GEOMETRY ON TOTALLY SEPARABLY CLOSED SCHEMES

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Abstract. We prove, for quasicompact separated schemes over ground fields, that Čech cohomology coincides with sheaf cohomology with respect to the Nisnevich topology. This is a partial generalization of Artin's result that for noetherian schemes such an equality holds with respect to the étale topology, which holds under the assumption that every finite subset admits an affine open neighborhood (AF-property). Our key result is that on the absolute integral closure of separated algebraic schemes, the intersection of any two irreducible closed subsets remains irreducible. We prove this by establishing general modification and contraction results adapted to inverse limits of schemes. Along the way, we characterize schemes that are acyclic with respect to various Grothendieck topologies, study schemes all local rings of which are strictly henselian, and analyze fiber products of strict localizations.

Contents

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Introduction

An integral scheme X that is normal and whose function field $k(X)$ is algebraically closed is called *totally algebraically closed*, or *absolutely algebraically closed*. These notion were introduced by Enochs [29] and Artin [3] and further studied, for example, in [35], [36] and [27].

Artin used such schemes to prove that Čech cohomology coincides with sheaf cohomology for any abelian sheaf in the étale topos of a noetherian scheme X that satisfies the *AF-property*, that is, every finite subset admits an affine open neighborhood. This is a rather large and very useful class of schemes, which includes all schemes that are quasiprojective of some affine scheme. Note, however, that there are smooth threefolds lacking this property (Hironaka's example, compare [31], Appendix B, Example 3.4.1).

In recent years, absolutely closed domains were studied in connection with tight closure theory. Hochster and Huneke [37] showed that the absolute algebraic closure is a big Cohen–Macaulay module in characteristic $p > 0$. This was further extended by Huneke and Lyubeznik [38]. Schoutens [55] and Aberbach [1] used the absolute closure to characterize regularity for local rings. Huneke [39] also gave a nice survey on absolute algebraic closure.

Other applications include Gabber's rigidity property for abelian torsion sheaves for affine henselian pairs [16], or Rydh's study of descent questions [53], or the proof by Bhatt and de Jong [8] of the Lefschetz Theorem for local Picard groups. Closely related ideas occur in the pro-étale site, introduced by Bhatt and Scholze [7], or the construction of universal coverings for schemes by Vakil and Wickelgren [63]. Absolute integral closure in characteristic $p > 0$ provides examples of flat ring extensions that are not a filtered colimit of finitely presented flat ring extensions, as Bhatt [6] noted. I have used absolute closure to study points in the fppf topos [57].

For aesthetic reasons, and also to stress the relation to strict localization and étale topology, I prefer to work with separable closure instead of algebraic closure: If X_0 is an integral scheme, the *total separable closure* $X = \mathrm{TSC}(X_0)$ is the integral closure of X_0 in some chosen separable closure of the function field. The goal of this paper is to make a systematic study of geometric properties of $X = \mathrm{TSC}(X_0)$, and to apply it to cohomological questions. One of our main results is the following rather counter-intuitive property:

Theorem (compare Thm. 12.1)**.** *Let X_0 be separated and of finite type over a ground field k, and $X = \mathrm{TSC}(X_0)$. Then for every pair of closed irreducible subsets $A, B \subset X$, the intersection $A \cap B$ remains irreducible.*

Note that for algebraic surfaces, this means that for all closed irreducible subsets $A \neq B$ in X, the intersection $A \cap B$ contains at most one point. Actually, I conjecture that our result holds true for arbitrary integral schemes that are quasicompact and separated.

Such geometric properties were crucial for for Artin to establish the equality $\check{H}^p(X_{\mathrm{et}}, F) = H^p(X_{\mathrm{et}}, F)$. He achieved this by assuming the AF-property. This condition, however, appears to be somewhat alien to the problem. Indeed, for locally factorial schemes, the AF-property is equivalent to quasiprojectivity, according to Kleiman's proof of the Chevalley conjecture ([42], Theorem 3 on page 327). Recently, this was extended to arbitrary normal schemes by Benoist [5].

Note, however, that nonnormal schemes may have the AF-property without being quasiprojective, as examples of Horrocks [40] and Ferrand [30] show.

For schemes X lacking the AF-property, Artin's argument break down at two essential steps: First, he uses the AF-property to reduce the analysis of fiber products $\mathrm{Spec}(\mathscr{O}^s_{X,x}) \times_X \mathrm{Spec}(\mathscr{O}^s_{X,y})$ of strict henselizations to the more accessible case of integral affine schemes $X = \mathrm{Spec}(R)$, in fact to the situation $A = \mathbb{Z}[T_1, \ldots, T_n]$, which then leads to the join construction $B = [A^h_p, A^h_q]$ inside some algebraic closure of the field of fractions $\mathrm{Frac}(A)$ (see [3], Theorem 2.2). Second, he needs the affine situation to form meaningful semilocal intersection rings $R = A^h_p \cap A^h_q$, in order to prove that B has separably closed residue field (see [3], Theorem 2.5).

My motivation to study totally separably closed schemes was to bypass the AF-property in Artin's arguments. In some sense, I was able to generalize half of his reasonings to arbitrary schemes X, namely those steps that pertain to henselization $\mathscr{O}^h_{X,x}$ for points $x \in X$ rather then strict localizations $\mathscr{O}^s_{X,a}$ for geometric points $a : \mathrm{Spec}(\Omega) \to X$, the latter having in addition separably closed residue fields.

In terms of Grothendieck topologies, we thus get results on the *Nisnevich topology* rather then the étale topology. This is one of the more recent Grothendieck topologies, which was considered in connection with motivic questions. Recall that the covering families of the Nisnevich topology on (Et/X) are those $(U_\lambda \to U)_\lambda$ so that each $U_\lambda \to U$ is *completely decomposed*, that is, over each point lies at least one point with the same residue field, and that $\bigcup U_\lambda \to U$ is surjective. We refer to Nisnevich [50] and the stacks project [60] for more details. Note that the Nisnevich topology plays an important role of Voevodsky's theory of sheaves with transfer and motivic cohomology, compare [64], Chapter 3 and [46]. Our second main result is:

Theorem (compare Thm. 13.1). *Let X be a quasicompact and separated scheme over a ground field k. Then $\check{H}^p(X_{\mathrm{Nis}}, F) = H^p(X_{\mathrm{Nis}}, F)$ for all abelian Nisnevich sheaves on X.*

It is quite sad that my methods apparently need a ground field, in order to use the geometry of contractions, which lose some of their force over more general ground rings. Again I conjecture that the result holds true for quasicompact and separated schemes, even for the étale topology. Indeed, this paper contains several general *reduction steps*, which reveal that it suffices to prove this conjecture merely for separated integral $\mathbb{Z}$-schemes of finite type. It seems likely that it suffices that the diagonal $\Delta : X \to X \times X$ is affine, rather than closed.

Along the way, it is crucial to characterize schemes that are acyclic with respect to various Grothendieck topologies. With analogous results for the Zariski and the étale topology, we have:

Theorem (compare Thm. 4.2). *Let X be a quasicompact scheme. Then the following are equivalent:*

(i) $H^p(X_{\mathrm{Nis}}, F) = 0$ *for every abelian Nisnevich sheaf F and every $p \geq 1$.*
(ii) *Every completely decomposed étale surjection $U \to X$ admits a section.*
(iii) *The scheme X is affine, and each connected component is local henselian.*
(iv) *The scheme X is affine, each irreducible component is local henselian, and the space* $\mathrm{Max}(X)$ *is at most zero-dimensional.*

The paper is organized as follows: In Section 1, we review some properties of schemes that are stable by integral surjections. These will be important for many reduction steps that follow. Section 2 contains a discussion of schemes all of whose local rings are strictly local. Such schemes X have the crucial property that any integral morphism $f : Y \to X$ with Y irreducible must be injective. In Section 3 we discuss the total separable closure $X = \mathrm{TSC}(X_0)$ of an integral scheme X_0, which are the most important examples of schemes that are everywhere strictly local.

Section 4 contains our characterizations of acyclic schemes with respect to three Grothendieck topologies, namely Zariski, Nisnevich and étale. In Section 5, we introduce technical conditions, namely the weak/strong Cartan–Artin properties, which roughly speaking means that the fiber products of strict localizations are acyclic with respect to the Nisnevich/étale topology. Note that such fiber products are almost always non-noetherian. Section 6 and 7 contain reduction arguments, which basically show that it suffices to check the Cartan–Artin properties on total separable closures $X = \mathrm{TSC}(X_0)$. On the latter, it translates into a simple, but rather counter-intuitive geometric condition on the intersection of irreducible closed subsets.

Section 8 reveals in the special case of algebraic surfaces this geometric condition indeed holds. Here one uses that one understands very well which integral curves on a normal surface are contractible to a point. The next three sections prepare the ground to generalize this to higher dimensions: In Section 9, we collect some facts on noetherian schemes concerning quasiprojectivity and connectedness of divisors. The latter is an application of Grothendieck's Connectedness Theorem. In Section 10 we show how to make a closed subset contractible on some modification. In Section 11, we introduce the technical notion of cyclic systems, which is better suited to understand contractions in inverse limits like $X = \mathrm{TSC}(X_0)$. Having this, Section 12 contains the Theorem that on total separable closures there are no cyclic systems, in particular the intersection of irreducible closed subsets remains irreducible. The application to Nisnevich cohomology appears in Section 13. There are also three appendices, discussing Lazard's observation on connected components of schemes, H. Cartan's argument on equality of Čech and sheaf cohomology, and some results on inductive dimension in general topology used in this paper.

