TOPOLOGICAL METHODS FOR COMPLEX-ANALYTIC BRAUER GROUPS

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Abstract. Using methods from algebraic topology and group cohomology, I pursue Grothendieck's question on equality of geometric and cohomological Brauer groups in the context of complex-analytic spaces. The main result is that equality holds under suitable assumptions on the fundamental group and the Pontrjagin dual of the second homotopy group. I apply this to Lie groups, Hopf manifolds, and complex-analytic surfaces.

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Introduction

The goal of this paper is to pursue Grothendieck's question on Brauer groups in the context of complex-analytic spaces, using methods from algebraic topology and group cohomology. This yields new results on Lie groups, Hopf manifolds, and surfaces.

Let me recall Grothendieck's question. Suppose X is a topological space endowed with a sheaf of rings $\mathcal{O}_X$. The *cohomological Brauer group* $\mathrm{Br}'(X)$ is defined as the torsion part of the cohomology group $H^2(X, \mathcal{O}_X^\times)$. One likes to have a geometric interpretation of such cohomology classes. A possible interpretation is in terms of principal PGL_r-bundles $P \to X$, via the nonabelian coboundary map $H^1(X, \mathrm{PGL}_r(\mathbb{C})) \to H^2(X, \mathcal{O}_X^\times)$. The group of equivalence classes of principal PGL_r-bundles is called the *Brauer group* $\mathrm{Br}(X)$. The coboundary map yields a canonical inclusion $\mathrm{Br}(X) \subset \mathrm{Br}'(X)$, and Grothendieck [22] asked whether this

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inclusion is actually a bijection. This is a major open problem in the theory of Brauer groups.

In algebraic topology, Serre solved Grothendieck's question if X is a finite CW-complex and the sheaf of rings is the sheaf of continuous complex functions $\mathcal{C}_X$. It turns out that here $\mathrm{Br}(X) = \mathrm{Br}'(X)$ equals the torsion part of $H^3(X, \mathbb{Z})$. In contrast, there are only few general results in algebraic geometry. In this context X is an algebraic scheme and $\mathcal{O}_X$ is its structure sheaf. Gabber's Theorem tells us that $\mathrm{Br}(X) = \mathrm{Br}'(X)$ for any affine scheme [18]. Grothendieck himself showed that equality holds for smooth algebraic surfaces [22], and I treated the case of algebraic surfaces with isolated singularities [50].

Grothendieck's question shows up in various areas. To mention a few: In moduli theory, Brauer groups are used in order to determine whether a coarse moduli space is actually a fine moduli space. In stack theory, Brauer groups are important to detect quotient stacks, as explained in the work of Edidin, Hassett, Kresch, and Vistoli [15]. In Homological Mirror Symmetry, Brauer groups are used for twisting derived categories.

This paper deals with complex-analytic spaces, which are not necessarily algebraic. Here methods of algebraic geometry frequently break down, largely due to extension problems involving coherent sheaves. In this context there are two general results: Elencwajg and Narasimhan showed $\mathrm{Br}(X) = \mathrm{Br}'(X)$ for complex tori [16], and Huybrechts and myself recently proved it for complex-analytic K3-surfaces with methods from differential geometry [30]. Here I prove a general result on complex-analytic Brauer groups that depends only on the homotopy type of the underlying topological space.

Theorem. *Let X be a complex-analytic space. Suppose $\pi_1(X)$ is a good group in Serre's sense, and that the subgroup of $\pi_1(X)$-invariants inside the Pontrjagin dual $\mathrm{Hom}(\pi_2(X), \mathbb{Q}/\mathbb{Z})$ is trivial. Then the inclusion $\mathrm{Br}(X) \subset \mathrm{Br}'(X)$ is an equality.*

Serre introduced the notion of *good groups*, which has to do with profinite completions, in the context of Galois cohomology [54]. I will recall this somewhat technical concept in Section 3. Note that free groups and polycyclic groups are good.

The conditions in the theorem appear bizarre, but it applies directly to complex Lie groups and Hopf manifolds:

Theorem. *Let X be a complex Lie group or a Hopf manifold. Then the inclusion $\mathrm{Br}(X) \subset \mathrm{Br}'(X)$ is an equality.*

This generalizes results of Iversen on characterfree algebraic groups [32], and Hoobler [28], Berkovič [3], and Elencwajg and Narasimhan [16] on abelian varieties and complex tori. Turning to surfaces, we obtain the second main result of this paper:

Theorem. *Let S be a smooth compact complex-analytic surface with $b_1 \neq 1$. Then the inclusion $\mathrm{Br}(S) \subset \mathrm{Br}'(S)$ is an equality.*

Here the main challenge is the case of elliptic surfaces. Usually, such surfaces do not satisfy the required conditions on $\pi_1(S)$ and $\pi_2(S)$, due to the presence of singular fibers in the elliptic fibration $S \to B$. However, there is always a Zariski open subset $U \subset S$ with the desired properties. Some additional arguments then show that this is enough for our purpose. A key ingredient is Hoobler's result [28],

see also Gabber [18], that any cohomology class $\beta \in \mathrm{Br}'(X)$ mapping into $\mathrm{Br}(Y)$ for some finite flat covering $Y \to X$ lies in the Brauer group.

My results for surfaces with $b_1 = 1$ are less definite. Such surfaces are also called of class VII. To date, this is the only class of surface resisting complete classification. Any known surface of class VII blows down to one of the three following types: Hopf surfaces, Inoue surfaces, and surfaces containing a global spherical shell. The latter is a holomorphic embedding of a thickened 3-sphere with connected complement.

Theorem. *We have* $\mathrm{Br}(S) = \mathrm{Br}'(S)$ *for any surface with* $b_1 = 1$ *whose minimal model is either a Hopf surface, Inoue surface, or a surface containing a global spherical shell.*

According to the *GSS-conjecture*, any surface of class VII should belong to one of these classes. If true, this would complete the Kodaira classification. The work of Dloussky, Oeljeklaus, and Toma gives considerable positive evidence [13], [14]. On the other hand, there are almost no results on fundamental groups of hypothetical surfaces of class VII. A notable exception is the work of Carlson and Toledo on representations of class VII fundamental group in fundamental groups of hyperbolic Riemannian manifolds [9].

Here is a plan for the paper: In Section 1, I set down some definitions concerning analytic Brauer groups, and also establish some Gaga type facts. In Section 2 we shall see that analytic Brauer groups might differ strongly from algebraic Brauer groups on noncompact surfaces. I relate this to the Shafarevich Conjecture, and give an application regarding the existence of nonalgebraic vector bundles on pointed algebraic surfaces. Section 3 contains a discussion of Serre's notion of good groups. I use good groups to prove $\mathrm{Br}(X) = \mathrm{Br}'(X)$ for certain complex-analytic spaces in Section 4. As application I discuss the case of complex Lie groups and Hopf manifolds. To apply the results to complex-analytic surfaces, we still have to improve them. This happens in Section 5. In Section 6, we then solve the case of elliptic surfaces. In Section 7, I analyze the case of surfaces of class VII. In Section 8, I combine our result with already known results. There is also a discussion of open problems.

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1. Analytic Brauer groups

In this section we shall introduce notation and establish some useful facts on analytic Brauer groups. Throughout, X denotes a complex-analytic space, and $\mathcal{O}_X$ is its sheaf of holomorphic functions. One way to define Brauer groups is in terms of holomorphic principal $\mathrm{PGL}_r(\mathbb{C})$-bundles, which is well-suited for our purposes. Examples of such bundles are projectivisations $P = \mathbb{P}(\mathcal{E})$ of locally free $\mathcal{O}_X$-modules $\mathcal{E}$ of rank $r > 0$. The Brauer group measures to which extent there are other principal bundles as follows:

Suppose $P \to X$ is a principal $\mathrm{PGL}_r(\mathbb{C})$-bundle, and $P' \to X$ is a principal $\mathrm{PGL}_{r'}(\mathbb{C})$-bundle. Using the homomorphism

$$\mathrm{PGL}_r(\mathbb{C}) \times \mathrm{PGL}_{r'}(\mathbb{C}) \longrightarrow \mathrm{PGL}_{rr'}(\mathbb{C}), \quad (A, A') \longmapsto A \otimes A'$$

we obtain another principal $\mathrm{PGL}_{rr'}(\mathbb{C})$-bundle $P \otimes P'$. One says that P and P' are equivalent if there are locally free $\mathcal{O}_X$-modules $\mathcal{E}', \mathcal{E}$ of rank $r', r > 0$ so that $P \otimes \mathbb{P}(\mathcal{E}')$ and $P' \otimes \mathbb{P}(\mathcal{E})$ are isomorphic. The group of equivalence classes is called the *Brauer group* $\mathrm{Br}(X)$. Addition is given by tensor products, and inverses come from taking dual bundles.

