stable because $g \geq 3$. It looks like in Figure 2.
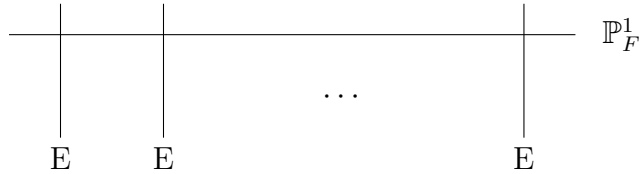

Figure 2: The stable curve X

Note that we may view X as an n-pointed curve, simply by choosing rational points in the rational component. The curve X has the following properties:

Proposition 2.4 *The curve X is of compact type, the abelian variety $\mathrm{Pic}^0_{X/F}$ is ordinary, and its Mordell–Weil group is $\mathrm{Pic}^0(X) = 0$.*

Proof. The normalization $\tilde{X} \to X$ is the disjoint union of g copies of E and a projective line. The canonical map $\mathrm{Pic}_{X/F} \to \mathrm{Pic}_{\tilde{X}/F}$ is an isomorphism, so $\mathrm{Pic}^0_{X/F} = E \times \ldots \times E$, and the result follows. $\square$

Proof of Proposition 2.1: Let $X = E_1 \cup \ldots \cup E_g \cup \mathbb{P}^1_F$ be the stable curve of genus g over the rational function field $F = k(T)$ constructed above. Pick n rational points $x_j \in \mathrm{Reg}(X)$ contained in the rational component $\mathbb{P}^1_F$. Let $\mathfrak{Y} \to \mathrm{Spf}(A)$ be the formal versal deformation of the n-pointed stable curve X. Then A is a formal power series ring in $3g - 3 + n$ variables with coefficients in F. Since $H^2(X, \mathcal{O}_X) = 0$, we may extend any ample invertible $\mathcal{O}_X$-module to an invertible $\mathcal{O}_{\mathfrak{Y}}$-module. By Grothendieck's Algebraization Theorem ([17], Theorem 5.4.5), the formal scheme $\mathfrak{Y}$ is the formal completion of a relative curve $Y \to \mathrm{Spec}(A)$.

Blowing up the closed point in $\mathrm{Spec}(A)$ and localizing at the generic point of the exceptional divisor, we obtain a discrete valuation ring B dominating

A. Its field of fractions $B \subset Q$ is also the field of fractions of A. The residue field $B_0 = B/\mathfrak{m}_B$ is a purely transcendental field extension of F. Let $Z \to \mathrm{Spec}(B)$ be the induced family of stable curves.

The classifying map $\mathrm{Spec}(B) \to \overline{M}_{g,n}$ is dominant. Hence $Z_Q = C_{\eta_n} \otimes Q$, which induces an injection $\mathrm{Pic}(C_{\eta_n}) \subset \mathrm{Pic}(Z_Q)$. We have $\mathrm{Pic}^0(Z_0) = 0$ by construction, so Lemma 1.9 applies, and we conclude that the group $\mathrm{Pic}^0(Z_Q)$ is torsion free. $\square$

2.2 Another stable curve

We keep the notation from the previous section, such that C_{η_n} is the generic n-pointed curve of genus g. Let $c_1, \ldots, c_n \in C_{\eta_n}$ be the marked points, and consider the free abelian group $P = \mathbb{Z}c_1 \oplus \ldots \oplus \mathbb{Z}c_n \oplus \mathbb{Z}K_{C_{\eta_n}}$. The following is a key step in proving Franchetta's Conjecture:

Proposition 2.5 *The cokernel of the map $P \to \mathrm{Pic}(C_{\eta_n})$ is torsion free.*

This again depends on the existence of certain elliptic curves over function fields. Let $F = k(T)$ be the rational function field, and E the ordinary elliptic curve with $\mathrm{Pic}^0(E) = 0$ from Proposition 2.4.

Proposition 2.6 *For each $m \geq 0$ there is a field extension $F \subset F'$ such that $E' = E \otimes F'$ satisfies $\mathrm{Pic}^0(E') = \mathbb{Z}^{\oplus t}$ for some $t \geq m$.*

Proof. The case $m = 0$ is trivial. We proceed by induction on m. Suppose we already have a field extension F' such that $\mathrm{Pic}^0(E') = \mathbb{Z}^{\oplus t}$ with $t \geq m$. Choose a basis $p_1, \ldots, p_t \in \mathrm{Pic}^0(E')$. Let $x \in E'$ be the origin, and $x_i \in E'$ the rational points with $x - x_i \sim p_i$.

Consider the henselization A of the localized polynomial algebra $F'[U]_{(U)}$, and the corresponding trivial family $Y = E' \otimes A$. Let $S, S_i \subset Y$ be the sections corresponding to the rational points $x, x_i \in E'$. Let $u, v \in \mathcal{O}_{Y,x}$ be the regular parameter system corresponding to $Y_0, S \subset Y$. Replacing v by a different parameter v', we obtain, locally around x, a curve $S' \neq S$ with $S' \cap Y_0 = \{x\}$. According to [20], Theorem 18.5.11, this locally defined curve S' defines a section $S_{t+1} \subset Y$ passing through x with $S_{t+1} \neq S$. Then the difference $S - S_{t+1}$ defines a nonzero point $p_{t+1} \in \mathrm{Pic}^0(Y_\eta)$ specializing to zero in $\mathrm{Pic}^0(Y_0)$. It follows that any nonzero multiple of p_{t+1} lies outside the span of the $p_1, \ldots, p_t$.

Next, choose a subalgebra $B \subset A$ that is a localization of a finite $F'[U]_{(U)}$-algebra at some F'-valued étale point, so that the section S_{t+1} is defined over B. Let F'' be the function field of B and set $E'' = E \otimes F''$. Then $\text{Pic}^0(E'')$ is torsion free according to Proposition 1.9. By construction, it contains a free group of rank $t + 1$.

It remains to check that $\text{Pic}^0(E'')$ is finitely generated. Let C be the normal curve over F' corresponding to the function field F'', and consider the regular proper surface $X = E' \times C$. Then $\text{Pic}^0_{X/F'} = \text{Pic}^0_{E'/F'} \times \text{Pic}^0_{C/F'}$, and the Néron–Severi group $\text{NS}(X) = \text{Pic}_{X/F'} / \text{Pic}^0_{X/F'}$ is finitely generated. Now view E'' as the generic fiber of the projection $X \to C$. The cokernel $\text{Pic}^0(E'')/\text{Pic}^0(E')$ injects into $\text{NS}(X)$, and we conclude that $\text{Pic}^0(E'')$ is finitely generated. $\square$

Remark 2.7 The proof shows that we may choose $F \subset F'$ as a finitely generated separable field extension of transcendence degree $\leq m$.

We now use such elliptic curves to construct a geometrically integral n-pointed stable curve of genus g. Choose a field extension $F \subset F'$ so that $\text{Pic}^0(E') = \mathbb{Z}^{\oplus t}$ for some $t \geq 2g - 2 + n$. Pick $2g - 2 + n$ rational points

$$p_1, p'_1, \ldots, p_{g-1}, p'_{g-1}, x_1, \ldots, x_n \in E'$$

that are part of a basis for $\text{Pic}(E')$. Let X be the stable curve of genus g obtained by identifying the pairs $p_i, p'_i \in E'$ for $i = 1, \ldots, g - 1$. The rational points $x_i \in E'$ define n rational points $x_i \in X$, which we denote by the same letter. The canonical morphism $\nu : E' \to X$ is the normalization map for X.

Proposition 2.8 *We have $\nu^*(K_X) = \sum_{j=1}^{g-1}(p_j + p'_j)$.*

Proof. This follows from duality theory for the finite morphism $\nu : E' \to X$. See, for example, [49], Proposition 2.3. $\square$

Proof of Proposition 2.5: Let X be the n-pointed stable curve of genus g over the function field F' constructed above. As in the proof of Proposition 2.1, we construct a discrete valuation ring B and a stable curve $Z \to \text{Spec}(B)$ with regular total space so that the following holds: (i) The residue field $B_0 = B/\mathfrak{m}_B$ is a purely transcendental field extension of F', and the closed fiber is $Z_0 = X \otimes B_0$. (ii) If $B \subset Q$ denotes the function field, then the classifying map $\text{Spec}(Q) \to \overline{M}_{g,n}$ is dominant.