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1. Integral surjections

Throughout the paper, integral surjections and inverse limits play an important role. We start by reexamining these concepts.

Let X_λ, $\lambda \in L$ be a filtered inverse system of schemes with affine transition morphisms $X_\mu \to X_\lambda$, $\lambda \leq \mu$. Then the corresponding inverse limit exists as a scheme. For its construction, one may tacitly assume that there is a smallest index $\lambda = 0$, and regards the $X_\lambda = \mathrm{Spec}(\mathscr{A}_\lambda)$ as relatively affine schemes over X_0. Then

$$X = \varprojlim(X_\lambda) = \mathrm{Spec}(\mathscr{A}), \quad \mathscr{A} = \varinjlim \mathscr{A}_\lambda.$$

Moreover, the underlying topological space of X is the inverse limit of the underlying topological spaces for X_λ, by [22], Proposition 8.2.9. In turn, for each point $x = (x_\lambda) \in X$ the canonical map $\varinjlim(\mathscr{O}_{X_\lambda, x_\lambda}) \to \mathscr{O}_{X,x}$ is bijective. Consequently, the residue field $\kappa(x)$ is the union of the $\kappa(x_\lambda)$, viewed as subfields.

Recall that a homomorphism $R \to A$ of rings is *integral* if each element in A is the root of a monic polynomial with coefficients in R. A morphism of schemes $f : X \to Y$ is called integral if there is a affine open covering $Y = \bigcup V_i$ so that $U_i = f^{-1}(V_i)$ is affine and $A_i = \Gamma(V_i, \mathscr{O}_X)$ is integral as algebra over $R_i = \Gamma(V_i, \mathscr{O}_Y)$. Note that the underlying topological space of the fibers $f^{-1}(y)$ are profinite.

Of particular interest are those $f : X \to Y$ that are integral and surjective. The following locution will be useful throughout: Let $\mathcal{P}$ be a class of schemes. We say that $\mathcal{P}$ is *stable under images of integral surjections* if for each integral surjection $f : X \to Y$ where the domain X belongs to $\mathcal{P}$, the range Y belongs to $\mathcal{P}$ as well. For example:

Theorem 1.1. *The class of affine schemes is stable under images of integral surjections.*

In this generality, the result is due to Rydh [54], who established it even for algebraic spaces. It generalizes Chevalley's Theorem ([20], Theorem 6.7.1), where Y is a noetherian scheme and f is finite surjective. See also [12], Corollary A.2, for the case that Y is an arbitrary scheme and f is finite surjective.

A scheme Y is called *local* if it is quasicompact and contains precisely one closed point. Equivalently Y is the spectrum of a local ring. We have the following permanence property:

Proposition 1.2. *The class of local schemes is stable under images of integral surjections*

Proof. Let $f : X \to Y$ be an integral surjection, with X local. We have to show that Y is local. According to Theorem 1.1, the scheme Y is affine. It follows that Y contains at least one closed point. Let $y, y' \in Y$ be two closed points. Since f is surjective, there are closed points $x, x' \in X$ mapping to y, y', respectively. Since X is local, we have $x = x'$, whence $y = y'$. $\square$

A scheme Y is called *local henselian* if it is local, and for every finite morphism $g : Y' \to Y$, the domain Y' is a sum of local schemes. Of course, there are only finitely many such summands, because $g^{-1}(b)$, where $b \in Y$ is the closed point, contains only finitely many points and g is a closed map.

Proposition 1.3. *The class of of local henselian scheme is stable under images of integral surjections.*

Proof. Let $f : X \to Y$ be an integral surjection, with X local henselian. By Proposition 1.2, the scheme Y is local. Now let $Y' \to Y$ be a finite morphism, and consider the induced finite morphism $X' = Y' \times_Y X \to X$. Then $X' = X'_1 \amalg \ldots \amalg X'_n$ for some local schemes X'_i, $1 \leq i \leq n$. Their images $Y'_i \subset Y'$ are closed, because f and the induced morphism $X' \to Y'$ are integral. Moreover, these images form a closed covering of Y, since f and the induced morphism $X' \to Y'$ is surjective. Regarding the Y_i as reduced schemes, the canonical morphism $X'_i \to Y'_i$ is integral and surjective. Using Proposition 1.2 again, we conclude that the Y'_i are local. It follows that there are at most n closed points on Y'. Denote them by $b_1, \ldots, b_m$, for some $m \leq n$. For each closed point b_j, let $C_j \subset Y'$ be the union of those Y'_i that contain b_j. Then the C_j are closed connected subsets, pairwise disjoint and finite in number. We conclude that the C_j are the connected components of Y'. By construction, each C_j is quasicompact and contains only one closed point, namely

b_j, such that C_j is local. It follows that Y' is a sum of local schemes, thus Y is local henselian. □

A scheme Y is called *strictly local*, if it is local henselian, and the residue field of the closed point is separably closed. The class of such schemes is not stable under images of surjective integral morphisms, for example $\operatorname{Spec}(\mathbb{C}) \to \operatorname{Spec}(\mathbb{R})$.

We say that a class $\mathcal{P}$ of schemes is stable under *images of integral surjections with radical residue field extensions* if for every morphism $f : X \to Y$ that is integral, surjective and whose field extensions $\kappa(y) \subset \kappa(x)$, $y = f(x)$, $x \in X$ are radical (that is, algebraic and purely inseparable), and with domain X belonging to $\mathcal{P}$, the domain Y also belongs to $\mathcal{P}$.

Proposition 1.4. *The class of strictly local schemes is stable under images of integral surjections with radical residue field extensions.*

Proof. This follows from Proposition 1.3, together with the fact that a field K is separably closed if it admits a radical extension $K \subset E$ that is separably closed. □

Recall that a ring R is called a *pm ring*, if every prime ideal $\mathfrak{p} \subset R$ is contained in exactly one maximal ideal $\mathfrak{m} \subset R$, compare [28]. Clearly, it suffices to check this for minimal prime ideals, which correspond to the irreducible components of $\operatorname{Spec}(R)$. Therefore, we call a scheme X a *pm scheme* if it is quasicompact, and each irreducible component is local. These notions will show up in Section 4. For later use, we observe the following fact:

Proposition 1.5. *The class of pm schemes is stable under images of integral surjections.*

Proof. Let $f : X \to Y$ be integral surjective, with X pm. Clearly, Y is quasicompact. Let $Y' \subset Y$ be an irreducible component. We have to check that Y' is local. Since f is surjective and closed, there is an irreducible closed subset $X' \subset Y'$ with $f(X') = Y'$. Replacing X, Y be these subschemes, we may assume that X, Y are irreducible. Let $y, y' \in Y$ be two closed points. Choose closed points $x, x' \in X$ mapping to them. Then $x = x'$, because X is pm, whence $y = y'$. □

2. Schemes that are everywhere strictly local

We now introduce a class of schemes that, in my opinion, seems rather natural with respect to the étale topology. Recall that a *strictly local* ring R is a local ring that is henselian and has separably closed residue field $\kappa = R/\mathfrak{m}$.

Definition 2.1. A scheme X is called *everywhere strictly local* if the local rings $\mathscr{O}_{X,x}$ are strictly local, for all $x \in X$.

Similarly, we call a ring R everywhere strictly local if the $R_\mathfrak{p}$ are strictly local for all prime ideals $\mathfrak{p} \subset R$. Note that strictly local rings are usually not everywhere strictly local. The relation between these classes of rings appears to be similar in nature to the relation between valuation rings and Prüfer rings (compare for example [14], §11). We have the following permanence property:

Proposition 2.2. *If X is everywhere strictly local, the same holds for each quasifinite X-scheme.*

Proof. Let $U \to X$ be quasifinite, and $u \in U$, with image point $x \in X$. Obviously, the residue field $\kappa(u)$ is separably closed. To check that $\mathscr{O}_{U,u}$ is henselian, we may replace X by the spectrum of $\mathscr{O}_{X,x}$. According to [23], Theorem 18.12.1 there is an étale morphism $X' \to X$, a point $u' \in U' = U \times_X X'$ lying over $u \in U$, and an open neighborhood $u' \in V' \subset U'$ so that the projection $V' \to X'$ is finite. Let $x' \in X'$ be the image of u'. After replacing X' by some affine open neighborhood of x', we may assume that $X' \to X$ is separated. There is a section $s : X \to X'$ with $s(x) = x'$ for the projection $X' \to X$, because $\mathscr{O}_{X,x}$ is strictly local, by [23], Proposition 18.8.1. Its image is an open connected component, according to [23], Corollary 17.9.4, so shrinking further we may assume that $X' = X$. By [23], Proposition 18.6.8, the local ring $\mathscr{O}_{U,u} = \mathscr{O}_{V',u'}$ is henselian. □