The Brauer group of a complex-analytic space is hard to handle. More approachable is the *cohomological Brauer group* $\mathrm{Br}'(X)$, which is defined as the torsion part of $H^2(X, \mathcal{O}_X^\times)$. The group extension

$$1 \longrightarrow \mathcal{O}_X^\times \longrightarrow \mathrm{GL}_r(\mathcal{O}_X) \longrightarrow \mathrm{PGL}_r(\mathcal{O}_X) \longrightarrow 1$$

yields a nonabelian coboundary map $H^1(X, \mathrm{PGL}_r(\mathbb{C})) \to H^2(X, \mathcal{O}_X)$, which induces an inclusion $\mathrm{Br}(X) \subset \mathrm{Br}'(X)$.

It is possible to compute $\mathrm{Br}'(X)$ as an abstract group using the exponential sequence $0 \to \mathbb{Z} \stackrel{2\pi i}{\to} \mathcal{O}_X \stackrel{\exp}{\to} \mathcal{O}_X^\times \to 1$. The corresponding long exact sequence reads

$$\mathrm{Pic}(X) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X^\times) \to H^3(X, \mathbb{Z}) \to H^3(X, \mathcal{O}_X).$$

Let $T = T(X)$ be the torsion part of $H^3(X, \mathbb{Z})$, which is also the image of the coboundary map $\mathrm{Br}'(X) \to H^3(X, \mathbb{Z})$. Furthermore, let $A = A(X)$ be the quotient $H^2(X, \mathbb{Z})/\mathrm{Pic}(X)$, which is the image of $H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$ as well. Note that the torsion free group A is sometimes called the *transcendental lattice*.

Proposition 1.1. *For any complex-analytic space X, the cohomological Brauer group canonically sits in a short exact sequence* $0 \to A \otimes \mathbb{Q}/\mathbb{Z} \to \mathrm{Br}'(X) \to T \to 0$.

Proof. We have an exact sequence

$$0 \longrightarrow A \longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow H^2(X, \mathcal{O}_X^\times) \longrightarrow H^3(X, \mathbb{Z}) \longrightarrow H^3(X, \mathcal{O}_X). \tag{1}$$

Set $K = H^2(X, \mathcal{O}_X)/A$. Applying the functor $\mathrm{Tor}_i(\cdot, \mathbb{Q}/\mathbb{Z})$, we obtain an exact sequence

$$0 \longrightarrow \mathrm{Tor}_1(K, \mathbb{Q}/\mathbb{Z}) \longrightarrow A \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow H^2(X, \mathcal{O}_X) \otimes \mathbb{Q}/\mathbb{Z}.$$

The term on the right vanishes, being the tensor product of a divisible group with a torsion group. Let M be the image of $H^2(X, \mathcal{O}_X^\times) \to H^3(X, \mathbb{Z})$. We clearly have $T \subset M$, and T equals the torsion part of M. As above, we have an exact sequence

$$0 \longrightarrow \mathrm{Tor}_1(K, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Tor}_1(H^2(X, \mathcal{O}_X^\times), \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Tor}_1(M, \mathbb{Q}/\mathbb{Z}) \longrightarrow K \otimes \mathbb{Q}/\mathbb{Z}$$

The term on the right vanishes, since K is divisible and $\mathbb{Q}/\mathbb{Z}$ is torsion. The assertion now follows from the fact that $\mathrm{Tor}_1(M, \mathbb{Q}/\mathbb{Z})$ is the torsion part of any abelian group M. □

The short exact sequence $0 \to A \otimes \mathbb{Q}/\mathbb{Z} \to \mathrm{Br}'(X) \to T \to 0$ splits, being an extension by a divisible and hence injective group. We call $A \otimes \mathbb{Q}/\mathbb{Z}$ the *analytic part* of the cohomological Brauer group, and T the *topological part*. This is a minor abuse of notation, because the short exact sequence has no canonical splitting, but it should not cause any confusion.

For compact complex-analytic spaces we therefore have a noncanonical isomorphism $\mathrm{Br}'(X) \simeq (\mathbb{Q}/\mathbb{Z})^{b_2 - \rho} \oplus T$, where b_2 is the second Betti number and ρ is the Picard number. The latter is defined as the rank of the image of the coboundary map $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$. Hodge theory gives additional information:

Corollary 1.2. *Let X be a smooth compact complex-analytic space. Assume that X is either Kähler or 2-dimensional. Then the analytic part $A \otimes \mathbb{Q}/\mathbb{Z}$ of the cohomological Brauer group $\mathrm{Br}'(X)$ vanishes if and only if $H^2(X, \mathcal{O}_X) = 0$.*

Proof. The condition is sufficient according to Proposition 1.1 and the exact sequence (1). For the converse we use the fact from Hodge theory that the complexification of the canonical map $H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$ is surjective. Compare [2], Chapter IV, Proposition 2.11 for surfaces, and [57], page 161 for Kähler manifolds. □

Let me point out that the topological part T of the Brauer group, which is the torsion part of the cohomology group $H^3(X, \mathbb{Z})$, is also isomorphic to the torsion part of the homology group $H_2(X, \mathbb{Z})$, by the Universal Coefficient Theorem.

We next discuss the relation between algebraic and analytic theories. The Brauer group and the cohomological Brauer group for schemes Y are defined as above, but one has to use the étale topology instead of the Zariski topology. If Y is an algebraic $\mathbb{C}$-scheme, we have an associated complex-analytic space $X = Y^{\mathrm{an}}$. For compact spaces, this does not influence cohomological Brauer groups:

Proposition 1.3. *Let Y be an algebraic $\mathbb{C}$-scheme, and $X = Y^{an}$ the associated complex-analytic space. Then the canonical map $\mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is surjective. It is even bijective provided X is compact.*

Proof. Fix an integer $n \geq 0$. The Kummer sequence $0 \to \mu_n \to \mathcal{O}_X^\times \xrightarrow{n} \mathcal{O}_X^\times \to 1$ gives a short exact sequence

$$0 \longrightarrow \mathrm{Pic}(X)_n \longrightarrow H^2(X, \mu_n) \longrightarrow {}_n\mathrm{Br}'(X) \longrightarrow 0.$$

Here $\mu_n = \mu_n(\mathbb{C})$ is the group of n-th roots of unity, and ${}_n\mathrm{Br}'(X)$ and $\mathrm{Pic}(X)_n$ are the kernel and cokernel for multiplication-by-n map. There is a similar short exact sequence for the étale topology on the algebraic $\mathbb{C}$-scheme Y, and we obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}(Y)_n & \longrightarrow & H^2(Y, \mu_n) & \longrightarrow & {}_n\mathrm{Br}'(Y) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Pic}(X)_n & \longrightarrow & H^2(X, \mu_n) & \longrightarrow & {}_n\mathrm{Br}'(X) & \longrightarrow & 0.
\end{array}$$

The vertical map in the middle is bijective by comparison results in étale cohomology ([24], Exposé XVI, Theorem 4.1). It follows that $\mathrm{Br}'(S) \to \mathrm{Br}'(X)$ is surjective.