Now suppose we have a class $L \in \mathrm{Pic}(C_{\eta_n})$ with $mL \in P$ for some $m \neq 0$. Write $mL = \lambda K_{C_{\eta_n}} + \sum \lambda_i c_i$ for certain coefficients $\lambda, \lambda_i \in \mathbb{Z}$. The task is to prove $L \in P$. Using the field extension $\kappa(\eta_n) \subset Q$, we obtain a class $L_Q \in \mathrm{Pic}(Z_Q)$. It extends to a divisor $D \in \mathrm{Div}(Z)$, and we have

$$mD = \lambda K_{Z/B} + \sum_{i=1}^{n} \lambda_i C_i$$

in $\mathrm{Pic}(Z)$, where $C_i \subset Z$ are the marked sections. This is because the divisor Z_0, being an integral fiber, supports only principal divisors. Pulling back to the normalization $\tilde{Z}_0 = E \otimes B_0$ of the closed fiber Z_0, we obtain

$$mD|_{\tilde{Z}_0} = \lambda \sum_{j=1}^{g-1}(p_j + p_j') + \sum_{i=1}^{n} \lambda_i x_i$$

in $\mathrm{Pic}(E \otimes B_0)$. Since the $p_j + p_j', x_i$ are part of a basis for $\mathrm{Pic}(E \otimes B_0)$ by Lemma 1.8, all coefficients λ, λ_i are multiples of m. Replacing L by $L - \frac{\lambda}{m} K_{C_{\eta_n}} - \sum \frac{\lambda_i}{m} c_i$, we reduce to the case that $mL = 0$. Then $L = 0$ by Proposition 2.1, and in particular $L \in P$. $\square$

2.3 The Strong Franchetta Conjecture

We come to the first main result of this paper:

Theorem 2.9 *Let k be a field, and $g \geq 3$ and $n \geq 0$ be integers. Let $\eta_n \in \overline{M}_{g,n}$ the generic point in the moduli space of n-pointed stable curves of genus $g \geq 3$, and $C = \overline{M}_{g,n+1}$ the tautological curve. Then the marked points $c_1, \ldots, c_n \in C_{\eta_n}$ and the canonical class $K_{C_{\eta_n}}$ freely generate $\mathrm{Pic}(C_{\eta_n}/\eta_n)$.*

Before we prove this, let us recall the definition of certain *tautological classes* in $\mathrm{Pic}(\overline{M}_{g,n}) \otimes \mathbb{Q}$. Let $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ be the projection sending an $(n+1)$-pointed stable curve $(X, x_1, \ldots, x_{n+1})$ to the n-pointed stable curve $(X', x_1', \ldots, x_n')$ obtained by contracting any component in $(X, x_1, \ldots, x_n)$ violating stability. The *Hodge class* $\lambda \in \mathrm{Pic}(\overline{M}_{g,n}) \otimes \mathbb{Q}$ is defined as the determinant of $\pi_*(\omega_\pi)$, where $\omega_\pi = \omega_{\overline{M}_{g,n+1}/\overline{M}_{g,n}}$ is the relative dualizing sheaf.

We also have canonical sections $s_i : \overline{M}_{g,n} \to \overline{M}_{g,n+1}$ for $i = 1, \ldots, n$ as follows. These morphisms are best described on geometric points: The

section s_i sends an n-pointed stable curve $(X', x_1', \ldots, x_n')$ to the $(n+1)$-pointed stable curve $(X, x_1, \ldots, x_{n+1})$ defined as follows: We have $X = X' \cup \mathbb{P}^1$, where the point $x_i' \in X'$ is identified with $\infty \in \mathbb{P}^1$. The marked points are $x_i = 0 \in \mathbb{P}^1$, $x_{n+1} = 1 \in \mathbb{P}^1$, and $x_j = x_j'$ for $j \neq i$. The *Witten classes* $\psi_i \in \mathrm{Pic}(\overline{M}_{g,n}) \otimes \mathbb{Q}$ are defined as $\psi_i = s_i^*(\omega_\pi)$.

There are also *boundary classes* $\delta_v \in \mathrm{Pic}(\overline{M}_{g,n}) \otimes \mathbb{Q}$, which are effective Weil divisors supported on $\overline{M}_{g,n} - M_{g,n}$. They correspond to various topological types of degeneration. The idea now is to restrict tautological classes on $\overline{M}_{g,n+1}$ to the generic curve $C_{\eta_n} \subset \overline{M}_{g,n+1}$.

Proposition 2.10 *The subgroup $P \subset \mathrm{Pic}(C_{\eta_n})$ generated by the marked sections and the canonical class contains the restriction of the tautological classes.*

Proof. First note that geometric points c in the generic n-pointed curve C_{η_n} correspond to $(n+1)$-pointed stable curves $(X, x_1, \ldots, x_{n+1})$. The curve X is smooth if c is not a marked point. On the other hand, if $c = c_i$ is a marked point, then $X = X' \cup \mathbb{P}^1$ with $x_i, x_{n+1} \in \mathbb{P}^1$. It now follows immediately from their definitions in [42], Section 1, that the restrictions $\delta_v|_{C_{\eta_n}}$ of the boundary classes are supported by the marked points $c_i \in C_{\eta_n}$.

How do Hodge classes and Witten classes restrict to the generic pointed curve? To see this, note that the fiber product $C_{\eta_n} \times_{\overline{M}_{g,n+1}} \overline{M}_{g,n+2}$ is isomorphic to the blowing up of $C_{\eta_n} \times_{\eta_n} C_{\eta_n}$ with respect to the centers (c_i, c_i) for $i = 1, \ldots, n$. This also follows from the modular interpretation of geometric points in $\overline{M}_{g,n+1}$. Indeed, the marked points $c_i \in C_{\eta_n}$, $i = 1, \ldots, n$ correspond to reducible stable curves of the form $(X' \cup \mathbb{P}^1, x_1, \ldots, x_{n+1})$ with $x_i, x_{n+1} \in \mathbb{P}^1$, and the exceptional curve in the blowing up with center $(c_i, c_i) \in C_{\eta_n} \times_{\eta_n} C_{\eta_n}$ is given by $(n+2)$-pointed stable curves

$$(X' \cup \mathbb{P}^1, x_1, \ldots, x_{n+2})$$

with $x_i, x_{n+1}, x_{n+2} \in \mathbb{P}^1$. As a consequence we have $\lambda|_{C_{\eta_n}} = 0$, and $\psi_i|_{C_{\eta_n}} = c_i$ for $i = 1, \ldots, n$, and $\psi_{n+1}|_{C_{\eta_n}} = K_{C_{\eta_n}}$. $\qquad\square$

Proof of Theorem 2.9: Set $P = \mathbb{Z}c_1 \oplus \ldots \oplus \mathbb{Z}c_n \oplus \mathbb{Z}K_{C_{\eta_n}}$. The canonical map $P \to \mathrm{Pic}(C_{\eta_n})$ is injective. Indeed, this is clear for $n = 0$. In case $n \neq 0$, we have $\mathrm{Pic}(C_{\eta_n}) = \mathrm{Pic}(C_{\eta_n}/\eta_n)$, and it suffices to construct an n-pointed stable curve X of genus g where the marked points and the canonical class are linearly independent. We constructed such a curve in Section 2.2.

The task is to prove surjectivity. Fix a point $L \in \mathrm{Pic}^d(C_{\eta_n}/\eta_n)$. Having only quotient singularities, the normal scheme $\overline{M}_{g,n+1}$ is $\mathbb{Q}$-factorial. Hence some multiple mL with $m > 0$ extends to a Cartier divisor class D on $\overline{M}_{g,n+1}$. According to [42], Theorem 5.1, we have $D = a\lambda + \sum_{i=1}^{n+1} b_i\psi_i + \sum_v d_v\delta_v$ for certain integral coefficients, at least after replacing m by a multiple. Recall that λ is the Hodge class, λ_i are the Witten classes, and δ_v are the boundary classes. Restricting to the generic curve and using Proposition 2.10, we infer $mL \in P$.

We now distinguish three case. First, suppose $n \geq 1$. Then $\mathrm{Pic}(C_{\eta_n}) = \mathrm{Pic}(C_{\eta_n}/\eta_n)$, and Proposition 2.5 implies $L \in P$. Second, suppose $n = 0$ and $d = 0$. Then there are only the two tautological classes λ and ψ_1 besides boundary classes. The equation

$$mL = (a\lambda + \sum_{i=1}^{n+1} b_i\psi_i + \sum_v d_v\delta_v)|_{C_\eta}$$

boils down to $mL = b_1 K_{C_{\eta_n}}$ and therefore $mL = 0$. Now Proposition 2.1 ensures $L = 0$.