Recall that a morphism $f : Y \to X$ is referred to as *radical* if it is universally injective. Equivalently, the map is injective, and the induced residue field extension $\kappa(x) \subset \kappa(y)$, are purely inseparable ([17], Section 3.7), for all $y \in Y$, $x = f(y)$. The following geometric property will play a crucial role later:

Lemma 2.3. *Let X be an everywhere strictly local scheme and Y be an irreducible scheme. Then every integral morphism $f : Y \to X$ is radical, in particular an injective map.*

Proof. Since the residue fields of X are separably closed and f is integral, it suffices to check that f is injective. Let $y, y' \in Y$ with same image $x = f(y) = f(y')$. Our task is to show $y = y'$. Replace X by the spectrum of the local ring $R = \mathscr{O}_{X,x}$ and Y by the corresponding fiber product. Then $Y = \operatorname{Spec}(A)$ becomes affine, and the morphism of schemes $f : Y \to X$ corresponds to a homomorphism of rings $R \to A$. Write $A = \bigcup A_\iota$ as the filtered union of finite R-subalgebras. Since R is henselian and A_ι are integral domains, the A_ι are local. Whence A is local, too. It follows that the closed points $y, y' \in \operatorname{Spec}(A)$ coincide □

Proposition 2.4. *Let X be everywhere strictly local. Then every étale morphism $f : U \to X$ is a local isomorphism with respect to the Zariski topology.*

Proof. Fix a point $u \in U$. We must find an open neighborhood on which f is an open embedding. Set $x = f(u)$. Consider first the special case that f admits a section $s : X \to U$ through u. Since $U \to X$ is unramified, such a section must be an open embedding by [23], Corollary 17.4.2. Replacing U by the image of this section, we reduce to the situation that f admits a right inverse s that is an isomorphism. Multiplying $f \circ s = \mathrm{id}_X$ with s^{-1} from the right yields $f = s^{-1}$, which is an isomorphism.

We now come to the general case. Since $\mathscr{O}_{X,x}$ is strictly local, the morphism $U \otimes_X \operatorname{Spec}(\mathscr{O}_{X,x}) \to \operatorname{Spec}(\mathscr{O}_{X,x})$ admits a section through $u \in U$, see [23], Proposition 18.5.11. According to [22], Theorem 8.8.2, such a section comes from a section defined on some open neighborhood of $x \in X$, and the assertion follows. □

Given a scheme X, we denote by (Et/X) the site whose objects are the étale morphisms $U \to X$, and (Zar/X) the site whose objects are the open subschemes $U \subset X$. In both cases, the covering families $(U_\alpha \to X)_\alpha$ are those where the map $\amalg U_\alpha \to X$ are surjective. In turn, we denote by X_{et} and X_{Zar} the corresponding topoi of sheaves within a fixed universe. The inclusion functor $i : (\mathrm{Zar}/X) \to (\mathrm{Et}/X)$ is cocontinuous, which means that the adjoint i^* on presheaves of the restriction functor i_* preserves the sheaf property. We thus obtain a morphism of topoi $i : X_{\text{et}} \to X_{\text{Zar}}$. The following is a direct consequence of the Comparison Lemma ([4], Exposé III, Theorem 4.1):

Corollary 2.5. *Let X be everywhere strictly local. Then the canonical morphism $i : X_{\text{et}} \to X_{\text{Zar}}$ of topoi is an equivalence. In particular, sheaves and cohomology groups for the étale site of X is essentially the same as for the Zariski site.*

We are mainly interested in irreducible schemes. Recall that a scheme X is called *unibranch* if it is irreducible, and the normalization map $X'_{\text{red}} \to X_{\text{red}}$ is bijective (compare [21], Section 23.2.1). Equivalently, the henselian local schemes $\operatorname{Spec}(\mathscr{O}^h_{X,x})$ are irreducible, for all $x \in X$, by [23], Corollary 18.8.16. We have the following characterization, which is close to Artin's original definition [3].

Proposition 2.6. *Let X be irreducible. Then X is everywhere strictly local if and only if it is unibranch and its function field $k(X) = \kappa(\eta)$ is separably closed.*

Proof. In light of [23] Corollary 18.6.13, the condition is necessary. To see that it is sufficient, we may assume that $X = \operatorname{Spec}(R)$ is a local integral scheme and have to check that R is henselian with separably closed residue field $k = R/\mathfrak{m}_R$.

Let us start with the latter. Seeking a contradiction, we assume that k is not separably closed. Then there is a finite separable field extension $k \subset L$ of degree $d \geq 2$. By the Primitive Element Theorem, we have $L = k[T]/(f)$ for some irreducible separable monic polynomial $f \in k[T]$. Choose a monic polynomial $F(T)$ with coefficients in R reducing to $f(T)$. Then $\partial F/\partial T \in R$ is a unit, and it follows from [47], Chapter I, Corollary 3.6 that the finite R-algebra $A = R[T]/(F)$ is étale. Since the field of fractions $\Omega = \operatorname{Frac}(R)$ is separably closed, we have a decomposition $A \otimes_R \Omega = \prod_{i=1}^d \Omega$. Set $S = \operatorname{Spec}(A)$, and fix one generic point $\eta_0 \in S$. Note that its residue field must be Ω. Its closure $S_0 \subset S$ is a local scheme, finite over R, and thus with residue field L.

Now choose a separable closure $k \subset k'$, consider the resulting strict henselization $R' = R^s$, and write $S'_0 = S_0 \otimes_R R'$ and $S' = S \otimes_R R'$ for the ensuing faithfully flat base-change. According to [18], Exposé VIII, Theorem 4.1, the preimage $S'_0 \subset S'$ is the closure of the point $\eta'_0 \in S'$ lying over $\eta_0 \in S$, and thus S'_0 is irreducible, in particular connected. The closed fiber $S'_0 \otimes_{R'} k' = \operatorname{Spec}(L) \otimes_k k'$ is a disjoint union of $d \geq 2$ closed points, which lie in the same connected component inside S'. But since R' is henselian, [23], Theorem 18.5.11 (a) ensures that no two points in the closed fiber $S' \otimes_{R'} k'$ lie in the same connected component of S', contradiction. Summing up, the residue field $k = R/\mathfrak{m}_R$ is separably closed.

It remains to see that R is henselian. Let $F \in R[T]$ be a monic polynomial. In light of [23], Theorem 18.5.11 (a') it suffices to see that the finite flat R-algebra $A = R[T]/(F)$ is a product of local rings. Let $d = \deg(F) = \deg(A)$ be its degree, and consider the R-scheme $Y = \operatorname{Spec}(A)$. Let $Y_1, \ldots, Y_r \subset Y$ be the closures of the connected components of the generic fiber $Y \otimes \Omega$. Using that R is unibranch, we infer that the closed fibers $Y_i \otimes k$ are local. By the Going-Down-Theorem for the integral ring extension $R \subset A$, every point $y \in Y \otimes k$ lies in some Y_i. It follows that $Y = \bigcup Y_i$. For each $y \in Y_k$, let $C_y \subset Y$ by the union of all those Y_i containing y. Clearly, the C_y are connected and local. Since the closed fibers of Y_i are local, the C_y, $y \in Y_k$ are pairwise disjoint. In turn, the C_y are the connected components of $Y = \operatorname{Spec}(A)$. It follows that A is a product of local rings. $\square$

3. Total separable closure

Let X be an integral scheme. We say that X is *totally separably closed* if it it is normal, and the function field $k(X)$ is separably closed. According to Proposition 2.6, these are precisely the integral normal schemes that are everywhere strictly local. From this we deduce:

Proposition 3.1. *If X is totally separably closed, so is every normal closed subscheme $X' \subset X$.*

Now let X_0 be an integral scheme, and choose a separable closure F^s of the function field $F = k(X_0)$. We define the *total separable closure*

$$X = \mathrm{TSC}(X_0)$$

to be the integral closure of X_0 inside F^s. We may regard it as filtered inverse limit: Let $F \subset F_\lambda \subset F^{\mathrm{alg}}$, $\lambda \in L$ be the set of intermediate fields that are finite over $F = k(X_0)$, ordered by the inclusion relation, and let $X_\lambda \to X_0$ be the corresponding integral closures. These form a filtered inverse system of schemes with finite surjective transition maps, and we get a canonical identification

$$X \longrightarrow \varprojlim X_\lambda,$$

We tacitly assume that the smallest element in the index set L is denoted by $\lambda = 0$. Note that the fibers of the map $X \to X_0$, viewed as a topological space, are *profinite*. Recall that such spaces are precisely totally disconnected compacta, which are also called *Stone spaces*.