Now suppose in addition that X is compact. It then easily follows from Serre's Gaga Theorems ([52], Proposition 18) that $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is bijective, hence the assertion. □

Brauer groups behave similarly:

Proposition 1.4. *Notation as above. Suppose that $X = Y^{an}$ is compact. Then the canonical map $\mathrm{Br}(Y) \to \mathrm{Br}(X)$ is bijective.*

Proof. Injectivity follows directly from Proposition 1.3. To check surjectivity, suppose we have a holomorphic $\mathrm{PGL}_r(\mathbb{C})$-principal bundle $P \to X$. It comes from a cocycle λ_{ij} for some open covering $U_i \subset X$. The problem here is that the open subsets U_i are not necessarily Zariski open, and the holomorphic maps $\lambda_{ij} \to \mathrm{PGL}_r(\mathbb{C})$ are not necessarily algebraic. We sidestep the problems as follows:

The cocycle λ_{ij} also defines, via the conjugacy action, a holomorphic *Azumaya algebra* $\mathcal{A}$ over $\mathcal{O}_X$. This means that $\mathcal{A}$ is a twisted form of the matrix algebra $\mathrm{Mat}_r(\mathcal{O}_X)$. According to Serre's Gaga Theorem, the underlying locally free $\mathcal{O}_X$-module is algebraic. Moreover, the structure map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ defining the algebra structure is algebraic ([52], Theorem 2). Summing up, the Azumaya $\mathcal{O}_X$-algebra $\mathcal{A}$ is isomorphic to $\mathcal{B} \otimes_{\mathcal{O}_Y} \mathcal{O}_X$ for some locally free $\mathcal{O}_Y$-algebra $\mathcal{B}$. Since the fibers $\mathcal{B}(x) = \mathcal{A}(x)$, $x \in X$ are matrix algebras over $\mathbb{C}$, the $\mathcal{O}_Y$-algebra $\mathcal{B}$ is Azumaya. It follows that the map on Brauer groups is surjective. □

2. Nonpurity results

So far we saw that analytic and algebraic Brauer groups coincide on compact algebraic spaces. In this section I discuss a striking difference between analytic and algebraic Brauer groups for noncompact spaces. For simplicity I confine the discussion to dimension two. Throughout the paper, a *surface* is a complex-analytic space S that is irreducible and of complex dimension two.

Proposition 2.1. *Let S be a noncompact surface. Then $H^2(S, \mathcal{O}_S^\times) = H^3(S, \mathbb{Z})$.*

Proof. We have $H^2(S, \mathcal{O}_S) = 0$ according to Siu's vanishing result for noncompact spaces [55], and $H^3(S, \mathcal{O}_S) = 0$ by dimension reasons. The exponential sequence gives an exact sequence

$$H^2(S, \mathcal{O}_S) \longrightarrow H^2(S, \mathcal{O}_S^\times) \longrightarrow H^3(S, \mathbb{Z}) \longrightarrow H^3(S, \mathcal{O}_S),$$

and the assertion follows. □

This means that the analytic part of the cohomological Brauer group vanishes upon restrictions. The topological part behaves differently:

Proposition 2.2. *Let S be a smooth surface, $A \subset S$ be a discrete subset, and $U = S - A$ the open complement. Then the restriction map $H^3(S, \mathbb{Z}) \to H^3(U, \mathbb{Z})$ induces a bijection on torsion parts.*

Proof. The long exact sequence for local cohomology groups gives an exact sequence

$$H^3_A(S, \mathbb{Z}) \longrightarrow H^3(S, \mathbb{Z}) \longrightarrow H^3(U, \mathbb{Z}) \longrightarrow H^4_A(S, \mathbb{Z}).$$

The term on the left vanishes and the term on the right is free. This follows from the Thom isomorphism $H^{p-4}(A, \mathbb{Z}) \to H^p_A(S, \mathbb{Z})$, compare [33], Chapter VIII, Proposition 2.3. □

It is more difficult to understand the behavior of the topological part if one removes more that just points. Recall that a complex-analytic space X is called *Stein* if $H^p(X, \mathcal{F}) = 0$ for all $p \geq 1$ and all coherent $\mathcal{O}_X$-modules $\mathcal{F}$. This is the analogue of affine schemes in complex-analytic geometry. Indeed, any complex-analytic space is covered by Stein open subsets, and affine algebraic spaces are Stein.

Proposition 2.3. *Suppose S is a complex-analytic surface that is Stein. Then we have $H^2(S, \mathcal{O}_S^\times) = 0$.*

Proof. Again we use the exact sequence

$$H^2(S, \mathcal{O}_S) \longrightarrow H^2(S, \mathcal{O}_S^\times) \longrightarrow H^3(S, \mathbb{Z}).$$

The term on the left vanishes by the Stein condition. The term on the right also vanishes: According to Hamm's result, any Stein space X of dimension n has the homotopy type of a CW-complex with cells of dimension $\leq n$ only [25]. $\square$

A complex-analytic space X is called *holomorphically convex* if there is a Stein space Y, together with a proper holomorphic map $X \to Y$. This notion is somewhere between compact spaces and Stein spaces. It is a rather important class of spaces: According to the *Shafarevich Conjecture*, the universal covering of any smooth projective space should be holomorphically convex.

Note that we may replace Y by the analytic spectrum of $f_*(\mathcal{O}_X)$. Then the map $f : X \to Y$ is surjective with connected fibers, and Y is called the *Stein reduction* of X.

Proposition 2.4. *Let S be a complex-analytic surface that is holomorphically convex with 2-dimensional Stein reduction. Then $H^2(S, \mathcal{O}_S^\times) = 0$.*

Proof. Let $f : S \to Y$ be the Stein reduction. The spectral sequence

$$H^p(Y, R^q f_*(\mathcal{O}_S)) \Longrightarrow H^{p+q}(S, \mathcal{O}_S)$$

together with $R^2 f_*(\mathcal{O}_S) = 0$ and Steinness of Y tells us that $H^2(S, \mathcal{O}_S)$ vanishes. In light of the exact sequence

$$H^2(S, \mathcal{O}_S) \longrightarrow H^2(S, \mathcal{O}_S^\times) \longrightarrow H^3(S, \mathbb{Z}),$$

it suffices to check that $H^3(S, \mathbb{Z}) = 0$. Consider the spectral sequence

$$H^p(Y, R^q f_*(\mathbb{Z})) \Longrightarrow H^{p+q}(S, \mathbb{Z}).$$

Note that the fibers $S_y = f^{-1}(y)$ are of complex dimension ≤ 1, and the base change maps $R^n f_*(\mathbb{Z})_y \to H^n(S_y, \mathbb{Z})$ are bijective (see [33], Section III, Theorem 6.2). This implies $R^q f_*(\mathbb{Z}) = 0$ for $q \geq 3$. Next we use the fact that $f : S \to Y$ is bijective over the complement of a discrete set $D \subset Y$. For $q \geq 1$, the sheaves $R^q f_*(\mathbb{Z})$ are supported on D, and hence $H^p(Y, R^q f_*(\mathbb{Z})) = 0$ for $p, q \geq 1$. We finally examine the terms $H^p(Y, f_*(\mathbb{Z}))$. Note that $f_*(\mathbb{Z}) = \mathbb{Z}$, because $f : S \to Y$ has connected fibers. According to Hamm's result, Y has the homotopy type of a CW-complex with cells of dimension ≤ 2 only [25]. This implies $H^p(Y, \mathbb{Z}) = 0$ for $p \geq 3$.

Summing up, the terms $H^p(Y, R^q f_*(\mathbb{Z}))$ in the spectral sequence vanish whenever $p + q = 3$, and therefore $H^3(S, \mathbb{Z}) = 0$. $\square$

Remark 2.5. As Fabrizio Catanese pointed out to me, the preceding result may be useful in connection with the Shafarevich Conjecture. Suppose we want to refute the Shafarevich Conjecture. Then we might try to find a smooth projective surface S whose universal covering $\tilde{S}$ has 2-dimensional Stein reduction, together with an Azumaya $\mathcal{O}_S$-algebra $\mathcal{A}$ that does not become a matrix algebra on $\tilde{S}$. I do not know whether this is feasible. But it reminds me about the Brauer–Manin obstruction, which was used to construct counterexamples to the Hasse principle.

Back to the comparison of algebraic and analytic Brauer groups. Suppose that Y is a smooth proper 2-dimensional $\mathbb{C}$-scheme, and $V \subset Y$ is an open subscheme. According to Grothendieck's Purity Theorems [22], Section 6, the restriction map $\mathrm{Br}'(Y) \to \mathrm{Br}'(V)$ is injective. In contrast, we just saw that the restriction maps on the corresponding analytic cohomological Brauer groups is usually not injective.

In other words, algebraic and analytic Brauer groups might differ dramatically on noncompact surfaces.