So only the case $n = 0$ and $d \neq 0$ remains. Now we argue as follows: The $\mathrm{Pic}^0_{C_\eta/\eta}$-torsor $\mathrm{Pic}^{d(2g-2)}_{C_\eta/\eta}$ contains both $(2g-2)L$ and dK_{C_η}. These rational points differ by a point in $\mathrm{Pic}^0(C_\eta/\eta) = 0$, so $(2g-2)L = dK_{C_\eta}$. In other word, L is a rational multiple of K_{C_η}. Now consider the affine surjection $C_{\eta_1} \to C_\eta$ defined on geometric points by $(X, x_1, x_2) \mapsto (X, x_2)$. We already saw that the canonical class $K_{C_{\eta_1}} \in \mathrm{Pic}(C_{\eta_1})$ is a primitive element. Since $\mathrm{Pic}(C_\eta/\eta) \to \mathrm{Pic}(C_{\eta_1})$ is injective, the canonical class $K_\eta \in \mathrm{Pic}(C_\eta/\eta)$ is primitive as well, and we infer that L is an integral multiple of K_{C_η}. $\qquad\square$

Remark 2.11 By definition, the generic curves C_{η_n} with $n \geq 1$ contain the rational point c_1, hence $\mathrm{Pic}(C_{\eta_n}) = \mathrm{Pic}(C_{\eta_n}/\eta_n)$. So a priori there is no difference between weak and strong Franchetta Conjectures.

Remark 2.12 Moriwaki's calculation [42] of $\mathrm{Pic}(\overline{M}_{g,n}) \otimes \mathbb{Q}$ depends on the existence of certain simply connected coverings of $\overline{M}_{g,n}$ due to Looijenga [39], Pikaart and de Jong [47], and Boggi and Pikaart [6].

Remark 2.13 To my knowledge, the group $\mathrm{Pic}(\overline{M}_{g,n})$ itself has not been calculate, neither in characteristic zero nor in positive characteristics.

2.4 Application to tame coverings

Let me give an application of the Strong Franchetta Conjecture to tame coverings. Belyi's Theorem [5] states that a complex curve is defined over a number field if and only if it admits a map to $\mathbb{P}^1_{\mathbb{C}}$ with at most three branch points. Fulton showed that any smooth curve X over a separably closed field of characteristic $p \neq 2$ admits a branched covering $X \rightarrow \mathbb{P}^1$ with tame ramification ([15], Proposition 8.1). Saïdi used this to prove an analog of Belyi's Theorem in odd characteristics: A smooth curve X over an algebraically closed field of characteristic $p > 2$ is defined over a finite field if and only if it is a tamely ramified covering of $\mathbb{P}^1$ with at most three branch points ([50], Theorem 5.6). It is unknown to what extent these facts hold true in characteristic $p = 2$. We have the following negative result for the generic n-pointed curve C_{η_n} of genus $g \geq 3$:

Theorem 2.14 *Suppose the ground field k has characteristic $p = 2$. Then any finite separable morphism $f : C_{\eta_n} \rightarrow \mathbb{P}^1_{\eta_n}$ has wild ramification.*

Proof. Suppose on the contrary that $f : C_{\eta_n} \rightarrow \mathbb{P}^1_{\eta_n}$ has only tame ramification. This means that, over the algebraic closure, every ramification point has odd ramification index.

Clearly, this implies that for all ramification points $x \in C_{\eta_n}$, say with image $y \in \mathbb{P}^1_{\eta_n}$, the field extension $\kappa(y) \subset \kappa(x)$ is separable. Moreover, the localization of the fiber $f^{-1}(y)$ at x is of the form dx for some odd integer $d \geq 2$. In turn, the relative canonical class $K_{C_{\eta_n}/\mathbb{P}^1_{\eta_n}} = \sum(d-1)x$ is 2-divisible. The same holds for the canonical class of $\mathbb{P}^1_{\eta_n}$. Consequently $K_{C_{\eta_n}}$ is 2-divisible, contradicting Theorem 2.9. $\square$

Remark 2.15 The proof actually shows that $f : C_{\eta_n} \rightarrow X$ has wild ramification if X is any curve whose canonical class is 2-divisible in the Picard scheme. The choice of an $L \in \text{Pic}(X/\eta_n)$ with $2L = K_X$ is sometimes called a *spin structure* or *theta characteristic*, compare [1], Chapter VI, Appendix B.

Remark 2.16 Consider the unique supersingular elliptic curve E in characteristic $p = 2$, which has Weierstrass equation $y^2 + y = x^3$ and invariant $j(E) = 0$. Its automorphism group is a semidirect product of $\mathbb{Z}/3\mathbb{Z}$ and the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$. Therefore, this curve even admits a Galois covering $f : E \rightarrow \mathbb{P}^1$ of degree $d = 3$, which has three ramification

points, all of ramification index $e = 3$. Amusingly, this is the only elliptic curve in characteristic $p = 2$ for which the existence of a tamely ramified function is a priori clear.

It seems to be open whether a given smooth curve X in characteristic $p \geq 0$ has a morphism $f : X \to \mathbb{P}^1$ whose ramification indices $e \geq 2$ are all odd. For example, we may ask this for the generic elliptic curve E over $\mathbb{Q}(j)$, which has Weierstrass equation

$$y^2 + xy = x^3 - \frac{1728}{j - 1728}x - \frac{1}{j - 1728}$$

and invariant $j(E) = j$. We can say at least the following:

Proposition 2.17 *There is a finite field extension $\mathbb{Q}(j) \subset F$ and a morphism $f : E \otimes F \to \mathbb{P}^1_F$ of degree $d = 4$ with four ramification points, all with ramification index $e = 3$.*

Proof. Let $\mathfrak{E} \to S$ be the universal elliptic curve over the punctured j-line $S = \mathbb{A}^1_{\mathbb{Q}} - \{0, 1728\}$. Consider the subscheme $U \subset \mathrm{Hom}(\mathfrak{E}, \mathbb{P}^1_S)$ of the relative Hom-scheme that parameterizes morphisms $\mathfrak{E}_s \to \mathbb{P}^1_s$ of degree $d = 4$ with four ramification points, all with index $e = 3$. It follows from [14], Theorem 1, that the projection $\mathrm{pr} : U \to S$ is dominant. Then the generic fiber $\mathrm{pr}^{-1}(\eta) \subset U$ contains a closed point u, and the residue field $F = \kappa(u)$ is the desired finite field extension of $\mathbb{Q}(j)$. $\qquad\square$

2.5 A stable curve with maximal index

Recall that the *index* $\mathrm{ind}(X) \in \mathbb{Z}$ of a proper curve X over an arbitrary field F is the positive generator for the image of the degree map $\mathrm{Pic}(X/F) \to \mathbb{Z}$. If X is geometrically integral, $\mathrm{ind}(X)$ is also the order of $\mathrm{Pic}^1_{X/F}$ in the Weil–Châtelet group $H^1(k, \mathrm{Pic}^0_{X/F})$.

The index of a stable curve of genus g is a divisor of degree $2g - 2$, because there is always the canonical class. It follows from Theorem 2.9 that the generic curve has maximal index $\mathrm{ind}(C_\eta) = 2g - 2$. It would be interesting to construct special curves with maximal index over fields of smaller transcendence degree. I do not know how to achieve this with smooth curves. The goal of this section is to produce an explicit stable curve with maximal index.

Throughout, we assume that our ground field k is algebraically closed, and let $F = k(T)$ be the rational function field. Note that, in our situation, we have $\text{Pic}(X/F) = \text{Pic}(X)$ by Lemma 1.1.

Proposition 2.18 *For each integer $d > 0$ there is a smooth curve Y of genus one over the rational function field F with $\text{ind}(Y) = d$.*

Proof. Let $F \subset F'$ be a cyclic field extension of degree d, say with Galois group $G \simeq \mathbb{Z}/d\mathbb{Z}$. Choose an elliptic curve E over F containing a rational point $x \in E$ of order d. Let $T \subset \text{Aut}(E)$ be the cyclic subgroup of order d generated by the translation $T_x(e) = e + x$.

Set $E' = E \otimes_F F'$. Then $g \in G$ acts on $\text{Aut}(E'/F')$ via $\phi \mapsto g\phi g^{-1}$. This action fixes the subgroup T pointwise, because $x \in E$ is a rational point. We get

$$H^1(G, T) = \text{Hom}(G, T) \simeq \mathbb{Z}/d\mathbb{Z}.$$

The latter identification depends on the choice of a generator $g_0 \in G$. The inclusion $T \subset \text{Aut}(E'/F')$ gives an inclusion $H^1(G, T) \subset H^1(G, \text{Aut}(E'/F'))$. As explained in [52], Chapter II, §1, Proposition 5, the set $H^1(G, \text{Aut}(E'/F'))$ may be viewed as the set of isomorphism classes of twisted forms Y of E whose preimage $Y' = Y \otimes F'$ is isomorphic to E'. Indeed, Y is the quotient of E' by the G-action $g \circ z_g$, where $z_g \in Z^1(G, \text{Aut}(E'/F'))$ is a cocycle representing a given cohomology class.

Now choose the generator $g_0 \mapsto T_x$ of $H^1(G, T)$ and consider the corresponding twisted form Y of E. Then $\text{ind}(Y)$ divides d. Indeed, the reduced divisor $D \subset E$ comprising the multiples of $x \in E$ defines a G-invariant divisor of degree d in E', hence an element in $\text{Pic}^d(Y/F)$.