Let me introduce the following notation as a general convention for this paper: Suppose that $C \subset X$ is a closed subscheme. Then the schematic images $C_\lambda \subset X_\lambda$ are closed, and the underlying set is just the image set. These form an filtered inverse system of schemes, again with affine transition maps by [20], Proposition 1.6.2. According to [9], Chapter I, §4, No. 4, Corollary to Prop. 9, we get a canonical identification $C = \varprojlim C_\lambda$ as sets, and it is easy to see that this is an equality of schemes. The fiber products $X \times_{X_\lambda} C_\lambda \subset X$, $\lambda \in L$ are closed subschemes and each one contains C as a closed subscheme, but is usually much larger. In fact, one has $C = \bigcap_{\lambda \in L}(X \times_{X_\lambda} C_\lambda)$ as closed subschemes inside X. Now if C is integral, then its function field $k(C)$ is separably closed. This yields:

Proposition 3.2. *If the closed subscheme $C \subset X$ is normal, then $C = \mathrm{TSC}(C_\lambda)$ for each index $\lambda \in L$.*

Now suppose that we have a ground field k. Let X_0 is an integral k-scheme and $X = \mathrm{TSC}(X_0)$ its total separable closure, as above. The following observation reduces the situation to the case that the ground field is separably closed: Let k' be the relative algebraic closure of k inside F^s, where $F = k(X_0)$. The scheme $X \otimes_k k'$ is not necessarily integral, but the schematic image $X' \subset X \otimes_k k'$ of the canonical morphism $X \to X \otimes_k k'$ is. Note that if the k-scheme X_0 is algebraic, quasiprojective, or proper, the respective properties hold for the k'-scheme X'. Clearly, the morphism $X \to X'$ is integral and dominant. Regarding $k(X)$ as an separable closure of $k(X')$, we get a canonical morphism $X \to \mathrm{TSC}(X')$.

Proposition 3.3. *The canonical morphism $X \to \mathrm{TSC}(X')$ is an isomorphism.*

Proof. The scheme X is integral, the morphism in question is integral and birational, and the scheme $\mathrm{TSC}(X')$ is normal, and the result follows. □

Finally, suppose that X_0 is an integral *algebraic space* rather than a scheme. Then one may define its total separable closure $X = \mathrm{TSC}(X_0) = \varprojlim X_\lambda$ in the analogous way. But here nothing interesting happens: Indeed, then some X_λ is a scheme ([42], Corollary 16.6.2 when X_0 is noetherian, and [53], Theorem B for X_0 quasicompact and quasiseparated), such that $\mathrm{TSC}(X_0)$ is a scheme.

4. Acyclic schemes

In this section we study acyclicity for quasicompact schemes with respect to the Zariski topology, the Nisnevich topology, and the étale topology. We thus take up the question in [25], Expose V, Problem 4.14, to study topoi for which every abelian sheaf has trivial higher cohomology.

First recall that a topological space is called *at most zero-dimensional* if its topology admits a basis consisting of subsets that are open-and-closed. Note that this is in the sense of dimension theory in general topology (confer, for example, [51]), rather than dimension theory in commutative algebra and algebraic geometry.

Given a ring R, we write $\mathrm{Max}(R) \subset \mathrm{Spec}(R)$ for the subspace of points corresponding to maximal ideals. Similarly, we write $\mathrm{Max}(X) \subset X$ for the set of closed points of a scheme X, endowed with the subspace topology.

Theorem 4.1. *Let X be a quasicompact scheme. Then the following are equivalent:*

(i) *We have $H^p(X, F) = 0$ for every abelian sheaf F and every $p \geq 1$, where cohomology is taken with respect to the Zariski topology.*
(ii) *Every surjective local isomorphism $U \to X$ admits a section.*
(iii) *The scheme X is affine, and each connected component is local.*
(iv) *The scheme X is affine, each irreducible component is local, and the space of closed points $\mathrm{Max}(X)$ is at most zero-dimensional.*
(v) *The scheme X is affine, and every element in $R = \Gamma(X, \mathscr{O}_X)$ is the sum of an idempotent and a unit.*

Proof. (i)⇒(ii) Seeking a contradiction, suppose that some surjective local isomorphism $U \to X$ does not admit a section. Since X is quasicompact, there are finitely many affine open subsets $U_1, \ldots, U_n \subset U$ so that each $U_i \to X$ is an open embedding, and $\amalg U_i \to X$ is surjective. Replace U by the direct sum $\amalg U_i$. Let F be the product of the extension-by-zero sheaves $(f_i)_!(\mathbb{Z}_{U_i})$, where $f_i : U_i \to X$ are the inclusion maps. Then $\Gamma(X, F) = 0$ for all $1 \leq i \leq n$. Consider the Čech complex

$$\Gamma(X, F) \longrightarrow \prod_{i=1}^n \Gamma(U_i, F) \longrightarrow \prod_{i,j=1}^n \Gamma(U_i \cap U_j, F)$$

The constant section (1_{U_i}) in the middle is a cocycle, but not a coboundary, and this holds true for all refinements of the open covering $X = \bigcup U_i$. In turn, we have $\check{H}^1(X, F) \neq 0$. On the other hand, the canonical map $\check{H}^p(X, F) \to H^p(X, F)$ from Čech cohomology to sheaf cohomology is injective and actually bijective for $p = 1$, whence $H^1(X, F) \neq 0$, contradiction.

(ii)⇒(i) Let F be an abelian sheaf. Every F-torsor becomes trivial on some $U \to X$ as above. Since the latter has a section, the torsor is already trivial on X. It follows $H^1(X, F) = 0$. In turn, the global section functor $F \mapsto \Gamma(X, F)$ is exact, hence $H^p(X, F) = 0$ for all F and $p \geq 1$.

(ii)⇒(iii) Choose an affine open covering $X = U_1 \cup \ldots \cup U_n$. Using that $\amalg U_i \to X$ has a section, we infer that X is quasiaffine, in particular separated. By the previous implication, $H^1(X, \mathscr{F}) = 0$ for each quasicoherent sheaf. According to Serre's Criterion [20], Theorem 5.2.1, the scheme X is affine. Next, let $C \subset X$ be a connected component, and suppose C is not local. Choose two different closed points $a_1 \neq a_2$ in C, and let U be the sum of $U_1 = X \smallsetminus \{a_2\}$ and $U_2 = X \smallsetminus \{a_1\}$. The ensuing local isomorphism $U \to X$ allows a section $s : X \to U$. Then $s(C) \subset U$ is connected, and intersects both U_1 and U_2. In turn, $s(C) \subset U_i$ for $i = 1, 2$. But U_1 and U_2 have empty intersection when regarded as subsets of U, contradiction.

(iii)⇒(iv) Let $A \subset X$ be an irreducible component, and $C \subset X$ be its connected component. Then A is local, because the subset $A \subset C$ is closed and nonempty. To see that $\operatorname{Max}(X)$ is at most zero-dimensional, write $X = \operatorname{Spec}(R)$. Let $x \in X$ be a closed point, $x \in U \subset X$ an open neighborhood of the form $U = \operatorname{Spec}(R_f)$, and $C \subset X$ be the connected component of x. Let $S \subset R$ be the multiplicative system of all idempotents $e \in R$ that are units on C. Then $C = \operatorname{Spec}(S^{-1}R)$, and the localization map $R \to S^{-1}R$ factors over R_f. In turn, there is some $e \in S$ so that the localization map $R \to R_e$ factors over R_f. In other words, there is an open-and-closed neighborhood $x \in V$ contained in U. It follows that the subspace $\operatorname{Max}(X) \subset X$ is at most zero-dimensional.

(iv)⇔(v) This equivalence is due to Johnstone [41], Chapter V, Proposition in 3.9.

(v)⇒(iii) Write $X = \operatorname{Spec}(R)$, and let $C = \operatorname{Spec}(A)$ be a connected component, $A = R/\mathfrak{a}$. Then every nonunit $f \in A$ is of the form $f = 1 + u$ for some unit $u \in A^\times$. In turn, the subset $A \smallsetminus A^\times \subset A$ comprises an ideal, because it coincides with the Jacobson radical $J = \{f \mid fg - 1 \in A^\times \text{ for all } g \in A\}$. Thus A is local.

(iii)⇒(ii) Let $f : U \to X$ a surjective local isomorphism. To produce a section, we may assume that U is a finite sum of affine open subschemes of X, in particular of finite presentation over X. Let $x \in X$ be a closed point. Obviously, there is a section after base-changing to $C = \operatorname{Spec}(\mathscr{O}_{X,x})$. Since this is a connected component, we may write $C = \bigcap V_\lambda$, where the V_λ are the open-and-closed neighborhoods of x, compare Appendix A. According to [22], Theorem 8.8.2, a section already exists over some V_λ. Using quasicompactness of X, we infer that there is an open-and-closed covering $X = X_1 \cup \ldots \cup X_n$ so that sections exists over each X_i. By induction on $n \geq 1$, one easily infers that a section exists over X. □

Let us call a scheme X *acyclic with respect to the Zariski topology* if it is quasicompact and satisfies the equivalent conditions of the Theorem. Note that some conditions in Theorem 4.1 already occurred in various other contexts:

Rings satisfying condition (v), that is, every element is a sum of an idempotent and a unit are also known as *clean rings*. Such rings have been extensively studied in the realm of commutative algebra (we refer to [49] for an overview).

Rings for which every prime ideal is contained in only one maximal ideal, one of the conditions occurring in (iv), are called *pm rings* or *Gelfand rings* (they were introduced in [28]).

Also note that in condition (iv) one cannot remove the assumption that $\operatorname{Max}(X)$ is at most zero-dimensional. In fact, if K is an arbitrary compact topological space, then the ring $R = \mathscr{C}(K)$ is pm, such that its spectrum has local irreducible components; on the other hand, $\operatorname{Max}(X) = K$ (compare [19], Section 4 for the latter, and Theorem 2.11 on p. 29 for the former).