We close this section with an amusing application to holomorphic vector bundles: Recall that an easy computation with Čech cocycle reveals that the restriction map of analytic Picard groups

$$\mathrm{Pic}(\mathbb{C}^2) \longrightarrow \mathrm{Pic}(\mathbb{C}^2 - \{0\})$$

has infinite cokernel. Such a behavior is rather typical:

Proposition 2.6. *Let S be a smooth compact algebraic surface with $H^2(S, \mathcal{O}_S) \neq 0$. Fix a point $s \in S$. Then there are infinitely many locally free coherent sheaves $\mathcal{E}_U$ on the analytic surface $U = S - \{s\}$ that do not extend to coherent sheaves on S.*

Proof. First note that $\mathrm{Br}(S) = \mathrm{Br}'(S)$ by Grothendieck's result on algebraic surfaces [22] and Proposition 1.4. The analytic part $A \otimes \mathbb{Q}/\mathbb{Z}$ of $\mathrm{Br}(S)$ is nontrivial and hence infinite according to Corollary 1.2. Pick a nonzero class $\beta \in \mathrm{Br}(S)$ from the analytic part and represent it by some Azumaya $\mathcal{O}_S$-algebra $\mathcal{A}$. The restriction $\beta_U \in \mathrm{Br}'(U)$ vanishes by Proposition 2.1, and this implies that there is a locally free $\mathcal{O}_U$-module $\mathcal{E}_U$ with $\mathcal{A}_U = \mathcal{E}nd(\mathcal{E}_U)$.

Suppose $\mathcal{E}_U$ extends to a coherent $\mathcal{O}_S$-module $\mathcal{E}$. Passing to double duals, we may assume that $\mathcal{E}$ is locally free. The bijection $\mathcal{E}nd(\mathcal{E})_U \to \mathcal{A}_U$ extends to a map $\mathcal{E}nd(\mathcal{E}) \to \mathcal{A}$. Its determinant vanishes either nowhere or in codimension one, and we infer that the map is bijective. A similar argument shows that this map is an isomorphism of algebras. This implies $\beta = 0$, contradiction. □

3. Serre's good groups

Serre's notion of good groups plays a crucial role in the sequel, and I want to recall this concept now. Let G be a group and $\hat{G} = \varprojlim G/N$ be its profinite completion. Here the inverse limit runs over all normal subgroups $N \subset G$ of finite index. We regard both G and $\hat{G}$ as topological groups: the group G carries the discrete topology, and $\hat{G}$ is endowed with the inverse limit topology where the factors G/N are discrete.

Now let M be a finite G-module. By this I understand a finite abelian group with discrete topology, and having a G-module structure. The map $G \to \mathrm{Aut}(M)$ factors over $\hat{G}$, and we may regard M as a topological $\hat{G}$-module as well. The canonical map $G \to \hat{G}$ induces a restriction map

$$H^p(\hat{G}, M) \longrightarrow H^p(G, M), \quad p \geq 0$$

on cohomology groups. Note that our cohomology groups are defined in terms of continuous cochains. Serre showed in [54], Chapter I, §2.6 that these restriction maps are bijective for $p = 0, 1$. He calls a group G *good* if the restriction maps $H^p(\hat{G}, M) \to H^p(G, M)$ are bijective for all integers $p \geq 0$ and all finite G-modules M. Finite groups are clearly good groups. Recall that a group is called *almost free* if it contains a free subgroup of finite index.

Proposition 3.1. *Almost free groups are good groups.*

Proof. First consider the case that G is a free group. Then $H^p(G, M) = 0$ for $p \geq 2$ and any G-module M, because G is the fundamental group of an Eilenberg–Maclane space $K(G, 1)$ with cells of dimension ≤ 1 only. On the other hand, we

have $H^p(\hat{G}, M) = 0$ for $p \geq 2$ and any torsion G-module M by [54], Chapter I, Proposition 16.

Now suppose that G is almost free. Then we find a normal subgroup $N \subset G$ that is free and with $Q = G/N$ finite. According to [49], Lemma in Section 5.1, the canonical map on profinite completions $\hat{N} \to \hat{G}$ is injective with $Q = \hat{G}/\hat{N}$.

The following is a variant of an argument due to Serre ([54], Section 2.6): Let M be a finite G-module, and consider the two Hochschild–Serre spectral sequences

$$H^p(Q, H^q(N, M)) \Longrightarrow H^{p+q}(G, M), \qquad H^p(Q, H^q(\hat{N}, M)) \Longrightarrow H^{p+q}(\hat{G}, M).$$

Using that N is free we obtain a long exact sequences

$$\ldots \to H^p(Q, H^0(N)) \to H^p(G) \to H^{p-1}(Q, H^1(N)) \to H^{p+1}(Q, H^0(N)) \to \ldots$$

and another long exact sequence

$$\ldots \to H^p(Q, H^0(\hat{N})) \to H^p(\hat{G}) \to H^{p-1}(Q, H^1(\hat{N})) \to H^{p+1}(Q, H^0(\hat{N})) \to \ldots,$$

where the coefficient groups are always M. Note that the canonical mappings $H^i(\hat{N}, M) \to H^i(N, M)$ are always bijective for $i = 0, 1$. Using the 5-lemma, we deduce that the canonical map $H^p(\hat{G}, M) \to H^p(G, M)$ is bijective. □

Recall that a group G is called *polycyclic* if there is a finite sequence of subgroups $0 = G_0 \subset G_1 \subset \ldots \subset G_n = G$ so that $G_{i-1} \subset G_i$ are normal with cyclic factors G_i/G_{i-1}. This are precisely the solvable groups all whose subgroups are finitely generated. Note that finitely generated nilpotent groups are polycyclic. A group is called *almost polycyclic* if it contains a polycyclic subgroup of finite index.

Proposition 3.2. *Almost polycyclic groups are good groups.*

Proof. Suppose first that G is a polycyclic group. Then the cohomology groups $H^p(G, M)$ are finite for all finite G-modules M and all integers $p \geq 0$. This is obvious for G cyclic, and follows by induction on the length n of the subnormal series $G_i \subset G$, together with the Hochschild–Serre spectral sequence.

To proceed, we use a general result of Serre: Let G be an arbitrary group containing a normal subgroup $N \subset G$. Suppose that N and $Q = G/N$ are good, and that $H^p(N, M)$ are finite for all finite A-modules M and $p \geq 0$. Then Serre outlined in [54], Section 2.6 that this implies that G is also good: He first checks that the sequence $1 \to \hat{N} \to \hat{G} \to \hat{Q} \to 1$ remains a group extension, and then compares the two Hochschild–Serre spectral sequences. Using this, we easily verify that our polycyclic group G is good, again by induction on the length n of the subnormal series $G_i \subset G$.

Now suppose that G is almost polycyclic. Then we find a normal subgroup $N \subset G$ that is polycyclic and with $Q = G/N$ finite. Then N, Q are good, and $H^q(N, M)$ are finite for all finite G-modules M. Repeating Serre's argument as above, we see that G is good. □

Good groups are very useful with respect to Grothendieck's question on Brauer groups. Let G be a group. The exact sequence of groups with trivial G-action

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \mathrm{GL}_r(\mathbb{C}) \longrightarrow \mathrm{PGL}_r(\mathbb{C}) \longrightarrow 1$$

induces a coboundary map in nonabelian cohomology

$$H^1(G, \mathrm{PGL}_r(\mathbb{C})) \longrightarrow H^2(G, \mathbb{C}^\times) \tag{2}$$

as explained in [54], Chapter I, §5. In representation theory of finite groups, $H^2(G, \mathbb{C}^\times)$ is also called the *Schur multiplier* (confer [35]). In this context, Grothendieck's question is: Given a torsion class $\beta \in H^2(G, \mathbb{C}^\times)$, does there exist some $r > 0$ so that β lies in the image of the coboundary map (2)?

Proposition 3.3. *Let G be a good group. Then any torsion class $\beta \in H^2(G, \mathbb{C}^\times)$ lies in the image of the coboundary map $H^1(G, \mathrm{PGL}_r(\mathbb{C})) \to H^2(G, \mathbb{C}^\times)$ for some integer $r > 0$.*

Proof. The idea is to reduce to the case that the group G is finite. Let $\mu_n = \mu_n(\mathbb{C})$ be the group of complex n-th roots of unity. Using the Kummer sequence, we infer that β lies in the image of $H^2(G, \mu_n) \to H^2(G, \mathbb{C}^\times)$, whenever $n\beta = 0$. Choose $\alpha \in H^2(G, \mu_n)$ mapping to β.