For the converse, suppose we have a class in $\text{Pic}^l(Y/F)$. It corresponds to an invertible $\mathcal{O}_Y$-module $\mathcal{L}$ of degree l by Lemma 1.1. The isomorphism class of the induced invertible $\mathcal{O}_{Y'}$-module $\mathcal{L}'$ is invariant under both g_0 and $g_0 \circ T_x$, hence also under the translation T_x. But $T_x^*(\mathcal{O}_{E'}(e)) \simeq \mathcal{O}_{E'}(e - x)$ for any rational point $e \in E'$, so $T_x^*(\mathcal{L}') \simeq \mathcal{L}'$ implies $lx = 0$. In turn, d divides l, and we conclude $\text{ind}(Y) = d$. $\square$

Remark 2.19 The cyclic field extensions used above exist. Indeed, if d is prime to the characteristic p, the cyclic extensions of degree d correspond to étale cohomology $H^1(F, \mathbb{Z}/d\mathbb{Z}) = F^*/(F^*)^d$ via the Kummer sequence

$$0 \longrightarrow \mathbb{Z}/d\mathbb{Z} \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow 1.$$

If $d = p^m$, they correspond to étale cohomology $H^1(F, \mathbb{Z}/d\mathbb{Z}) = F/\wp(F)$ via the Artin–Schreier sequence

$$0 \longrightarrow \mathbb{Z}/p^m\mathbb{Z} \longrightarrow W_m(F) \longrightarrow W_m(F) \longrightarrow 0,$$

where $W_m(F)$ is the sheaf of Witt vectors of length m. In general, decompose $d = lp^m$ with l prime to p, and let F' be the subextension inside the maximal abelian extension $F \subset F^{\mathrm{ab}}$ generated by the linearly disjoint cyclic extensions corresponding to l and p^m ([8], Chapter V, §10, no. 4, Corollary 2 of Theorem 1).

Now we assemble the desired stable curve:

Proposition 2.20 *There is a geometrically integral stable curve X of genus g over the rational function field F with $\mathrm{ind}(X) = 2g - 2$.*

Proof. Let Y be a smooth curve of genus one with $\mathrm{ind}(Y) = 2g - 2$, as in Proposition 2.18. By Lemma 1.1, there is an invertible $\mathcal{O}_Y$-module $\mathcal{L}$ of degree $2g - 2$. The Riemann–Roch Theorem gives $h^0(Y, \mathcal{L}) = 2g - 2$, hence $\mathcal{L}$ comes from an effective divisor $D \subset Y$ of length $2g - 2$. By Bertini's Theorem, the very ample sheaf $\mathcal{L}$ has a global section such that the corresponding divisor $D \subset Y$ is smooth (see [31], Corollary 6.11). Such a subscheme is necessarily of the form $D = \mathrm{Spec}(F')$ for some separable field extension $F \subset F'$ of degree $2g - 2$ because $\mathrm{ind}(Y) = \deg(\mathcal{L})$.

Let $F \subset F'' \subset F'$ be a subextension of degree $g - 1$, such that $F'' \subset F'$ has degree two. The cocartesian diagram

$$
\begin{array}{ccc}
\mathrm{Spec}(F') & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\mathrm{Spec}(F'') & \longrightarrow & X
\end{array}
$$

defines an integral curve X. The curve X has a single cuspidal singularity, which breaks up into $g - 1$ nodal singularities over the algebraic closure. Therefore X is stable and geometrically integral. The exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F'' \oplus \mathcal{O}_Y \longrightarrow F' \longrightarrow 0$$

gives an exact sequence

$$0 \to H^0(\mathcal{O}_X) \to F'' \oplus H^0(\mathcal{O}_Y) \to F' \to H^1(X, \mathcal{O}_X) \to H^1(Y, \mathcal{O}_Y) \to 0,$$

so $h^1(X, \mathcal{O}_X) = g$. Since $Y \to X$ is birational, the map $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is surjective, and its kernel consists of numerically trivial sheaves. We conclude that $\mathrm{ind}(X) = \mathrm{ind}(Y) = 2g - 2$ holds. $\qquad\square$

Remark 2.21 Unfortunately, the curve X is not of compact type and contains no rational point. It is therefore difficult to analyze specialization of points in the Picard scheme of the versal deformation for X. In particular, the indices of curves occurring in the versal deformation are hard to control.

3 Hilbert sets

Our next goal is to find, beside the generic point $\eta_n \in M_{g,n}$, other nonclosed points $x \in M_{g,n}$ whose curve C_x has the property that the marked section and the canonical class generate $\mathrm{Pic}(C_x)$, at least up to torsion.

3.1 Geometrically unibranch singularities

In this section we shall remove a problem in characteristic $p > 0$, which occurs over and over again, namely: Normality is not necessarily preserved under inseparable field extensions. An annoying consequence is, for instance, that the Jacobian $\mathrm{Pic}^0_{X/F}$ of a proper normal scheme X is not necessarily proper.

We will sidestep such problems using geometrically unibranch schemes instead of normal schemes. This property is stable under field extensions and, as we shall see, almost as good as normality. Recall that a local ring A is called *geometrically unibranch* if it is irreducible, and the normalization $\tilde{A}$ of the reduction A_{red} induces a bijection $\mathrm{Spec}(\tilde{A}) \to \mathrm{Spec}(A)$ whose residue field extension is purely inseparable. Equivalently, the strict henselization $A \subset A^{\mathrm{sh}}$ is irreducible ([20], Proposition 18.8.15). A scheme is geometrically unibranch if all its local rings are geometrically unibranch. The nice thing about such schemes is that we may change the structure sheaf $\mathcal{O}_X$ as long as the topological space remains the same:

Lemma 3.1 *Let $f : X \to Y$ be an integral universal homeomorphism of schemes. Then Y is geometrically unibranch if and only if X is geometrically unibranch.*

Proof. To check this we may assume that $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ are local. We have $B^{\mathrm{sh}} = B \otimes_A A^{\mathrm{sh}}$ according to [20], Proposition 18.8.10 and Remark 18.8.11. Hence A^{sh} is irreducible if and only if B^{sh} is irreducible, because $A \to B$ is a universal homeomorphism. In turn, A is geometrically unibranch if and only if B is geometrically unibranch. $\square$

For the rest of this section, we shall fix a ground field F of characteristic $p > 0$. We first consider Jacobians. Surely, Jacobians of geometrically unibranch proper schemes may be nonproper. However, things are not too bad:

Proposition 3.2 *Let X be proper F-scheme. If X is geometrically unibranch, then there is an integer $n \geq 0$ such that the schematic image of the multiplication map $[p^n] : \operatorname{Pic}^0_{X/F} \to \operatorname{Pic}^0_{X/F}$ is proper.*

Proof. Let $H_n \subset \operatorname{Pic}^0_{X/F}$ be the image of the multiplication map $[p^n]$. Being the image of a homomorphism of algebraic group schemes, the embedding $H_n \subset \operatorname{Pic}^0_{X/F}$ is closed by [26], Exposé VI$_B$, Proposition 1.2.

Note that we may replace the ground field F by its algebraic closure $\bar{F}$. Indeed, the scheme $\bar{X} = X \otimes \bar{F}$ is geometrically unibranch by [18], Proposition 6.15.7. Moreover, the image H_n of the multiplication map $[p^n]$ on $\operatorname{Pic}^0_{X/F}$ is proper if and only if the induced image $\bar{H}_n = H_n \otimes \bar{F}$ is proper, according to [25], Exposé VIII, Corollary 4.8.