We now turn to the *Nisnevich topology* on the category (Et/X). Recall that a morphism $U \to V$ of schemes is called *completely decomposed* if for each $v \in V$, there is a point $u \in U$ with $f(u) = v$ and $\kappa(v) = \kappa(u)$. The covering families $(U_\alpha \to U)$ for the Nisnevich topology are those families for which each $U_\alpha \to U$ is completely decomposed, and $\bigcup U_\alpha \to U$ is surjective. Sheaves on the ensuing site are referred to as *Nisnevich sheaves*, and we write $H^p(X_{\mathrm{Nis}}, F)$ for their cohomology groups, compare [50]. Note that each point $x \in X$ yields a point in the sense of topos-theory, and that the corresponding local ring is the henselization $\mathscr{O}_{X,x}^h$.

Theorem 4.2. *Let X be a quasicompact scheme. Then the following are equivalent:*

(i) $H^p(X_{\mathrm{Nis}}, F) = 0$ *for every abelian Nisnevich sheaf F and every $p \geq 1$.*
(ii) *Every completely decomposed étale surjection $U \to X$ admits a section.*
(iii) *The scheme X is affine, and each connected component is local henselian.*
(iv) *The scheme X is affine, each irreducible component is local henselian, and the space* $\mathrm{Max}(X)$ *is at most zero-dimensional.*

The arguments are parallel to the ones for Theorem 4.1, and left to the reader. For the implication (iv)⇒(iii), one needs the following:

Lemma 4.3. *Let Y be a local scheme. Then Y is henselian if and only if each irreducible component $Y_i \subset Y$, $i \in I$ is henselian.*

Proof. The condition is necessary, because $X_i \subset X$ are closed subsets. Conversely, suppose that each X_i is henselian. Let $f : X \to Y$ be a finite morphism, and let $a_1, \ldots, a_n \in X$ be the closed points. Each of them maps to the closed point $b \in Y$. The closed subsets $X_i = f^{-1}(Y_i)$ contain $a_1, \ldots, a_n$ and are finite over X_i, whence the canonical map $\amalg_{j=1}^n \mathrm{Spec}(\mathscr{O}_{X_i, a_j}) \to X_i$ is an isomorphism. For each $1 \leq j \leq n$, consider the subset

$$C_j = \bigcup_{i \in I} \mathrm{Spec}(\mathscr{O}_{X_i, a_j}) \subset X.$$

Then the $C_j \subset X$ are stable under generization and contain a single closed point a_j, thus $C_j = \mathrm{Spec}(\mathscr{O}_{X, a_j})$. Furthermore, the C_j form a a partition of X. It is not a priori clear that $C_j \subset X$ is closed if the index set I is infinite. But this nevertheless holds: Write $X = \mathrm{Spec}(A)$, where A is a semilocal ring. Let $\mathfrak{m}_j \subset A$ be the maximal ideal corresponding to $a_j \in X$. Since the C_j are pairwise disjoint, the ideals $\mathfrak{m}_j$ are pairwise coprime. By the Chinese Reminder Theorem, the canonical map $A \to \prod_{j=1}^n A_{\mathfrak{m}_j}$ is bijective. Whence $C_j \subset X$ are closed, and X is the sum of local schemes. □

We finally come to the étale topology. This is the Grothendieck topology on the category (Et/X) whose covering families $(U_\alpha \to U)$ are those families with $\bigcup U_\alpha \to U$ surjective. Sheaves on this site are called *étale sheaves*, and we write $H^p(X_{\mathrm{et}}, F)$ for their cohomology groups. In the following, the equivalence (ii)⇔(iii) is due to Artin [3]:

Theorem 4.4. *Let X be a quasicompact scheme. Then the following are equivalent:*

(i) *We have* $H^p(X_{\mathrm{et}}, F) = 0$ *for every abelian étale sheaf F and every $p \geq 1$.*
(ii) *Every étale surjective $U \to X$ admits a section.*
(iii) *The scheme X is affine, and each connected component is strictly local.*
(iv) *The scheme X is acyclic, each irreducible component is strictly local, and the space* $\mathrm{Max}(X)$ *is at most zero-dimensional.*

The remaining arguments are parallel to the ones for Theorem 4.1, and left to the reader.

I conjecture that the class of schemes that are acyclic with respect to either the Zariski or the Nisnevich topology are stable under images of integral surjections. Indeed, if $f : X \to Y$ is integral and surjective, with X acyclic, then Y is affine by Theorem 1.1, and it follows from Proposition 1.2 and 1.3 that each irreducible component is local or local henselian, respectively. It only remains to check that $\mathrm{Max}(Y)$ is at most zero-dimensional, and here lies the problem: We have a closed continuous surjection $\mathrm{Max}(X) \to \mathrm{Max}(Y)$, but cannot conclude from this that the image is at most zero-dimensional, in light of the existence of dimension-raising maps (confer [51], Chapter 7, Theorem 1.8). The following weaker statement will suffice for our applications:

Proposition 4.5. *Let Y be a scheme, and $Y = Y_1 \cup \ldots \cup Y_n$ be a closed covering. If each Y_i is acyclic with respect to the Zariski topology, so is Y.*

Proof. As discussed above, it remains to show that the space of closed points $\mathrm{Max}(Y)$ is at most zero-dimensional. The canonical morphism $Y_1 \amalg \ldots \amalg Y_n \to Y$ is surjective and integral, whence Y is a pm scheme by Proposition 1.5. Note that this ensures that the topological space Y is *normal*, that is, disjoint closed subsets can be separated by disjoint open neighborhoods, according to [11], Proposition 2. Furthermore, the subspace $\mathrm{Max}(Y)$ is *compact*, which means quasicompact and Hausdorff ([58], Corollary 4.7 together with Proposition 4.1). Since our covering is closed, we have $\mathrm{Max}(Y) = \mathrm{Max}(Y_1) \cup \ldots \cup \mathrm{Max}(Y_n)$, and this is again a closed covering.

To proceed, we now use the notion of small and large inductive dimension from general topology. Since this material is perhaps not so well-known outside general topology, I have included a brief discussion in Appendix C. The relevant results are tricky, but rely only on basic notations of topology. Proposition 16.3, which is a consequence of a general sum theorem on the large inductive dimension, ensures that the compact space $\mathrm{Max}(Y) = \mathrm{Max}(Y_1) \cup \ldots \cup \mathrm{Max}(Y_n)$ is at most zero-dimensional. □

5. The Cartan–Artin properties

Let X be a topological space and F be an abelian sheaf. Then there is a spectral sequence

$$E_2^{pq} = \check{H}^p(X, \underline{H}^q(F)) \Longrightarrow H^{p+q}(X, F)$$

computing sheaf cohomology in terms of Čech cohomology, where $\underline{H}^q(F)$ denotes the presheaf $U \mapsto H^p(U, F)$. A result of H. Cartan using this spectral sequence tells us that Čech and sheaf cohomology coincide on spaces admitting a basis $\mathscr{B}$ for the topology that is stable under finite intersections and satisfies $H^p(U, F) = 0$ for all $U \in \mathscr{B}$, $p \geq 1$, F.

The same holds, cum grano salis, for arbitrary sites (compare the discussion in Appendix B). For the étale site of a scheme X, there seems to be no candidate for such a basis $\mathscr{B}$ of open sets. However, there is the following substitute: Let $a = (a_1, \ldots, a_n)$ be a sequence of geometric points $a_i : \mathrm{Spec}(\Omega_i) \to X$. Following Artin, we define

$$X_a = X_{a_1,\ldots,a_n} = \mathrm{Spec}(\mathscr{O}^s_{X,a_1}) \times_X \ldots \times_X \mathrm{Spec}(\mathscr{O}^s_{X,a_n}).$$

The schemes X_a can be viewed as inverse limits of finite intersections of small and smaller neighborhoods, and therefore appears to be a good substitute for the $U \in \mathscr{B}$. It was M. Artin's insight [3] that for the collapsing of the Cartan–Leray spectral sequence in the étale topology it indeed suffices to verify that the inverse limits X_a are acyclic.

In order to treat the Nisnevich topology, we use the following: If $x = (x_1, \ldots, x_n)$ is a sequence of ordinary points $x_i \in X$, we likewise set

$$X_x = X_{x_1,\ldots,x_n} = \operatorname{Spec}(\mathscr{O}^h_{X,x_1}) \times_X \ldots \times_X \operatorname{Spec}(\mathscr{O}^h_{X,x_n}),$$

using the henselizations rather then the strict localizations. Usually, one writes $\overline{x}_i : \operatorname{Spec}(\Omega_i) \to X$ for geometric points with image point $x_i \in X$. In order to keep notation simple, and since we are mainly interested in geometric points, I prefer to write a_i for geometric points. Although this is slightly ambiguous, it should cause no confusion. Let us now introduce the following terminology:

Definition 5.1. We say that the scheme X has the *weak Cartan–Artin property* if for each sequence $x = (x_1, \ldots, x_n)$ of points on X, the scheme X_x is acyclic with respect to the Nisnevich topology. We say that X has the *strong Cartan–Artin property* if for each sequence $a = (a_1, \ldots, a_n)$ of geometric points on X, the scheme X_a is acyclic with respect to the étale topology.