Since G is good, we have

$$H^2(G, \mu_n) = H^2(\hat{G}, \mu_n) = \varinjlim H^2(G/N, \mu_n).$$

Replacing G by some suitable G/N, we may assume that the group G is finite. It is then a fact from representation theory that any factor system comes from a projective representation (see for example [29], Chapter V, Hilfssatz 24.2), hence the result. □

4. Applications to complex spaces

We now come back to geometry. Fix a complex-analytic space X. We seek conditions under which the canonical inclusion $\mathrm{Br}(X) \subset \mathrm{Br}'(X)$ is an equality. The following result is interesting because it makes only assumptions on the homotopy type of X:

Theorem 4.1. *Let X be a connected complex-analytic space. Suppose that the fundamental group $\pi_1(X)$ is good, and that the subgroup of $\pi_1(X)$-invariants in the Pontrjagin dual $\mathrm{Hom}(\pi_2(X), \mathbb{Q}/\mathbb{Z})$ vanishes. Then $\mathrm{Br}(X) = \mathrm{Br}'(X)$.*

Proof. Let $\beta \in \mathrm{Br}'(X)$, say with $n\beta = 0$, and choose a class $\alpha \in H^2(X, \mu_n)$ mapping to β. Let $\tilde{X} \to X$ be the universal covering. Then $\pi_2(X) = \pi_2(\tilde{X})$, and the canonical map $H_2(\tilde{X}, \mathbb{Z}) \to \pi_2(\tilde{X})$ is bijective by the Hurewicz Theorem. Moreover, the canonical map $H^2(\tilde{X}, \mu_n) \to \mathrm{Hom}(H_2(\tilde{X}, \mathbb{Z}), \mu_n)$ is bijective by the Universal Coefficient Theorem. The universal covering yields a spectral sequence

$$H^p(G, H^q(\tilde{X}, \mu_n)) \Longrightarrow H^{p+q}(X, \mu_n),$$

where $G = \pi_1(X)$. Since $H^1(\tilde{X}, \mu_n) = 0$, this yields a short exact sequence

$$0 \longrightarrow H^2(G, \mu_n) \longrightarrow H^2(X, \mu_n) \longrightarrow H^0(G, H^2(\tilde{X}, \mu_n)).$$

The term on the right vanishes, in light of our assumption on $\pi_2(X)$ and the inclusion $H^2(\tilde{X}, \mu_n) \subset \mathrm{Hom}(\pi_2(X), \mathbb{Q}/\mathbb{Z})$. The upshot is that any 2-cohomology class on X with values in μ_n comes from group cohomology.

To finish the proof we use the assumption that $G = \pi_1(X)$ is good, which implies

$$H^2(G, \mu_n) = H^2(\hat{G}, \mu_n) = \varinjlim H^2(G/N, \mu_n).$$

Hence there is an finite étale covering $g : X' \to X$ with $g^*(\beta) = 0$, and in particular $g^*(\alpha) = 0$. Then a result of Hoobler ([28], Section 3), see also Gabber ([18], Lemma 4), tells us that $\alpha \in \mathrm{Br}(X)$. Alternatively, we could construct a principal $\mathrm{PGL}_r(\mathbb{C})$-bundle $P \to X$ representing β as the principal bundle associated

with the representation $\pi_1(X) \to \mathrm{PGL}_r(\mathbb{C})$ constructed in the end of the proof for Proposition 3.3. □

The preceding Theorem applies in particular if $\pi_2(X) = H_2(\tilde{X}, \mathbb{Z}) = 0$. This is made to measure for complex Lie groups:

Corollary 4.2. *Let X be any complex Lie group. Then* $\mathrm{Br}(X) = \mathrm{Br}'(X)$.

Proof. We may assume that X is connected. The fundamental group $\pi_1(X)$ is a finitely generated abelian group, hence good. Indeed: it is abelian because X is a H-space, and it is finitely generated because X is homotopy equivalent to a compact differentiable manifold (see [34], Theorem 6). The homotopy group $\pi_2(X)$ vanishes, according to Browder results on torsion in homology of H-spaces ([8], Theorem 6.11). Consequently Theorem 4.1 applies. □

This generalizes a result of Iversen on characterfree linear algebraic groups [32]. It also generalizes results of Hoobler [28] and Berkovič [3] on abelian varieties, and of Elencwajg and Narasimhan on complex tori [16].

It is instructive to look at the case of connected abelian Lie groups. They have the form $X = \mathbb{C}^n/\Gamma$ for some lattice $\Gamma \subset \mathbb{C}^n$, and are studied in the book of Abe and Kopfermann [1]. We have $H^i(X, \mathbb{Z}) = \mathrm{Hom}(\Lambda^i\Gamma, \mathbb{Z})$, and from this it is in principle possible to compute the Brauer group $\mathrm{Br}(X) = \mathrm{Br}'(X)$ as a quotient of $H^2(X, \mathbb{Q}/\mathbb{Z})$. The full cohomology group $H^2(X, \mathcal{O}_X^\times)$, however, can be tremendously large: As explained in [38], there are lattices Γ such that $H^2(X, \mathcal{O}_X)$ are infinite dimensional complex vector spaces.

We next apply our result to Hopf manifolds. Recall that a complex space X is called a *Hopf manifold* if it is compact and its universal covering $\tilde{X}$ is biholomorphic to $\mathbb{C}^n - \{0\}$ with $n \geq 2$. Hopf manifolds are the simplest examples of compact complex manifolds that are not Kähler.

Corollary 4.3. *Let X be a Hopf manifold. Then* $\mathrm{Br}(X) = \mathrm{Br}'(X)$.

Proof. We obviously have $H_2(\tilde{X}, \mathbb{Z}) = 0$. Furthermore, Kodaira proved in [40], Theorem 30 that the fundamental group sits in a central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(X) \longrightarrow G \longrightarrow 1 \tag{3}$$

for some finite group G. Actually, Kodaira treated the 2-dimensional case, but his arguments work in all dimensions $n \geq 2$. So $\pi_1(X)$ is almost free hence good by Proposition 3.1, and we conclude with Theorem 4.1. □

Actually, we may compute the Brauer group of a n-dimensional Hopf manifold X as above: Let $c \in H^2(G, \mathbb{Z})$ be the extension class of the central extension (3). It defines homomorphisms $c : H^i(G, \mathbb{Z}) \to H^{i+2}(G, \mathbb{Z})$ via the cup product.

Proposition 4.4. *Let X be a Hopf manifold. With the preceding notation, we have an exact sequence $H^1(G, \mathbb{Z}) \xrightarrow{c} H^3(G, \mathbb{Z}) \to \mathrm{Br}(X) \to H^2(G, \mathbb{Z}) \xrightarrow{c} H^4(G, \mathbb{Z})$.*

Proof. First note that the analytic part of the Brauer group vanishes, because $H^2(X, \mathcal{O}_X) = 0$. The latter is due to Kodaira for Hopf surfaces ([40], Theorem 26), and follows for higher dimensional Hopf manifolds from a result of Mall ([44], Theorem 3).

We have to compute the topological part of the Brauer group. To simplify notation, set $\pi_1 = \pi_1(X)$ and $m = 2n-1$. Note that $n \geq 2$ and $m \geq 3$. The spectral

sequence $H^p(\pi_1, H^q(\tilde{X}, \mathbb{Z})) \Rightarrow H^{p+q}(X, \mathbb{Z})$ reduces to a long exact sequence

$$\to H^{p-1-m}(\pi_1, H^m(\tilde{X})) \to H^p(\pi_1, H^0(\tilde{X})) \to H^p(X) \to H^{p-m}(\pi_1, H^m(\tilde{X})) \to .$$

In particular, the edge maps of the spectral sequence $H^p(\pi_1, H^0(\tilde{X}, \mathbb{Z})) \to H^p(X, \mathbb{Z})$ are bijective for $p < m$. For $p = m$, we obtain an exact sequence

$$0 \longrightarrow H^m(\pi_1, H^0(\tilde{X})) \longrightarrow H^m(X) \longrightarrow H^0(\pi_1, H^m(\tilde{X})).$$

The term on the right equals $H^m(\tilde{X})^{\pi_1} = \mathbb{Z}$, because $H^m(\tilde{X}) = \mathbb{Z}$ and the action of π_1 on $\tilde{X}$ is orientation preserving. Hence the torsion in $H^m(X)$ is contained in $H^m(\pi_1, H^0(\tilde{X}))$.