Next, let $\tilde{X} \to X$ be the normalization of X_{red}. The $\tilde{X}$ is geometrically normal, hence $\operatorname{Pic}^0_{\tilde{X}/F}$ is proper by [22], Theorem 2.1. Let $\tilde{X} \to \tilde{X}^{(p^m)}$ be the m-fold relative Frobenius map, which is a finite universal homeomorphism. It admits a factorization

$$\tilde{X} \longrightarrow X \longrightarrow \tilde{X}^{(p^m)}$$

for some $m \geq 0$ by [35], Proposition 6.6, because $\tilde{X} \to X$ is a finite universal homeomorphism. Applying this result again, this time to the finite universal homeomorphism $X \to \tilde{X}^{(p^m)}$, we obtain a factorization

$$X \longrightarrow \tilde{X}^{(p^m)} \longrightarrow X^{(p^n)}$$

for some $n \geq 0$. Consequently, the image of $\operatorname{Pic}^0_{\tilde{X}^{(p^m)}/F}$ in $\operatorname{Pic}^0_{X/F}$ contains the image of $\operatorname{Pic}^0_{X^{(p^n)}/F}$, which is nothing but H_n. Using that $\tilde{X}^{(p^m)}$ is normal, we conclude that H_n is proper. $\square$

Corollary 3.3 *Let X be a proper F-scheme and $J = \operatorname{Pic}^0_{X/F}$ be its Jacobian. If X is geometrically unibranch, then there is an abelian subvariety $A \subset J$ such that A is the image of $[p^m] : J \to J$ for all integers $m \gg 0$.*

Proof. Choose $n \geq 0$ from Proposition 3.2 so that the image $H \subset J$ of $[p^n] : J \to J$ is proper. Then H is connected because J is connected. Like in the proof of [22], Proposition 3.1, there is an integer $m \geq n$ such that the image $A \subset J$ of $[p^m] : J \to J$ is smooth, and the result follows. $\square$

The next result tells us that geometrically unibranch schemes behave well in families:

Proposition 3.4 *Let S be an integral noetherian scheme, and $f : X \to S$ a morphism of finite type whose generic fiber X_η is geometrically unibranch. Then there is a nonempty open subset $U \subset S$ such that all fibers X_s, $s \in U$ are geometrically unibranch.*

Proof. Using generic flatness ([19], Theorem 11.1.1), we may replace S by an open subset and assume that f is flat. By the usual limit argument, there is a finite morphism $X' \to X$ such that $X'_\eta \to X_\eta$ coincides with the generic fiber of the normalization for X_{red}. By assumption, $X'_\eta \to X_\eta$ is a universal homeomorphism, hence its geometric fibers are connected. According to [19], Proposition 15.5.1, the whole map $X' \to X$ has geometrically connected fibers, hence is a universal homeomorphism. We may replace X by X' and assume that X_η is normal.

Next, we define for each integer $n \geq 0$ a scheme X_n as follows. The underlying topological space for X_n is the underlying topological space for X. If $W \subset X$ is an affine open subset, we set

$$\Gamma(W, \mathcal{O}_{X_n}) = \Gamma(W, \mathcal{O}_X) \cap Q(L^{p^n}).$$

Here Q is the function field of S and L is the function field of X. Then each $(X_n)_\eta$ is a normal scheme and the projection morphisms $X_n \to S$ are proper. The descending sequence of fields $L \supset L^p \supset \ldots$ defines a sequence of universal homeomorphisms $X_0 \to X_1 \to \ldots$ with $X_0 = X$.

According to [30], Theorem 1, there is an integer $n \geq 0$ such that the geometric fiber Y_η of $Y = X_n$ is geometrically normal over Q. By [19], Theorem 12.2.4, the set $W \subset Y$ of geometrically normal points is open. According to Chevalley's Theorem ([16], Theorem 7.1.4), the image $f(W) \subset S$ is constructible. We have $\eta \in f(W)$ by construction. Being constructible, the set $f(W)$ contains a nonempty open subset $U \subset S$. The fibers Y_s, $s \in U$ are geometrically unibranch. Since $X_s \to Y_s$ is a finite universal homeomorphism, we may apply Lemma 3.1 and conclude that the X_s, $s \in U$ are geometrically unibranch as well. $\square$

The other direction is simpler:

Proposition 3.5 *Let S be an integral noetherian scheme, and $f : X \to S$ a morphism of finite type whose generic fiber X_η is not geometrically unibranch. Then there is a nonempty open subset $U \subset S$ such that all fibers X_s, $s \in U$ are not geometrically unibranch.*

Proof. Clearly, we may assume that S is affine and that X is reduced. Like in the proof for Proposition 3.4, there is a finite morphism $X' \to X$ such that $X'_\eta \to X_\eta$ is the normalization. Shrinking S, we may assume that the fibers X'_s have no embedded component by [19], Proposition 9.9.2. According to [19], Corollary 9.7.9, the set $Z \subset X$ over which the geometric fibers of $X' \to X$ are not connected is constructible. Replacing X by some open subset, we may assume that $Z \subset X$ is closed.

The set $x \in X$ where $\mathrm{codim}_x(Z_{f(x)}, X_{f(x)}) = 0$ is constructible by [19], Proposition 9.9.1, and disjoint from X_η. By Chevalley's Theorem, there is a nonempty open subset $U \subset S$ such that $\mathrm{codim}_x(Z_{f(x)}, X_{f(x)}) > 0$ holds for all $x \in f^{-1}(U)$. In other words, Z_s contains no generic point from X_s for $s \in U$. We infer that the fibers X_s are not geometrically unibranch for $s \in U$. $\square$

Remark 3.6 In Grothendieck's terminology, the preceding two result tell us that the property of being geometrically unibranch is *constructible* (compare [19], Definition 9.2.1). It then follows from [19], Corollary 9.2.4, that given any morphism $f : X \to S$ of finite presentation, the set $E \subset S$ of points whose fiber X_t is geometrically unibranch is locally constructible.

3.2   Picard numbers in families and Hilbert sets

Let S be a scheme and $f : X \to S$ a proper morphism. How do the Picard numbers $\rho(X_s)$ vary in this family? The function $s \mapsto \rho(X_s)$ can be very nasty. To explore its nature we need the notion of Hilbert sets. Suppose for simplicity that S is integral and noetherian. Let T be an integral scheme and $T \to S$ an étale dominant morphism of finite type. The generic fiber T_η is an integral scheme given by a finite separable field extension of $\kappa(\eta)$. We define $H_T \subset S$ as the set of all points $s \in S$ where the fiber T_s is integral and nonempty. A subset $H \subset S$ is called a *separable Hilbert set* if there are finitely many $T_i \to S$, $1 \le i \le n$ as above with $H = H_{T_1} \cap \ldots \cap H_{T_n}$. Clearly, open sets are separable Hilbert sets. Moreover, separable Hilbert subsets are stable under generization.

Usually, Hilbert sets are defined in terms of irreducible polynomials. They occur in Galois theory in connection with *Hilbert's Irreducibility Theorem*. The latter states, in its original form, that an irreducible polynomial $P(T, X) \in \mathbb{Q}[T, X]$ remains irreducible for infinitely many rational specializations $T = a/b$. Hilbert sets are studied in detail in [13], Chapter 11 and 12. Our definition follows Lang [37] and Serre [53], which is better suited for

algebraic geometry. For example, Hilbert sets are closely related to Néron–Severi groups:

Proposition 3.7 *Let S be an integral noetherian scheme, $\eta \in S$ its generic point, and $f : X \to S$ a proper morphism. Then there are countably many nonempty separable Hilbert subsets $H_i \subset S$ such that $\rho(X_s) = \rho(X_\eta)$ for all $s \in \bigcap H_i$.*

Proof. Replacing S by some nonempty open subset, we may assume that the relative Picard functor $\mathrm{Pic}_{X/S}$ is representable by a scheme, according to [27], Exposé XII, Corollary 1.2. After shrinking S further, we may also assume that there is a group scheme $J \to S$ of finite type, together with an open embedding $J \subset \mathrm{Pic}_{X/S}$, such that $J_s = \mathrm{Pic}^0_{X_s/s}$ for all $s \in S$. This holds by [22], Lemma 1.2. By generic flatness, we may also assume that $J \to S$ is flat.

Consider the group scheme $\mathrm{NS}_{X/S}$. By Chow's Lemma, there is a projective surjective morphism $X' \to X$ such that the composite map $f' : X' \to S$ is projective. Shrink S so that $\mathrm{NS}_{X'/S}$ is representable by a group scheme. Then $\mathrm{NS}_{X'/S}$ contains only countably many irreducible component. Indeed, this follows from the existence of the Hilbert scheme (see [7], Chapter 8, Theorem 5). Using that $\mathrm{Pic}_{X/S} \to \mathrm{Pic}_{X'/S}$ is of finite type ([27], Exposé XIII, Theorem 3.5), we conclude that $\mathrm{NS}_{X/S}$ has only countably many irreducible components as well.

Let $A_i \subset \mathrm{NS}_{X/S}$ be such an irreducible component. If $A_i \to S$ is not dominant, choose a nonempty open subset $H_i \subset S$ disjoint from the image of A_i. If $A_i \to S$ is dominant, let $U_i \subset S$ be an open subset over which A_i becomes flat, hence étale, and let $H_i \subset U_i$ be the corresponding Hilbert subset.