In the latter case, this means that the schemes X_a are affine, their irreducible components are strictly local, and the subspace $\operatorname{Max}(X_a) \subset X_a$ of closed points is at most zero-dimensional. In the former case, however, there is no condition on the residue fields of the closed points. Our interest in this property comes from the following result, which was proved by Artin [3] for the strong Cartan–Artin property. For the weak Cartan–Artin property, the arguments are virtually the same.

Theorem 5.2 (**M. Artin**). *Let X be a noetherian scheme. If it has the strong Cartan–Artin property, then $\check{H}^p(X_{\text{et}}, \underline{H}^q(F)) = 0$ for all abelian étale sheaves F and all $p \geq 0$, $q \geq 1$. If it has the weak Cartan–Artin property, then $\check{H}^p(X_{\text{Nis}}, \underline{H}^q(F)) = 0$ for all abelian Nisnevich sheaves F and all $p \geq 0$, $q \geq 1$. In particular, Čech cohomology coincides with sheaf cohomology.*

To avoid tedious repetitions, we will work in both cases with the schemes X_a constructed with the strict localizations. This is permissible, in light of the following observation:

Proposition 5.3. *The scheme X has the weak Cartan–Artin property if and only if for each sequence $a = (a_1, \ldots, a_n)$ of geometric points on X, the scheme X_a is acyclic with respect to the Nisnevich topology.*

Proof. Let $x_i \in X$ be the image of the geometric point a_i, and set $x = (x_1, \ldots, x_n)$. Consider the canonical morphism $X_a \to X_x$. The rings $\mathscr{O}^s_{X,a}$ are filtered direct limits of finite étale $\mathscr{O}^h_{X,x}$-algebras, whence , the scheme X_a is an filtered inverse limit $X_a = \varprojlim V_\lambda$ of X_x-schemes whose structure morphism $V_\lambda \to X_x$ are étale coverings. In particular, the morphism $f : X_a \to X_x$ and the projections $X_a \to V_\lambda$ are a flat integral surjections.

We now make the following observation: Suppose $B \subset X_a$ and $C \subset X_x$ are connected components, with $f(B) \subset C$. We claim that the induced map $f : B \to C$ is surjective. To see this, let $B_\lambda \subset V_\lambda$ be the connected components containing the images of B. These B_λ form a filtered inverse system of connected affine schemes with $B = \varprojlim B_\lambda$, and the structure maps $B_\lambda \to X_x$ factor over C. Since C is connected and the induced projections $V_\lambda \times_{X_x} C \to C$ are étale coverings, the preimages $V_\lambda \times_{X_x} C$ are sums of finitely many connected components, each being an étale covering of C. This follows, for example, from the fact that the category of finite Galois coverings of a connected scheme is a Galois category admitting a fiber functor. This was established for locally noetherian schemes in [18], Expose V; for the general case see [45], Theorem 5.24. One of the connected components must be B_λ, and we conclude that the $B_\lambda \to C$ are surjective. Likewise, the transition maps $B_\lambda \to B_\mu$, $\lambda \geq \mu$ must be surjective, and we infer that $B = \varprojlim B_\lambda \to C$ is surjective.

Now suppose that X_a is acyclic with respect to the Nisnevich topology. Then it is affine by Theorem 4.2. It follows from Proposition 1.1 that X_x is affine. Let $C \subset X_x$ be a connected component. We have to show that C is local. Since $f : X_a \to X_x$ is surjective, there is a connected component $B \subset X_a$ with $f(B) \subset C$, and we saw in the preceding paragraph that $f : B \to C$ is an integral surjection. Since X_a is acyclic, the scheme B is local henselian. The same then holds for C, by Proposition 1.3.

Conversely, suppose that X_x is acyclic. Then X_x is affine, and the same holds for X_a, because f is an affine morphism. Let $B \subset X_a$ be a connected component, and $C \subset X_x$ the connected component containing $f(B)$. Then B is local, and we saw above that $f : B \to C$ is an integral surjection. Moreover, the B_λ are local, and the transition maps are local, whence $B = \varprojlim B_\lambda$ is local. □

6. Reduction to normal schemes

The goal of this section is to reduce checking the Cartan–Artin properties to the case of irreducible, or even normal schemes. Artin [3] bypassed this by assuming the AF-property, together with the fact that any affine scheme is a subscheme of some integral scheme. Without assuming the AF-property, or knowing that any scheme is a subscheme of some integral scheme, a different approach is necessary.

Let X be a scheme. We now take a closer look at the functoriality of the X_a with respect to X. Let $f : X \to Y$ be a morphism of schemes, and $a = (a_1, \ldots, a_n)$ a sequence of geometric points on X, and $b = (b_1, \ldots, b_n)$ the sequence of geometric image points on Y. We then obtain an induced map between strictly local rings, and thus a morphism $X_a \to Y_b$.

Lemma 6.1. *Let $\mathcal{P}$ be a class of morphisms that is stable under fiber products, and that contains all closed embeddings. If $f : X \to Y$ is separated, and the induced morphisms $\operatorname{Spec}(\mathscr{O}^s_{X,a_i}) \to \operatorname{Spec}(\mathscr{O}^s_{Y,b_i})$ belong to $\mathcal{P}$, then $X_a \to Y_b$ belongs to $\mathcal{P}$.*

Proof. By assumption, the induced morphism

$$\prod_{i=1}^n \operatorname{Spec}(\mathscr{O}^s_{X,a_i}) \longrightarrow \prod_{i=1}^n \operatorname{Spec}(\mathscr{O}^s_{Y,b_i})$$

belongs to $\mathcal{P}$, where the products designate fiber products over Y. It remains to check the following: If U, V are two X-schemes, than the morphism $U \times_X V \to U \times_Y V$ is a closed embedding. This indeed holds by [17], Proposition 5.2, because $f : X \to Y$ is separated. □

Proposition 6.2. *Notation as above. If $f : X \to Y$ is integral then the same holds for $X_a \to Y_b$. If moreover $f : X \to Y$ is finite, a closed embedding, radical or with radical field extensions, the respective property holds for $X_a \to Y_b$.*

Proof. In light of Lemma 6.1, it suffices to treat the case that the sequence $a = (a_1, \ldots, a_n)$ consists of a single geometric point, which by abuse of notation we also denote by a. Write $X = \varprojlim X_\lambda$ as a filtered inverse limit of finite Y-schemes X_λ, and let a_λ be the geometric point on X_λ induced by a. Let $x \in X$ and $x_\lambda \in X_\lambda$ be their respective images. Then $\mathscr{O}_{X,x} = \varinjlim \mathscr{O}_{X_\lambda,x_\lambda}$, and it follows from [23], Corollary 18.6.14 that $\mathscr{O}^s_{X,a} = \varinjlim \mathscr{O}^s_{X_\lambda,a_\lambda}$. On the other hand, the base change $X_\lambda \times_Y \operatorname{Spec}(\mathscr{O}^s_{Y,b})$ is a finite sum of strictly local schemes. Let C_λ be the connected component corresponding to $a'_\lambda = (a_\lambda, b)$. It follows from [23], Proposition 18.6.8 that $\mathscr{O}^s_{X_\lambda,a_\lambda} = \mathscr{O}_{C_\lambda,a'_\lambda}$. In turn, $\mathscr{O}^s_{X,a}$ is a filtered inverse limit of finite $\mathscr{O}^s_{Y,b}$-algebras, thus integral. From this description, the additional assertions follow as well. $\square$

Our goal now is to reduce checking the Cartan–Artin properties to the case of integral normal schemes. Recall that $a = (a_1, \ldots, a_n)$ is a fixed sequence of geometric points on X.

Proposition 6.3. *Suppose there are only finitely many irreducible components $X_j \subset X$, $1 \leq j \leq r$ that contain all geometric points $a_1, \ldots, a_n$. Then the closed subschemes*

$$X_{j,a} = (X_j)_a = \prod_{i=1}^n \operatorname{Spec}(\mathscr{O}^s_{X_j,a_i}), \quad 1 \leq j \leq r$$

form a closed covering of X_a. Here the product means fiber product over X.

Proof. Let $x_i \in X$ be the images of the geometric points a_i. Clearly, the structure morphism $X_a \to X$ factors through $\bigcap_{i=1}^n \operatorname{Spec}(\mathscr{O}_{X,x_i})$, and this intersection is the set of points $x \in X$ containing all $x_1, \ldots, x_n$ in their closure. Thus the inclusion $\bigcap_{i=1}^n \operatorname{Spec}(\mathscr{O}_{X',x_i}) \subset \bigcap_{i=1}^n \operatorname{Spec}(\mathscr{O}_{X,x_i})$ is an equality, where $X' = X_1 \cup \ldots \cup X_r$. It then follows that the closed embedding $X'_a \subset X_a$ is bijective. We thus may assume that $X = X'$.