To proceed, we view $\mathbb{Z} = H^m(\tilde{X})$ as a π_1-module with trivial action. The Hochschild–Serre spectral sequence $H^p(G, H^q(\mathbb{Z}, \mathbb{Z})) \Rightarrow H^{p+q}(\pi_1, \mathbb{Z})$ for the central extension (3) reduces to a long exact sequence

$$H^{p-2}(G, H^1(\mathbb{Z}, \mathbb{Z})) \to H^p(G, H^0(\mathbb{Z}, \mathbb{Z})) \to H^p(\pi_1, \mathbb{Z}) \to H^{p-1}(G, H^1(\mathbb{Z}, \mathbb{Z})),$$

because $H^p(\mathbb{Z}, M) = 0$ for $p \geq 2$ and any $\mathbb{Z}$-module M. Since $H^p(G, M)$ are torsion groups for $p > 0$ and any G-module M, the groups $H^p(\pi_1, \mathbb{Z})$ are torsion for $p \geq 2$.

Combining the preceding two paragraphs, we infer that $H^3(\pi_1, \mathbb{Z})$ is torsion, and equals the torsion part in $H^3(X, \mathbb{Z})$. Using Proposition 1.1, we infer

$$\mathrm{Br}(X) = \mathrm{Br}'(X) = H^3(\pi_1, \mathbb{Z}).$$

By the Hochschild–Serre spectral sequence, this group sits in an exact sequence

$$H^1(G, \mathbb{Z}) \longrightarrow H^3(G, \mathbb{Z}) \longrightarrow H^3(\pi_1, \mathbb{Z}) \longrightarrow H^2(G, \mathbb{Z}) \longrightarrow H^4(G, \mathbb{Z}). \tag{4}$$

According to [27], Section 6 the outer maps are taking cup products with the extension class $-c \in H^2(G, \mathbb{Z})$, hence the assertion. □

To my knowledge, there has been no attempt to classify such group extensions occurring in Hopf manifolds, except for dimension two: Kato determined all possible groups $\pi_1(S)$ for Hopf surfaces S in [36], [37]. There are two cases: In the *linear case* $\pi_1(S)$ is conjugate to a subgroup of $\mathrm{GL}_2(\mathbb{C})$, and then G is an extension of a subgroup $K \subset \mathrm{SL}_2(\mathbb{C})$ by a cyclic group. Such subgroups are classified (cyclic, dihedral, tetrahedral, octahedral, icosahedral). In the nonlinear case $\pi_1(S)$ is not conjugate to a subgroup of $\mathrm{GL}_2(\mathbb{C})$, and G must be cyclic.

5. From open to compact surfaces

In order to tackle smooth compact surfaces, we have to improve Theorem 4.1. The key result is the following, which works in dimension 2 only:

Theorem 5.1. *Let S be a smooth compact surface, and $V \subset S$ be a nonempty Zariski open subset. Suppose that the fundamental group $\pi_1(V)$ is good, and that the subgroup of $\pi_1(V)$-invariants in the Pontrjagin dual $\mathrm{Hom}(\pi_2(V), \mathbb{Q}/\mathbb{Z})$ vanishes. Then $\mathrm{Br}(S) = \mathrm{Br}'(S)$.*

Proof. Fix a class $\beta \in \mathrm{Br}'(S)$, say with $n\beta = 0$, and choose a class $\alpha \in H^2(S, \mu_n)$ mapping to β. As in the proof of Theorem 4.1, there is a finite étale covering $U \to V$ with $\alpha_U = 0$. By [11], Theorem 3.4 there is a compactification $U \subset X$ with some

compact complex-analytic surface X, so that we have a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ V & \longrightarrow & S. \end{array}$$

Replacing X by a suitable blowing-up with center in the boundary $C = X - U$, we may assume that the compact surface X is smooth, and that $C = C_1 + \ldots + C_m$ is a normal crossing divisor with smooth irreducible components (see, for example, [41], Chapter I and II).

In the next step we use local cohomology groups $H^2_C(X, \mu_n)$. Combining the Kummer sequence with the local cohomology sequence on X, we obtain a diagram

$$\begin{array}{ccccccccc} & & & & H^2_C(X,\mu_n) & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & \mathrm{Pic}(X)_n & \longrightarrow & H^2(X,\mu_n) & \longrightarrow & {}_n\mathrm{Br}'(X) & \longrightarrow & 0. \\ & & & & \downarrow & & & & \\ & & & & H^2(U,\mu_n) & & & & \end{array}$$

Here ${}_n\mathrm{Br}'(X)$ and $\mathrm{Pic}(X)_n$ are the kernel and cokernel for multiplication-by-n map. We remark that that the upper vertical map factors over the left horizontal map. To see this, note that according to [33], Chapter 10, Proposition 2.5, the canonical map between local cohomology groups

$$\bigoplus_{i=1}^{m} H^2_{C_i}(X, \mu_n) \longrightarrow H^2_C(X, \mu_n)$$

is bijective. The summands $H^2_{C_i}(X, \mu_n)$ are freely generated by the cycle class $\mathrm{cl}_X(C_i)$, as explained in [10], Proposition 2.2.6 on page 141 (this class is called the *Thom class* in topology). Moreover, the image of $\mathrm{cl}_X(C_i)$ and $\mathcal{O}_X(C_i)$ in $H^2(X, \mu_n)$ coincide by [17], Chapter II, Proposition 2.2. The upshot is that $\alpha_U = 0$ implies that $\alpha_X \equiv 0$ modulo $\mathrm{Pic}(X)_n$, and hence $\beta_X = 0$. To finish the proof, it therefore suffices to verify the following statement:

Lemma 5.2. *Let S be a smooth complex-analytic surface, X a normal complex-analytic surface, and $f : X \to S$ be a proper holomorphic surjection. Then* $\mathrm{Br}(S)$ *contains the kernel of the induced map* $\mathrm{Br}'(S) \to \mathrm{Br}'(X)$.

Proof. First, consider the special case that the proper holomorphic map $X \to S$ is finite. It then follows that it is flat as well, because S is smooth and X is Cohen–Macauley (this follows from [53], Chapter IV, Theorem 9). Then the result of Hoobler ([28], Section 3) and Gabber ([18], Lemma 4) tells us that any cohomology class $\beta \in \mathrm{Br}'(S)$ with $\beta_X = 0$ lies in $\mathrm{Br}(S)$.

Now consider the general case. There exist a proper bimeromorphic mapping $f : S' \to S$ so that the 2-dimensional integral component $Y' \subset X \times_S S'$ is flat over S'. This is a special case of Hironaka's general result on flattening [26]. Here we need only the 2-dimensional case, which also Maurer [45] worked out. Actually we only need that $Y' \to S'$ is finite. Replacing S' by a suitable blowing-up, we may assume that S' is smooth. The normalization X' of Y' is finite over S' as well.

According to the special case, the preimage $\beta_{S'}$ lies in $\mathrm{Br}(S')$. The statement now follows from the following observation:

Lemma 5.3. *Let S, S' be two smooth surfaces, and $f : S' \to S$ a proper bimeromorphic map. Then any class $\beta \in \mathrm{Br}'(S)$ with $\beta_{S'} \in \mathrm{Br}(S')$ lies in $\mathrm{Br}(S)$.*

Proof. As in the proof for Proposition 1.4, it is more convenient to work with Azumaya algebras than with principal bundles. Let $\mathcal{B}$ be an Azumaya $\mathcal{O}_{S'}$-algebra representing $\beta_{S'}$. Then the double dual $\mathcal{A} = f_*(\mathcal{B})^{\vee\vee}$ is coherent reflexive $\mathcal{O}_S$-algebra, which is actually locally free, because any reflexive sheaf on a complex manifold is locally free in codimension two. I claim that $\mathcal{A}$ is Azumaya. For this, consider the $\mathcal{O}_X$-algebra homomorphism

$$\varphi : \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{E}nd(\mathcal{A}), \quad A \otimes A' \longmapsto (B \mapsto ABA').$$

Both sheaves are locally free, and the map is bijective on $S - T$, where $T \subset S$ is the discrete subset of all points $s \in S$ with 1-dimensional fiber $f^{-1}(s)$. The set of points where the map is not bijective is a Cartier divisor (choose local holomorphic bases and take determinants) hence our map φ is bijective everywhere on S. This means that $\mathcal{A}$ is an Azumaya algebra by [21], Theorem 5.