Then $H = \bigcap H_i$ is the desired countable intersection of Hilbert subsets. Indeed, for each $s \in H$, the points $a \in \mathrm{NS}(X_s/s)$ correspond to the points $b \in \mathrm{NS}(X_\eta/\eta)$ via some dominant irreducible component A_i. Moreover, a is rational if and only if b is rational. In other words, we have constructed a bijective specialization map $\mathrm{NS}(X_\eta/\eta) \to \mathrm{NS}(X_s/s)$ and conclude $\rho(X_s) = \rho(X_\eta)$. $\square$

How do countable intersections of Hilbert sets look like? There seems to be no general answer. However, over certain ground fields we can say more;

Lemma 3.8 *Let k be an uncountable field, $k \subset F$ a finitely generated field extension of transcendence degree ≥ 1, and S an integral F-scheme of finite type of dimension $n \geq 1$. Then any countable intersection $\bigcap H_i$ of nonempty separable Hilbert subsets $H_i \subset S$ contains uncountably many closed points.*

Proof. We may assume that $S = \mathrm{Spec}(R)$ is affine. Let $R \subset Q$ be the field of fractions. Replacing F by its perfect closure in Q and shrinking S, we may assume that S is geometrically reduced. Clearly, we may assume that our separable Hilbert subsets $H_i \subset S$ are given by étale morphisms $T_i \to S$.

Next, we reduce the problem to a special case. Choose a separating transcendence basis $t_1, \ldots, t_n \in Q$ over F. Shrinking S, we obtain a separable morphism $f : S \to \mathbb{A}_F^n$. Choose nonempty open subsets $T_i' \subset T_i$ so that the composition $T_i' \to \mathbb{A}_F^n$ is étale, and let $H_i' \subset \mathbb{A}_F^n$ be the corresponding separable Hilbert subsets. Then $f^{-1}(H_i') \subset H_i$, and we see that it suffices to treat the special case $S = \mathbb{A}_F^n$.

We now proceed by induction on $n \geq 1$. First, suppose $S = \mathbb{A}_F^1$. According to [13], Theorem 12.9, there are nonempty open subsets $U_i \subset \mathbb{A}_k^2$ with

$$\{a + tb \mid a, b \in U_i(k)\} \subset H_i.$$

Since k is uncountable, the countable intersection $\bigcap U_i$ contains uncountable many rational points (choose a line not contained in any $\mathbb{A}_k^2 - U_i$). Consequently $\bigcap H_i$ contains uncountable many rational points as well.

Now suppose the result is already true for $n \geq 1$. Let F' be the function field of $\mathbb{A}_F^n$, and consider the projection $\mathbb{A}_F^{n+1} \to \mathbb{A}_F^n$. Its generic fiber is isomorphic to the affine line $\mathbb{A}_{F'}^1$. Set $H = \bigcap H_i$. We just saw that the intersection $H \cap \mathbb{A}_{F'}^1$ contains a rational point $z \in \mathbb{A}_{F'}^1$. Its closure $S' \subset \mathbb{A}_F^{n+1}$ is n-dimensional. Applying our preliminary reduction and the induction hypothesis, we conclude that $H = \bigcap H_i$ contains uncountably many closed points. $\square$

Remark 3.9 Under the additional hypothesis that S is geometrically reduced, the preceding proof shows that there are uncountably many closed points $s \in \bigcap H_i$ whose residue field extension $F \subset \kappa(s)$ are separable.

Summing up, we obtain the following result about families of Néron–Severi groups:

Theorem 3.10 *Let k be an uncountable field, $k \subset F$ a nonalgebraic finitely generated field extension, S an integral F-scheme of finite type of dimension*

at least one, and $f : X \to S$ a proper morphism. Then there are uncountably many closed points $s \in S$ with $\rho(X_s) = \rho(X_\eta)$.

Proof. Combine Proposition 3.7 and Lemma 3.8. $\square$

The situation simplifies if we look at Picard numbers of geometric fibers $X_{\bar{s}} = X \otimes \bar{\kappa}(s)$ instead of schematic fibers X_s. More precisely:

Theorem 3.11 *Let S be an integral noetherian scheme and $f : X \to S$ a proper morphism. Then there are countably many nonempty open subsets $U_i \subset S$ such that $\rho(X_{\bar{s}}) = \rho(X_{\bar{\eta}})$ for all $s \in \bigcap U_i$.*

Proof. As in the proof for Proposition 3.7, we reduce to the case that $\mathrm{Pic}_{X/S}$ and $J \subset \mathrm{Pic}_{X/S}$ are representable by group schemes, and set $\mathrm{NS}_{X/S} = \mathrm{Pic}_{X/S}/J$. For each irreducible component $A_i \subset \mathrm{NS}_{X/S}$, choose a nonempty open subset $U_i \subset S$ over which A_i is finite and flat. Then $\bigcap U_i$ is the desired countable intersection of open subsets.

To see this, fix a point $s \in \bigcap U_i$. Let $R = \mathcal{O}_{S,s}$ be the corresponding local ring, $F = \kappa(s)$ its residue field, $F \subset F^{\mathrm{sep}}$ a separable closure, and $R \subset R^{\mathrm{sh}}$ be the corresponding strict henselization. Then $A_i \otimes R$ is étale over R, and $A_i \otimes R^{\mathrm{sh}}$ decomposes into disjoint sections by [20], Proposition 18.5.19. We conclude that $\mathrm{NS}(X \otimes F^{\mathrm{sep}}/F^{\mathrm{sep}})$ is canonically isomorphic to $\mathrm{NS}(X \otimes Q/Q)$, where Q is the function field of R^{sh}. The canonical mappings $\mathrm{NS}(X \otimes F^{\mathrm{sep}}/F^{\mathrm{sep}}) \to \mathrm{NS}(X \otimes \bar{F})$ and $\mathrm{NS}(X \otimes Q/Q) \to \mathrm{NS}(X \otimes \bar{Q})$ are obviously bijective, so $\rho(X_{\bar{s}}) = \rho(X_{\bar{\eta}})$. $\square$

3.3 Fibered surfaces with small Picard number

Fix an uncountable algebraically closed ground field k. The task now is to construct fibered surfaces with small Picard number defined over small transcendental extension fields $k \subset F$. The Tate–Shioda formula then implies that their generic fibers have small Picard group as well. The upshot is that there is a dense set of points $x \in M_{g,n}$ with $\dim \overline{\{x\}} \le 2$ such that the marked points and the canonical class of the corresponding curve C_x generate $\mathrm{Pic}(C_x) \otimes \mathbb{Q}$.

We start with some notation. A *surface* over F is a proper 2-dimensional F-scheme Y with $\Gamma(Y, \mathcal{O}_Y) = F$. A *fibration* on a surface Y is curve B together with a proper morphism $f : Y \to B$ satisfying $\mathcal{O}_B = f_*(\mathcal{O}_Y)$. The *Tate–Shioda formula* relates the Picard number $\rho(Y)$ of a fibered surface

with the rank of the Mordell–Weil group of the generic fiber (confer [57] and [54]):

Proposition 3.12 *Let Y be a normal surface whose singularities are $\mathbb{Q}$-factorial, and $f : Y \to B$ be a fibration. The we have*

$$\rho(Y) - \operatorname{rank} \operatorname{Pic}^0(Y_\eta)/\operatorname{Pic}^0(Y) = 2 + \sum_{b \in B}(\rho(Y_b) - 1).$$

Proof. The intersection form is negative semidefinite on the group of Weil divisors supported by a given fiber $f^{-1}(b) \subset Y$, and the fiber $f^{-1}(b)$ generates the radical over $\mathbb{Q}$. Consequently, the vertical Weil divisors generate a subgroup of rank $1 + \sum_{b \in B}(\rho(Y_b) - 1)$ inside the Néron–Severi group.

Next, choose a horizontal curve $H \subset Y$. Given any Weil divisor C, we may subtract a suitable rational multiple of H until C_η has degree zero. Since $\operatorname{Pic}(Y) \to \operatorname{Pic}(Y_\eta)$ has finite cokernel, we conclude that the horizontal Weil divisors generate a subgroup of rank $1 + \operatorname{rank} \operatorname{Pic}^0(Y_\eta)/\operatorname{Pic}^0(Y)$ inside the Néron–Severi group. This easily implies the formula. $\square$

This formula often allows us to control the rank of the Jacobian $\operatorname{Pic}^0(Y_\eta)$ in terms of the Picard number $\rho(Y)$.

Now fix an integer $n \geq 0$ and a genus $g \geq 2$, and let X be an n-pointed stable curve of genus g over k. For simplicity, we also assume $\operatorname{Aut}(X) = 0$. Then the closed point $x \in \overline{M}_{g,n}$ corresponding to X lies in the smooth locus [38] . As in the proof of Proposition 2.1, we blow up the center $x \in \overline{M}_{g,n}$ and localize at the generic point of the resulting exceptional divisor. This produces a discrete valuation ring A whose residue field $F = A/\mathfrak{m}_A$ is a purely transcendental field extension of degree $3g - 4 + n$, and whose field of fractions $A \subset Q$ is the function field of the moduli space $\overline{M}_{g,n}$.