Let $\eta \in X_a$ be a generic point. The structure morphism $X_a \to X$ is flat. By going-down, the point η maps to the generic point $\eta_j \in X_j$ for some $1 \leq j \leq r$. In turn, it is contained in $(X_j)_a = X_a \times_X X_j$. Consequently the $(X_j)_a \subset X$ form a covering, because they are contain every generic point and are closed subsets. $\square$

Proposition 6.4. *Suppose the irreducible components of X are locally finite, and that the diagonal $X \to X \times X$ is an affine morphism. Then X has the weak/strong Cartan–Artin property if and only if the respective property holds for each irreducible component, regarded as an integral scheme.*

Proof. Let $a = (a_1, \ldots, a_n)$ be a sequence of geometric points on X, and denote by $X_1, \ldots, X_r \subset X$ the irreducible components over which all a_i factors. According to 6.3, we have a closed covering $X_a = (X_1)_a \cup \ldots \cup (X_r)_a$. Now Proposition 4.5 ensures that the space of closed points $\operatorname{Max}(X_a)$ is at most zero-dimensional. Let $S = (X_1)_a \amalg \ldots \amalg (X_r)_a$ be the sum. Then $S \to X_a$ is integral and surjective. By the assumption on the diagonal morphism of X, the scheme S is affine, and the scheme X_a must be affine as well, by Theorem 1.1. Finally, let $C \subset X_a$ be an irreducible component. Choose some $1 \leq j \leq r$ with $C \subset (X_j)_a$. Then C is an irreducible component of $(X_j)_a$. Since $(X_j)_a$ is acyclic, the scheme C is strictly local, according to Theorems 4.2 and 4.4, and it follows that X_a is acyclic. $\square$

Concerning the Cartan–Artin properties, we now may restrict our attention to integral schemes. Our next task is to reduce to normal schemes. Suppose that $f : X \to Y$ is a finite birational morphism between integral schemes, and let $b = (b_1, \ldots, b_n)$ be a sequence of geometric points in Y. As a shorthand, we write

$$f^{-1}(b) = f^{-1}(b_1) \times \ldots \times f^{-1}(b_n)$$

for the finite set of sequences $a = (a_1, \ldots, a_n)$ of geometric points with $f(a_i) = b_i$. Note that one may regard $f^{-1}(b)$ as the product of the points in the finite schemes $X \times_Y \operatorname{Spec} \kappa(b_i)$. For each $a \in f^{-1}(b)$, we obtain an induced morphism $X_a \to Y_b$, which is finite. Its image is denote, for the sake of simplicity, by $f(X_a) \subset Y_b$, which is a closed subset. Now recall that an integral scheme X is called *geometrically unibranch* if the normalization map $X' \to X$ is radical. This obviously holds if X is already normal.

Proposition 6.5. *Assumptions as above. Suppose that X is geometrically unibranch. Then the finite morphisms $X_a \to Y_b$ are radical, and the closed subsets*

$$f(X_a) \subset Y_b, \quad a \in f^{-1}(b)$$

form a closed covering of Y_b.

Proof. First, let us check that $X_a \to Y_b$ is radical. In light of Proposition 6.2, it suffices to treat the case that the sequence $a = (a_1, \ldots, a_n)$ consists of a single geometric point. Let $x \in X$, $y \in Y$ be the points corresponding to the geometric points $a = a_1$ and $b = b_1$, respectively. Set $R = \mathscr{O}_{Y,y}$ and write $X \times_Y \operatorname{Spec}(\mathscr{O}_{Y,y}) = \operatorname{Spec}(A)$. Then A is a semilocal, integral, and a finite R-algebra. According to [23], Proposition 18.6.8, we have $A \otimes_R R^s = A^s$. This is a finite R^s-algebra, whence splits into $A^s = B_1 \times \ldots \times B_m$, where the factors are local and correspond to the geometric points in $f^{-1}(y)$. In fact, the factors are the strict localizations of X at the these geometric points, and $\mathscr{O}_{X,a}$ is one of them. Since X is geometrically unibranch, all factors B_i are integral, according to [23], 18.6.12.

As $f : X \to Y$ is birational, the inclusion $R \subset A$ becomes bijective after localizing with respect to the multiplicative system $S = R \smallsetminus 0$. In turn, the inclusion on the left

$$S^{-1}R^s \subset S^{-1}A^s = S^{-1}(A \otimes_R R^s) = S^{-1}A \otimes_{S^{-1}R} S^{-1}R^s = S^{-1}R^s$$

is bijective, such that we have a canonical identification $S^{-1}A^s = S^{-1}R^s$. Let $R_i \subset B_i$ be the image of the composite map $R^s \to B_i$, which is also a residue class ring of R^s. Hence the $R_i \simeq R^s$ are integral strictly local rings, in particular geometrically unibranch, and $R_i \subset B_i$ induces a bijection on fields of fractions. It follows that $\operatorname{Spec}(B_i) \to \operatorname{Spec}(R_i)$ are radical. It follows that $\operatorname{Spec}(\mathscr{O}^s_{X,a}) \to \operatorname{Spec}(\mathscr{O}^s_{Y,b})$ is radical.

It remains to check that $Y_b = \bigcup_a f(X_a)$. Let $\eta \in Y_b$ be a generic point. The images $\eta_i \in \operatorname{Spec}(\mathscr{O}_{Y,b_i})$ are generic points as well, because the projections $Y_b \to \operatorname{Spec}(\mathscr{O}^s_{Y,b_i})$ are flat, whence satisfy going-down. We saw in the preceding paragraph that the generic points in $\operatorname{Spec}(\mathscr{O}^s_{Y,b_i})$ correspond to the points in $f^{-1}(b_i)$. Let $a_i \in f^{-1}(b_i)$ be the point corresponding to η_i, and form the sequence $a = (a_1, \ldots, a_n)$. Using the $f : X \to Y$ is birational, we infer that η lies in the image of $X_a \to Y_b$. $\square$

If Y is an integral scheme, its normalization map $f : X \to Y$ is integral, but not necessarily finite. However, the latter holds for the so-called *japanese schemes*, confer [21], Chapter 0, §23.

Proposition 6.6. *Suppose that Y is an integral scheme whose normalization morphism $f : X \to Y$ is finite. If X has the weak/strong Cartan–Artin property, then the respective property holds for Y.*

Proof. Let $b = (b_1, \ldots, b_n)$ be a sequence of geometric points on Y. One argues as for the proof of Proposition 6.4. Using that $f : X_a \to Y_b$ are radical, one sees that the closed points on Y_b have residue fields that are separably closed. Whence Y_b is acyclic with respect to the Nisnevich/étale topology. □

7. Reduction to TSC schemes

I conjecture that the class of schemes having the weak Cartan–Artin property is stable under images of integral surjections. The goal of this section is to establish a somewhat weaker variant that suffices for our applications. Recall that a scheme X is called *geometrically unibranch* if it is irreducible, and the normalization morphism $Y \to X_{\text{red}}$ is radical. Equivalently, the strictly local rings $\mathscr{O}^s_{X,a}$ are irreducible, for all geometric points a on X, according to [23], Proposition 18.8.15.

Theorem 7.1. *Let $f : X \to Y$ be an integral surjection between schemes that are geometrically unibranch and have affine diagonal. If X has the weak Cartan–Artin property, so does Y.*

This relies on a precise understanding of the connected components inside the X_a, for sequences $a = (a_1, \ldots, a_n)$ of geometric points. We start by collecting some useful facts.

Proposition 7.2. *Suppose X is geometrically unibranch and quasiseparated. Then the connected components of X_a are irreducible.*

Proof. Clearly, the scheme

$$X_a = X_{a_1,\ldots,a_n} = \operatorname{Spec}(\mathscr{O}^s_{X,a_1}) \times_X \ldots \times_X \operatorname{Spec}(\mathscr{O}^s_{X,a_n})$$

is a filtered inverse limit of étale X-schemes that are quasicompact and quasiseparated. Let $U \to X$ be an étale morphism, with U affine. Since $U \to X$ satisfies going-down, each generic point on U maps to the generic point in X. Thus U contains only finitely many irreducible components, because $U \to X$ is quasifinite. Whence for each connected component $U' \subset U$, there is a sequence of irreducible components $U_1, \ldots, U_n \subset U$, possibly with repetitions, so that $U' = \bigcup U_i$ and $U_i \cap U_{i+1} \neq \emptyset$. Choose such a sequence with $n \geq 1$ minimal. Seeking a contradiction, we suppose $n \geq 2$. Choose $u \in U_1 \cap U_2$. Let $x \in X$ be its image. Clearly, $\mathscr{O}_{X,x} \to \mathscr{O}_{U,u}$ becomes bijective upon passing to strict henselizations. Since $\mathscr{O}^s_{X,x}$ is irreducible and $\mathscr{O}_{U,u} \subset \mathscr{O}^s_{U,u}$ satisfies going-down, we conclude that $\mathscr{O}_{U,u}$ is irreducible, contradiction. The upshot is that the connected components of U are irreducible. It remains to apply the Lemma below with $X_a = V$. □

Lemma 7.3. *Let $V = \varprojlim V_\lambda$ be a filtered inverse limit with flat transition maps of schemes V_λ that are quasicompact and quasiseparated, and whose connected components irreducible. Then each connected component of V is irreducible.*