It remains to check that the Azumaya algebra $\mathcal{A}$ represents the cohomology class β. This is easy for algebraic surfaces with Grothendieck's purity results from [23], Section 6. We already saw, however, that such purity results do not hold true in the analytic situation. We shall use methods from Čech cohomology instead. The argument runs as follows:

By Lemma 5.4 below, it suffices to check that the Azumaya $\mathcal{O}_{S'}$-algebras $\mathcal{A}' = f^*(\mathcal{A})$ and $\mathcal{B}$ have the same cohomology class. We check this by construction explicit 2-cocycles. To do this, we first have to settle some technical points regarding the existence of 2-cocycles for 2-cohomology classes. Choose an open covering $U_i \subset S$, $i \in I$ so that $\beta_{U_i} = 0$. After passing to a refinement, we may assume that every $s \in T$ is contained in precisely one U_i. The Azumaya $\mathcal{O}_{S'}$-algebra $\mathcal{B}$ represents the zero class on each preimage $U_i' = f^{-1}(U_i)$, hence there is a locally free $\mathcal{O}_{U_i'}$-module $\mathcal{E}_i$ and an isomorphism $s_i : \mathcal{B}_{U_i'} \to \mathcal{E}nd(\mathcal{E}_i)$. The double duals $\mathcal{F}_i = f_*(\mathcal{E}_i)^{\vee\vee}$ are locally free $\mathcal{O}_{U_i}$-modules, and the bijections s_i induce bijections $t_i : \mathcal{A}_{U_i} \to \mathcal{E}nd(\mathcal{F}_i)$, by the same argument as above.

On the overlaps $U_{ij} = U_i \cap U_j$ there are invertible sheaves $\mathcal{L}_{ij}$ defined by the condition $\mathcal{F}_i|_{U_{ij}} \otimes \mathcal{L}_{ij} \simeq \mathcal{F}_j|_{U_{ij}}$. We regard $U_{ij} \mapsto \mathcal{L}_{ij}$ as a 1-cochain with respect to the open covering $\mathfrak{U} = (U_i)_{i \in I}$ taking values in the presheaf $\mathcal{H}^1\mathcal{O}_S^\times$. The latter is defined by $\Gamma(U, \mathcal{H}^1\mathcal{O}_S^\times) = \mathrm{Pic}(U)$. As explained in [51], Lemma 3.1 the cochain $\mathcal{L}_{ij}$ is actually a cocycle. Things would be particularly nice if the $\mathcal{L}_{ij} \in \mathrm{Pic}(U_{ij})$ are trivial. We can achieve this, for example, by first refining the covering to a covering with U_{ij} contractible (compare [6], Corollary I.5.2), and then further refining to a covering that is Stein. However, the following argument works in more general situations as well: By definition, complex-analytic spaces have a countable basis for the topology, hence are paracompact. This implies $\check{H}^1(X, \mathcal{H}^1\mathcal{O}_X^\times) = 0$, as explained in [20], Chapter II, Proposition 5.10.1. So after passing to a refinement, we find invertible $\mathcal{O}_{U_i}$-modules $\mathcal{L}_i$ with $\mathcal{L}_i|_{U_{ij}} \otimes \mathcal{L}_{ij} \simeq \mathcal{L}_j|_{U_{ij}}$. Replacing $\mathcal{E}_i$ by $\mathcal{E}_i \otimes f^*(\mathcal{L}_i)$, which replaces $\mathcal{F}_i$ by $\mathcal{F}_i \otimes \mathcal{L}_i$, we may assume that all $\mathcal{L}_{ij}$ are trivial.

Summing up, we placed ourselves into a situation in which there are isomorphisms $t_{ij} : \mathcal{F}_j|_{U_{ij}} \to \mathcal{F}_i|_{U_{ij}}$. Choose such bijections subject to the conditions

$t_{ii} = \mathrm{id}$ and $t_{ij}t_{ji} = \mathrm{id}$. Now consider the 2-cochain $\lambda \in Z^2(\mathfrak{U}, \mathcal{O}_S^\times)$ defined by

$$t_{jk}t_{ij} = \lambda_{ijk}t_{ik}$$

on the triple overlaps $U_{ijk} = U_i \cap U_j \cap U_k$. According to Giraud ([19], Chapter IV, Section 3.5, see also Chapter V, Section 4) this is a 2-cocycle whose cohomology class represents the cohomology class of $\mathcal{A}$.

By our construction, the holomorphic map $U'_{ij} \to U_{ij}$ is biholomorphic whenever $i \neq j$. For such pair of indices, the isomorphisms t_{ij} may be regarded as isomorphisms $t'_{ij} : \mathcal{E}_i|_{U'_{ij}} \to \mathcal{E}_j|_{U'_{ij}}$. In case $i = j$ we define $t'_{ii} = \mathrm{id}$. This yields another 2-cocycle $\lambda' \in Z^2(\mathfrak{U}', \mathcal{O}_{S'}^\times)$ via

$$t'_{jk}t'_{ij} = \lambda'_{ijk}t'_{ik},$$

whose cohomology class represents the cohomology class of $\mathcal{B}$. By construction, the local sections $\lambda_{ijk} \in \Gamma(U_{ijk}, \mathcal{O}_S)$ map to the local section $\lambda'_{ijk} \in \Gamma(U'_{ijk}, \mathcal{O}_{S'})$ under the canonical map. It follows that the Azumaya $\mathcal{O}_{S'}$-algebras $\mathcal{A}'$ and $\mathcal{B}$ have the same class. □

It remains to check the following observation:

Lemma 5.4. *Let $f : S' \to S$ be a proper bimeromorphic map between smooth surfaces. Then the canonical map $H^2(S, \mathcal{O}_S^\times) \to H^2(S', \mathcal{O}_{S'}^\times)$ is bijective.*

Proof. Compare the long exact sequence

$$\mathrm{Pic}(S) \longrightarrow H^2(S, \mathbb{Z}) \longrightarrow H^2(S, \mathcal{O}_S) \longrightarrow H^2(S, \mathcal{O}_S^\times) \longrightarrow H^3(S, \mathbb{Z})$$

with the corresponding sequence for S', and use the vanishing $R^p f_*(\mathcal{O}_{S'}) = 0$ for $p > 0$ and $R^p f_*(\mathbb{Z}) = 0$ for $p \neq 0, 2$. □

6. Elliptic surfaces

In this section we apply Theorem 5.1 to elliptic surfaces. Recall that an *elliptic surface* is a smooth compact surface S, together with a holomorphic map $f : S \to B$ onto a smooth compact curve B so that almost all fibers $S_b = f^{-1}(b)$, $b \in B$ are elliptic curves. Elliptic surfaces might have any Kodaira dimension $\kappa(S) \leq 1$, and any algebraic dimension $a(S) \leq 2$. Recall that the *algebraic dimension* $a(S)$ is the transcendence degree of the field of meromorphic functions on S. Surfaces with $a(S) = 2$ are algebraic, surfaces with $a(S) = 1$ are elliptic, and surfaces with $a(S) = 0$ contain only finitely many curves. An algebraic surface might have several elliptic structures, even infinitely many. In contrast, surfaces with $a(S) \leq 1$ have at most one elliptic structure, if any.

Proposition 6.1. *Let S be an elliptic surface. Then* $\mathrm{Br}(S) = \mathrm{Br}'(S)$.

Proof. Fix an elliptic fibration $f : S \to B$. It is possible to compute the fundamental group $\pi_1(S)$ in terms of this fibration, see [58]. This fundamental group, however, might be rather small, due to the influence of singular fibers. Things simplify much if one throws away the singular fibers:

Choose finitely many points $b_1, \ldots, b_m \in B$ such that all singular fibers occur among the fibers $S_{b_1}, \ldots, S_{b_m}$, and that the complement $V = B - \{b_1, \ldots b_m\}$ is *hyperbolic*. The latter means that its universal covering space is the upper half plane $\tilde{V} = \mathfrak{H}$. Set $U = S - \bigcup_{i=1}^m S_{b_i}$. Then the induced map $U \to V$ is proper and

smooth with elliptic fibers. Let $F \subset U$ be any fiber. The long homotopy sequence reads

$$\pi_2(V) \longrightarrow \pi_1(F) \longrightarrow \pi_1(U) \longrightarrow \pi_1(V) \longrightarrow 0.$$

By construction, the group $\pi_1(V)$ is free and $\pi_2(V) = \pi_2(\tilde{V})$ vanishes. We conclude that $\pi_1(U)$ is an extension of a free group by $\mathbb{Z}^{\oplus 2}$. Arguing as in the proof for Proposition 3.2, we see that $\pi_1(U)$ is good. Moreover, the universal covering $\tilde{U} = \mathfrak{H} \times \mathbb{C}$ is contractible. This means that Proposition 5.1 applies to our situation, and we conclude $\mathrm{Br}(S) = \mathrm{Br}'(S)$. □

Let me close this section with some remarks about the relation of the elliptic surface $S \to B$ to the corresponding jacobian elliptic surface $X \to B$. According to Kodaira [39], any elliptic surface $f : S \to B$ (without exceptional curves and multiple fibers) comes along with two invariants: The *homological invariant* $\mathcal{G} = R^1 f_*(\mathbb{Z})$, and the *functional invariant* $j : B \to \mathbb{P}^1$. The latter attaches to each point with smooth fiber the j-invariant of its fiber. Kodaira showed that the jacobian fibration $X \to B$ of $S \to B$ has the same homological and functional invariant. By construction, the jacobian fibration has a section, so X is an algebraic surface. It is not difficult to see that the topological part of $\mathrm{Br}(S)$ and $\mathrm{Br}(X)$ coincide. In contrast, the analytic part might differ drastically due to jumps in Picard numbers. The situation is simpler for algebraic surfaces S: Here Nori [48] showed that $\rho(S) = \rho(X)$, and hence $\mathrm{Br}(S) \simeq \mathrm{Br}(X)$.