Write $F = k(t_i)$ for some transcendence basis $t_i \in F$, and choose lifts $t_i \in A$. We obtain an inclusion $k[t_i] \subset A$, and in turn a lift $F \subset A$. Then $F \subset Q$ is a finitely generated field extension of transcendence degree one, which corresponds to a proper normal curve B over F. The discrete valuation ring $A \subset Q$ defines a rational point $b_0 \in B$, hence $F = \Gamma(B, \mathcal{O}_B)$, and B is geometrically integral. The tautological curve $C \to \overline{M}_{g,n}$ defines a fibered regular surface $f : Y \to B$. By construction, we have $f^{-1}(b_0) = X \otimes F$ and $f^{-1}(\eta) = C_\eta$. To apply the Tate–Shioda formula, we have to ensure the following:

Lemma 3.13 *Suppose the normalization $\tilde{X}$ of X has only rational components. Then the canonical map $\mathrm{Pic}^0(Y) \to \mathrm{Pic}^0(Y_\eta)$ vanishes.*

Proof. Let $F \subset \bar{F}$ be an algebraic closure. The normalization $\tilde{Y}$ of $\bar{Y} = Y \otimes \bar{F}$ is geometrically normal, so $\mathrm{Pic}^0_{\tilde{Y}/\bar{F}}$ is proper by [22], Theorem 2.1. Moreover, it contains a unique abelian subscheme with the same underlying topological space by [22], Corollary 3.2. Its dual abelian variety is the Albanese variety $\mathrm{Alb}_{\tilde{Y}/\bar{F}}$. The resulting morphism $\tilde{Y} \to \mathrm{Alb}_{\tilde{Y}/\bar{F}}$ factors over $\tilde{B}$, because the normalization of X has genus zero.

According to [27], Exposé XII, Corollary 1.5, the map $\mathrm{Pic}^0_{Y/\bar{F}} \to \mathrm{Pic}^0_{\tilde{Y}/\bar{F}}$ is affine. It follows that the cokernel of $\mathrm{Pic}^0_{B/F} \to \mathrm{Pic}^0_{Y/F}$ is affine. Using that $\mathrm{Pic}^0_{Y_\eta/\eta}$ is abelian, we infer that the map $\mathrm{Pic}^0_{Y/F} \to \mathrm{Pic}^0_{Y_\eta/\eta}$ factors over the origin, hence $\mathrm{Pic}^0(Y) \to \mathrm{Pic}^0(Y_\eta)$ vanishes. $\qquad\square$

From now on, we shall assume that the normalization of X has only rational components. Then Theorem 2.9, together with the Tate–Shioda formula gives

$$\rho(Y) - (n+1) = 2 + \sum_{b \in B}(\rho(Y_b) - 1).$$

We now use this formula to specialize Y and maintain control over $\mathrm{Pic}(Y_\eta)$. Recall that $t_i \in F$, $1 \le i \le 3g - 4 + n$ is a transcendence basis over k. Consider the rational function field $L = k(t_1)$. Then we may view $L \subset F$ as the function field of $\mathbb{P}^{3g-5+n}_L$. Our F-schemes X and B extend to proper flat morphisms $\mathfrak{Y} \to S$ and $\mathfrak{B} \to S$, respectively, over some open subsets $S \subset \mathbb{P}^{3g-5+n}_L$. Shrinking S, we also have a morphism $f : \mathfrak{Y} \to \mathfrak{B}$ such that $\mathcal{O}_{\mathfrak{B}} \to f_*(\mathcal{O}_{\mathfrak{Y}})$ is bijective, and remains bijective after any base change. Let $\mathfrak{Y}_{\eta_s} \subset \mathfrak{Y}_s$ be the generic fiber of the induced fibration $\mathfrak{Y}_s \to \mathfrak{B}_s$.

Proposition 3.14 *There is an uncountable dense subset of closed points $s \in S$ such that $\mathfrak{Y}_{\eta_s}$ is a stable n-pointed curve of genus g, and that the marked points and the canonical class generate $\mathrm{Pic}(\mathfrak{Y}_{\eta_s}) \otimes \mathbb{Q}$.*

Proof. Replacing S by some nonempty open subset, we may assume that the following holds: First, the fibers of $\mathfrak{Y} \to S$ and $\mathfrak{B} \to S$ are geometrically integral by [19], Theorem 9.7.7, and geometrically unibranch by Proposition 3.4. Second, the maps $\mathrm{Pic}^0_{\mathfrak{Y}_s/s} \to \mathrm{Pic}^0_{\mathfrak{Y}_b/b}$ factor over the origin for almost all closed points $b \in \mathfrak{B}_s$. It then follows that $\mathrm{Pic}^0(\mathfrak{Y}_s) \to \mathrm{Pic}^0(\mathfrak{Y}_{\eta_s})$ vanishes. Third, the generic fibers $\mathfrak{Y}_{\eta_s}$ are n-pointed smooth curves of genus g.

By Theorem 3.10, there are uncountably many closed points $s \in S$ with $\rho(\mathfrak{Y}_s) = \rho(Y)$. Let $\tilde{\mathfrak{Y}}_s \to \mathfrak{Y}_s$ be the normalization and $\bar{\mathfrak{Y}}_s \to \tilde{\mathfrak{Y}}_s$ a resolution of singularities. Then $\bar{\mathfrak{Y}}_s$ is a fibered surface with generic fiber $\bar{\mathfrak{Y}}_{\eta_s} = \mathfrak{Y}_{\eta_s}$. We have $\rho(\tilde{\mathfrak{Y}}_s) = \rho(\mathfrak{Y}_s)$ by Proposition 1.4. The Picard number $\rho(\bar{\mathfrak{Y}}_s)$ of the desingularization is usually larger. However, any additionally classes lie in the fibers of $\bar{\mathfrak{Y}}_s \to \mathfrak{B}_s$. The Tate–Shioda formula therefore implies that $\mathrm{Pic}(Y_\eta)$ and $\mathrm{Pic}(\mathfrak{Y}_{\eta_s})$ have the same rank. Recall that Y_η is the generic n-pointed curve of genus g. Using Theorem 2.9, we infer that the canonical class and the marked points generate $\mathrm{Pic}(\mathfrak{Y}_{\eta_s}) \otimes \mathbb{Q}$. $\qquad\square$

We come to the second main result of this paper:

Theorem 3.15 *Let k be an uncountable algebraically closed field. Then there is an uncountable dense set of points $x \in M_{g,n}$ with $\dim \overline{\{x\}} \leq 2$ such that the canonical class and the marked sections generate $\mathrm{Pic}(C_x) \otimes \mathbb{Q}$, where C_x is the curve corresponding to the point $x \in M_{g,n}$.*

Proof. Consider the family $\mathfrak{Y} \to \mathfrak{B}$ of fibered surfaces over S constructed in Proposition 3.14. We then have a rational map $\mathfrak{B} \dashrightarrow M_{g,n} \otimes L$ whose image is a divisor, because the composition $\mathfrak{B} \dashrightarrow M_{g,n}$ is dominant. After shrinking S, we find an open subset $\mathfrak{U} \subset \mathfrak{B}$ whose fibers over S are nonempty, such that $\mathfrak{U} \to M_{g,n} \otimes L$ is everywhere defined and quasifinite. This is because $\dim(\mathfrak{B}) = \dim(M_{g,n}) - 1$.

Consider the uncountable dense set of points $s \in S$ from Proposition 3.14, such that $\mathrm{Pic}(\mathfrak{X}_{\eta_s}) \otimes \mathbb{Q}$ is generated by the canonical class and the marked sections. Let $x \in M_{g,n}$ be the images of the points $\eta_s \in \mathfrak{B}$. By construction, the residue fields $\kappa(x)$ have transcendence degree ≤ 2 over k. Hence the $x \in M_{g,n}$ constitute the desired uncountable dense subset. $\qquad\square$

3.4 Open problems

We close the paper by listing some open problems:

(1) Theorem 3.15 states that the marked points and the canonical class generate $\mathrm{Pic}(C_x)$. If $l \in \mathrm{Pic}(C_x/x)$ is a rational point, what denominators are necessary to write l as a linear combination of the canonical class and the marked points? Is it sufficient to allow powers of the characteristic exponent $p \geq 1$ as denominators?

(2) Is it possible to choose a dense set of nonclosed points $x \in M_{g,n}$ as in Theorem 3.15 so that all closures $\overline{\{x\}}$ are 1-dimensional? This seems to rely on improved versions of Hilbert's Irreducibility Theorem for prime fields.

(3) Can we say more about the structure of countable intersections of nonempty Hilbert sets in Theorem 3.10? Obviously, we cannot expect such sets to have a reasonable algebraic structure. Is it possible to write such sets as a countable union of sets with some sort of algebraic structure?

(4) Does the Strong Franchetta Conjecture generalize to other moduli problems, and if so, in what form? What about polarized abelian varieties, or surfaces of general type, or canonically polarized varieties of higher dimensions? For example, Silverberg [55] showed that generic complex abelian varieties have finite Mordell–Weil groups.