7. Surfaces of class VII

In this section we analyze smooth compact surfaces S of class VII. By definition, this means $b_1(S) = 1$. Such surfaces are not algebraic, not even Kähler. They form the only class of surfaces resisting complete classification. We first observe that the cohomological Brauer group is rather small:

Lemma 7.1. *For surfaces S of class VII we have $H^2(S, \mathcal{O}_S^\times) = H^3(S, \mathbb{Z})$*

Proof. The exponential sequence gives an exact sequence

$$H^2(S, \mathcal{O}_S) \longrightarrow H^2(S, \mathcal{O}_S^\times) \longrightarrow H^3(S, \mathbb{Z}) \longrightarrow H^3(S, \mathcal{O}_S).$$

The term on the right vanishes for dimension reason. and the term on the left also vanishes, as Kodaira showed in [40], Theorem 26. □

Recall that a *global spherical shell* consists of an holomorphic open embedding of some $U = \left\{ z \in \mathbb{C}^2 \mid 1 - \epsilon < |z| < 1 + \epsilon \right\}$ with $0 < \epsilon < 1$ into S so that the complement $S - U$ is connected. A surface admitting a global spherical shell is of class VII. Such surfaces are not very interesting with respect to Grothendieck's question on Brauer groups:

Proposition 7.2. *Let S be a smooth compact surface containing a global spherical shell. Then* $\mathrm{Br}(S) = \mathrm{Br}'(S) = 0$.

Proof. We just saw that the analytic part of $\mathrm{Br}'(S)$ vanishes. Using a Mayer–Vietoris argument, Dloussky showed in [12], Lemma 1.10 that $H_2(S, \mathbb{Z})$ is torsion free. In other words, $H^3(S, \mathbb{Z})$ is torsion free, so the topological part of the Brauer group vanishes as well. □

We now turn to surfaces of class VII with $b_2 = 0$. Such surfaces are indeed classified. If there is a curve $C \subset S$, then Kodaira proved that S is a Hopf surface

[40], Theorem 34. We already settled this case in Corollary 4.3. If there is no curve at all on S, then the classification results of Inoue apply [31]. He showed that the universal covering is $\tilde{S} = \mathfrak{H} \times \mathbb{C}$ and that $\pi_1(S)$ is polycyclic. Actually, Inoue made an additional technical assumption, namely the existence of a twisted vector field. Later, the work of Bogomolov [4], Li, Yau, and Zheng [42], [43], and Teleman [56] revealed that this assumption automatically holds. One refers to surfaces with $b_1 = 1$ and $b_2 = 0$ without curves as *Inoue surfaces*. Summing up, we may apply Theorem 4.1 and deduce:

Proposition 7.3. *Let S be a smooth compact surface of class VII whose minimal model has $b_2 = 0$. Then* $\mathrm{Br}(S) = \mathrm{Br}'(S)$.

Inoue showed that there are precisely three types of Inoue surfaces, which are denoted by S_M, $S^+_{N,p,q,r;t}$, and $S^-_{N,p,q,r}$. Let us take a closer look at the first type S_M. The parameter $M = (m_{ij})$ is a matrix $M \in \mathrm{SL}_3(\mathbb{Z})$ that, viewed as a complex matrix, has one real eigenvalue $\alpha > 1$ and two nonreal eigenvalues $\beta, \bar{\beta} \in \mathbb{C}$. The surfaces $S = S_M$ has universal covering $\mathfrak{H} \times \mathbb{C}$, and the fundamental group $\pi_1(S)$ is a split extension

$$0 \longrightarrow \mathbb{Z}^{\oplus 3} \longrightarrow \pi_1(S) \longrightarrow \mathbb{Z} \longrightarrow 0. \tag{5}$$

The generator $g \in \mathbb{Z}$ of the quotient acts on elements $x \in \mathbb{Z}^{\oplus 3}$ via $gxg^{-1} = Mx$. It requires some notation to describe the action of $\pi_1(S)$ on the universal covering $\tilde{S} = \mathfrak{H} \times \mathbb{C}$. I do not want to write this down here, and refer to [31], Section 2. However, note that the condition on the eigenvalues of M ensures that this action is properly discontinuous.

One easily computes that $H_1(S_M, \mathbb{Z}) = \pi_1^{\mathrm{ab}}(S_M)$ is isomorphic to the abelian group $\mathbb{Z} \oplus \mathrm{coker}(M - \mathrm{id})$, hence $\mathrm{Br}(S_M) = \mathrm{Br}'(S_M) = \mathrm{coker}(M - \mathrm{id})$. To obtain examples of matrices $M \in \mathrm{SL}_3(\mathbb{Z})$ with required properties, just choose a polynomial $p(T) = T^3 + a_2T^2 + a_1T + a_0$ with integral coefficients admitting only one real root α, and $\alpha > 1$. This holds, for example, if $a_0 \ll 0$ with a_1, a_2 fixed. Now

$$M = \begin{pmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{pmatrix}$$

is a matrix with characteristic polynomial $\det(T - M) = p(T)$, hence has the desired properties. The greatest common divisor δ_i of the i-minors of the matrix $M - \mathrm{id}$ are $\delta_1 = \delta_2 = 1$ and $\delta_3 = p(1) = 1 + a_0 + a_1 + a_2$. Hence the invariant factors for the submodule $\mathrm{im}(M - \mathrm{id}) \subset \mathbb{Z}^{\oplus 3}$ are $1, 1, p(1)$. The upshot is that $\mathrm{Br}(S_M)$ is cyclic of order $|1 + a_0 + a_1 + a_2|$, which could be arbitrarily large.

8. Main result and open questions

In this section I gather our results on complex-analytic surfaces. In the following, let me call a compact smooth surface S *hypothetical* if it is of class VII, but its minimal model is neither Hopf, Inoue, nor contains a global spherical shell. According to the *GSS-Conjecture* such surface should not exist.

Theorem 8.1. *Let S be a smooth compact surface. Suppose that S is not hypothetical. Then* $\mathrm{Br}(S) = \mathrm{Br}'(S)$.

Proof. If S is algebraic, this is Grothendieck's result [22], Section 2. For nonalgebraic surfaces we have to go through Kodaira's classification [2], Section IV: If the algebraic dimension is $a(S) = 1$, then S is elliptic and Proposition 6.1 applies. If $a(S) = 0$, then S is either a K3-surface, a 2-dimensional complex torus, or a surface of class VII. Huybrechts and myself settled the case of K3-surfaces in [30], whereas Elencwajg and Narasimhan treated complex tori [16], compare also Corollary 4.2. We treated nonhypothetical class VII surfaces in Section 7. □

I want to finish the paper by stating some open problems:

(1) Suppose S is a minimal surface of class VII with $b_2 > 0$. Is it possible to prove that $H^3(S, \mathbb{Z})$, or equivalently $H_2(S, \mathbb{Z})$ are torsion free, without referring to the GSS-conjecture? This would entail $\mathrm{Br}'(S) = 0$.

(2) Does $\mathrm{Br}(S) = \mathrm{Br}'(S)$ hold true for singular compact surfaces?

(3) Suppose S is a smooth compact surface, and $P \to S$ is a topological principal $\mathrm{PGL}_r(\mathbb{C})$-bundle. Under what conditions does there exist a holomorphic structure on P? There is a lot of work on the corresponding question for principal $\mathrm{GL}_r(\mathbb{C})$-bundles. We refer to Brînzănescu's book [7].

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