References

[1] E. Arbarello, M. Cornalba, P. Griffiths, J. Harris: Geometry of algebraic curves. I. Grundlehren Math. Wiss. 267. Springer, New York, 1985.

[2] E. Arbarello, M. Cornalba: The Picard groups of the moduli spaces of curves. Topology 26 (1987), 153–171.

[3] E. Arbarello, M. Cornalba: Calculating cohomology groups of moduli spaces of curves via algebraic geometry. Inst. Hautes Etudes Sci. Publ. Math. 88 (1998), 97–127.

[4] W. Baily: Some results on the moduli of Riemann surfaces. Proc. Nat. Acad. Sci. U.S.A. 47 (1961) 325–327.

[5] G. Belyi: Galois extensions of a maximal cyclotomic field. Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), 267–276.

[6] M. Boggi, M. Pikaart: Galois covers of moduli of curves. Compositio Math. 120 (2000), 171–191.

[7] S. Bosch, W. Lütkebohmert, M. Raynaud: Néron models. Ergeb. Math. Grenzgebiete (3) 21. Springer, Berlin, 1990.

[8] N. Bourbaki: Algebra II. Chapters 4–7. Springer, Berlin, 1990.

[9] P. Deligne, D. Mumford: The irreducibility of the space of curves of given genus. Publ. Math. Inst. Hautes Étud. Sci. 36 (1969), 75–109.

[10] P. Deligne, M. Rapoport: Les schémas de modules de courbes elliptiques. In: P. Deligne, W. Kuyk (eds.), Modular Functions of one Variable II, pp. 143–316. Lect. Notes Math. 349. Springer, Berlin, 1973.

[11] P. Deligne: Le lemme de Gabber. Astérisque 127 (1985), 131–150.

[12] A. Franchetta: Sulle serie lineari razionalmente determinate sulla curva a moduli generali di dato genere. Matematiche (Catania) 9 (1954), 126–147.

[13] M. Fried, M. Jarden: Field arithmetic. Ergeb. Math. Grenzgebiete (3) 11. Springer, Berlin, 1986.

[14] M. Fried, E. Klassen, Y. Kopeliovich: Realizing alternating groups as monodromy groups of genus one covers. Proc. Amer. Math. Soc. 129 (2001), 111–119.

[15] W. Fulton: Hurwitz schemes and irreducibility of moduli of algebraic curves. Ann. Math. 90 (1969), 542–575.

[16] A. Grothendieck, J.A. Dieudonné: Éléments de géométrie algébrique I: Le langage de schémas. Grundlehren Math. Wiss. 166. Springer, Berlin, 1970.

[17] A. Grothendieck: Éléments de géométrie algébrique III: Étude cohomologique des faiscaux cohérent. Publ. Math., Inst. Hautes Étud. Sci. 11 (1961).

[18] A. Grothendieck: Éléments de géométrie algébrique IV: Étude locale des schémas et des morphismes de schémas. Publ. Math., Inst. Hautes Étud. Sci. 24 (1965).

[19] A. Grothendieck: Éléments de géométrie algébrique IV: Étude locale des schémas et des morphismes de schémas. Publ. Math., Inst. Hautes Étud. Sci. 28 (1966).

[20] A. Grothendieck: Éléments de géométrie algébrique IV: Étude locale des schémas et des morphismes de schémas. Publ. Math., Inst. Hautes Étud. Sci. 32 (1967).

[21] A. Grothendieck: Les schémas de Picard: théorémes d'existence. Séminaire Bourbaki, Exp. 232 (1962), 143–161.

[22] A. Grothendieck: Les schémas de Picard: propriétés générales. Séminaire Bourbaki, Exp. 236 (1962), 221–243.

[23] A. Grothendieck: Le groupe de Brauer II. In: J. Giraud (ed) et al., Dix exposes sur la cohomologie des schemas, pp. 88–189. North-Holland, Amsterdam, 1968.

[24] A. Grothendieck: Le groupe de Brauer III. In: J. Giraud (ed.) et al., Dix exposes sur la cohomologie des schemas, pp. 88–189. North-Holland, Amsterdam, 1968.

[25] A. Grothendieck et al.: Revêtements étales et groupe fondamental. Lect. Notes Math. 224. Springer, Berlin, 1971.

[26] A. Grothendieck et al.: Schemas en groupes I. Lect. Notes Math. 151. Springer, Berlin, 1970.

[27] A. Grothendieck et al.: Théorie des intersections et théorème de Riemann-Roch. Lect. Notes Math. 225. Springer, Berlin, 1971.

[28] J. Harer: The second homology group of the mapping class group of an orientable surface. Invent. Math. 72 (1983), 221–239.

[29] R. Hartshorne: Algebraic geometry. Grad. Texts Math. 52. Springer, Berlin, 1977.

[30] R. Hübl, R. Kunz: On algebraic varieties over fields of prime characteristic. Arch. Math. (Basel) 62 (1994), 88–96.

[31] J.-P. Jouanolou: Théorèmes de Bertini et applications. Progress in Mathematics 42. Birkhäuser, Boston, 1983.

[32] S. Kleiman: Toward a numerical theory of ampleness. Ann. Math. 84 (1966), 293–344.

[33] F. Knudsen: The projectivity of the moduli space of stable curves. III. Math. Scand. 52 (1983), 200–212.

[34] K. Kodaira: On compact analytic surfaces II. Ann. of Math. 77 (1963), 563–626.

[35] J. Kollár: Quotient spaces modulo algebraic groups. Ann. of Math. 145 (1997), 33–79.

[36] A. Kouvidakis: The Picard group of the universal Picard varieties over the moduli space of curves. J. Differential Geom. 34 (1991), 839–850.

[37] S. Lang: Fundamentals of diophantine geometry. Springer, Berlin, 1983.

[38] K. Lønsted: The singular points on the moduli spaces for smooth curves. Math. Ann. 266 (1984), 397–402.

[39] E. Loojienga: Smooth Deligne–Mumford compactifications by means of Prym level structures. J. Alg. Geom. 3 (1994), 283–293.

[40] N. Mestrano: Conjecture de Franchetta forte. Invent. Math. 87 (1987), 365–376.

[41] J. Milne: Étale cohomology. Princeton Mathematical Series, 33. Princeton University Press, Princeton, 1980.

[42] A. Moriwaki: The $\mathbb{Q}$-Picard group of the moduli space of curves in positive characteristic. Internat. J. Math. 12 (2001), 519–534.

[43] D. Morrison: The birational geometry of surfaces with rational double points. Math. Ann. 271 (1985), 415–438.

[44] D. Mumford: Lectures on curves on an algebraic surface. Princeton University Press, Princeton, 1966.

[45] D. Mumford, J. Fogarty, F. Kirwan: Geometric invariant theory. Third edition. Ergeb. Math. Grenzgebiete (3) 34. Springer, Berlin, 1993.

[46] D. Mumford: Stability of projective varieties. Enseignement Math. 23 (1977), 39–110.

[47] M. Pikaart, A. de Jong: Moduli of curves with non-abelian level structure. In: R. Dijkgraaf, C. Faber, G. van der Geer (eds.), The moduli space of curves, pp. 483–509. Progress in Math. 129. Birkhäuser, Boston, 1995.

[48] B. Poonen: Varieties without extra automorphisms I. Curves. Math. Res. Lett. 7 (2000), 67–76.

[49] M. Reid: Nonnormal del Pezzo surfaces. Publ. Res. Inst. Math. Sci. 30 (1994), 695–727.

[50] M. Saïdi: Revêtements modérés et groupe fondamental de graphe de groupes. Compositio Math. 107 (1997), 319–338.

[51] S. Schröer: On non-projective normal surfaces. Manuscr. Math. 100 (1999), 317–321.

[52] J.-P. Serre: Cohomologie galoisienne. Fifth edition. Lect. Notes Math. 5. Springer, Berlin, 1994.

[53] J.-P. Serre: Lectures on the Mordell–Weil theorem. Vieweg, Braunschweig, 1989.

[54] T. Shioda: Mordell–Weil lattices for higher genus fibration over a curve. In: K. Hulek (ed.) et al., New trends in algebraic geometry, pp. 359–373. London Math. Soc. Lecture Note Ser. 264. Cambridge Univ. Press, Cambridge, 1999.

[55] A. Silverberg: Mordell–Weil groups of generic abelian varieties. Invent. Math. 81 (1985), 71–106.

[56] E. Spanier: Algebraic topology. Springer-Verlag, New York-Berlin, 1981

[57] J. Tate: On the conjectures of Birch and Swinnerton-Dyer and a geometric analog. Séminaire Bourbaki, Exp. 306 (1966).

Mathematische Fakultät, Ruhr-Universität, 44780 Bochum, Germany

Email address: s.schroeer@ruhr-uni-bochum.de