Proof. Let $C \subset V$ be a connected component. Then the closure of the images $\operatorname{pr}_\lambda(C) \subset V_\lambda$ are connected, whence are contained in a connected component $C_\lambda \subset V_\lambda$. These connected components form an inverse system, and we have canonical maps $C \to \varprojlim C_\lambda \subset \varprojlim V_\lambda = V$. The inclusion is a closed embedding, and $\varprojlim C_\lambda$ is connected, since the C_λ are quasicompact and quasiseparated. In turn $C = \varprojlim C_\lambda$. By assumption, the closed subsets $C_\lambda \subset V_\lambda$ are irreducible. According to a result of Lazard discussed in Appendix A, the connected component $C_\lambda \subset V_\lambda$ is the intersection of its neighborhoods that are open-and-closed. This ensures that the transition maps $C_\mu \to C_\lambda$ remain flat. If follows that the projections $C \to C_\lambda$ are flat as well. By going-down, each generic point $\eta \in C$ maps to the generic point $\eta_\lambda \in C_\lambda$, whence $\eta = (\eta_\lambda)$ is unique, and C must be irreducible. □

Suppose that X is geometrically unibranch and quasiseparated, such that the connected components of X_a coincide with the irreducible components. We may describe the subspace $\operatorname{Min}(X_a) \subset X_a$ of generic points as follows: Let $K = k(X)$ be the function field, and $\mathscr{O}^s_{X,a_i} \subset L_i$ be the field of fractions of the strictly local ring. Since $f : X_a \to X$ is flat and thus satisfies going-down, each generic point of X_a maps to the generic point $\eta \in X$. Since $X_a \to X$ is an inverse limit of quasifinite X-scheme, each point in $f^{-1}(\eta)$ is indeed a generic point of X_a. Thus:

Proposition 7.4. *Assumptions as above. Then the subspace of generic points* $\operatorname{Min}(X_a) \subset X_a$ *is canonically identified with* $\operatorname{Spec}(L_1 \otimes_K \ldots \otimes_K L_n)$.

In particular, the space $\operatorname{Min}(X_a)$ is profinite. We may describe it in terms of Galois theory as follows: Choose a separable closure $K \subset K^s$ and embeddings $L_i \subset K^s$. Let $G = \operatorname{Gal}(K^s/K)$ be the Galois group, and set $H_i = \operatorname{Gal}(K^s/L_i)$. Then G is a profinite group, the $H_i \subset G$ are closed subgroups, and the quotients G/H_i are profinite. Then:

Proposition 7.5. *Assumptions as above. Then the space of generic points* $\operatorname{Min}(X_a)$ *is homeomorphic to the orbit space for the canonical left G-action on the space* $G/H_1 \times \ldots \times G/H_n$.

Proof of Theorem 7.1. Suppose that X, Y are geometrically unibranch, with affine diagonal, and let $f : X \to Y$ be a integral surjection. Assume X satisfies the weak Cartan–Artin property. We have to show that the same holds for Y. Without restriction, it suffices to treat the case that Y, X are integral.

Let $b = (b_1, \ldots, b_n)$ be a sequence of geometric points on Y, and $C \subset Y_b$ be a connected component. Our task is to check that C is local henselian. According to Proposition 7.2, the scheme C is integral. Let $\eta \in C$ be the generic point. Since f is surjective and integral, we may lift b to a sequence of geometric points a on X. Since X, Y are geometrically unibranch, the strictly local rings $\mathscr{O}^s_{X,a_i}, \mathscr{O}^s_{Y,b_i}$ remain integral, and we have a commutative diagram

$$\begin{array}{ccc} \mathscr{O}_{X,a_i} & \longrightarrow & K_i \\ \uparrow & & \uparrow \\ \mathscr{O}_{Y,b_i} & \longrightarrow & L_i, \end{array}$$

where the terms on the right are the fields of fractions. Let $K = k(X)$ and $L = k(Y)$. Then the induced morphism

$$K_1 \otimes_K \ldots \otimes_K K_n \subset L_1 \otimes_L \ldots \otimes_L L_n$$

is faithfully flat and integral. It follows that each generic point $\eta \in Y_b$ is in the image of $X_a \to Y_b$. The latter morphism is integral by Proposition 6.2, whence also surjective. In particular, there is an irreducible closed subscheme $B \subset X_a$ surjecting onto $C \subset Y_b$. By assumption, B is local henselian, and the morphism $B \to C$ is surjective and integral. Whence C is local henselian, according to Proposition 1.3. □

In particular, an integral normal scheme X_0 has the weak Cartan–Artin property if and only this holds for the total separable closure $X = \mathrm{TSC}(X_0)$. The latter is a scheme that is everywhere strictly local. For such schemes, the Cartan–Artin properties reduce to a striking geometric property with respect to the Zariski topology:

Theorem 7.6. *Let X be an irreducible quasiseparated scheme that is everywhere strictly local. Then the following are equivalent:*

(i) *The scheme X has the Cartan–Artin property.*
(ii) *The scheme X has the weak Cartan–Artin property.*
(iii) *For every pair of irreducible closed subset $A, B \subset X$ the intersection $A \cap B$ is irreducible.*

Proof. Given a sequence $x = (x_1, \ldots, x_n)$ of geometric points, which we may regard as ordinary points $x_i \in X$, we have $\mathscr{O}_{X,x_i} = \mathscr{O}^s_{X,x_i}$, whence

$$X_x = \mathrm{Spec}(\mathscr{O}_{X,x_1}) \cap \ldots \cap \mathrm{Spec}(\mathscr{O}_{X,x_n}),$$

and this is the set of points in X containing each x_i in their closure. Since X is quasiseparated, the scheme X_x is quasicompact, so every point specializes to a closed point. It follows that

$$X_x = \bigcup_z \mathrm{Spec}(\mathscr{O}_{X,z}),$$

where the union runs over all closed points $z \in X_x$.

The implication (i)⇒(ii) is trivial.

To see (iii)⇒(i), it suffices, in light of Theorem 4.4, to check that each X_x contains at most one closed point, that is, X_x is either empty or local. By induction on $n \geq 1$, it is enough to treat the case $n = 2$, with $x_1 \neq x_2$. Now suppose there are two closed points $a \neq b$ in $X_x = \mathrm{Spec}(\mathscr{O}_{X,x_1}) \cap \mathrm{Spec}(\mathscr{O}_{X,x_2})$, and let $A, B \subset X$ be their closure. Then $x_1, x_2 \in A \cap B$. The intersection is irreducible by assumption. Whence the generic point $\eta \in A \cap B$ lies in X_x. Since it is in the closure of $a, b \in X_x$, and these points are closed in X_x, we must have $a = \eta = b$, contradiction.

It remains to verify (ii)⇒(iii). Seeking a contradiction, we suppose that there are irreducible closed subsets $A, B \subset X$ with $A \cap B$ reducible. Choose generic points $x_1 \neq x_2$ in this intersection, and consider the pair $x = (x_1, x_2)$. The resulting scheme $X_x = \mathrm{Spec}(\mathscr{O}_{X,x_1}) \cap \mathrm{Spec}(\mathscr{O}_{X,x_2})$ is irreducible, because this holds for X. In light of Theorem 4.2, X_x must be local. Let $z \in X_x$ be the closed point. Write η_A, η_B for the generic points of A and B, respectively. By construction, $\eta_A, \eta_B \in X_x$, whence $z \in A \cap B$. Using that $x_1, x_2 \in A \cap B$ are generic and contained in the closure of $\{z\} \subset X$, we infer $x_1 = z = x_2$, contradiction. □

8. Algebraic surfaces

Let k be a ground field, and S_0 a separated scheme of finite type, assumed to be integral and 2-dimensional. Note that we do not suppose that S_0 is quasiprojective. Examples of normal surfaces that are proper but not projective appear in [56]. In this section we investigate geometric properties of the absolute integral closure $S = \mathrm{TSC}(S_0)$. Our result is:

Theorem 8.1. *Assumptions as above. Let $A, B \subset S$ be two integral, 1-dimensional closed subschemes with $A \neq B$. Then $A \cap B$ contains at most one point.*

Proof. Clearly, we may assume that S_0 is proper and normal, and that the ground field k is algebraically closed. Employing the notation from Section 3, we write

$$S = \varprojlim S_\lambda \quad \text{and} \quad A = \varprojlim A_\lambda \quad \text{and} \quad B = \varprojlim B_\lambda, \quad \lambda \in L.$$

Seeking a contradiction, we assume that there are two closed points $x \neq y$ in $A \cap B$. Let $x_\lambda, y_\lambda \in A_\lambda \cap B_\lambda$ be their images on S_λ. Enlarging $\lambda = 0$, we may assume that $A_0 \neq B_0$ and $x_0 \neq y_0$.

The integral Weil divisor $B_0 \subset S_0$ has a rational selfintersection number $(B_0)^2 \in \mathbb{Q}$, defined in the sense of Mumford ([48], Section II (b)). Now choose finitely many closed points $z_1, \ldots, z_n \in B_0$ neither contained in A_0 nor the singular locus $\mathrm{Sing}(S_0)$, and let $S'_0 \to S_0$ be the blowing-up with reduced center given by these points. If we choose $n \gg 0$ large enough, the selfintersection number of the strict transform $B'_0 \subset S'_0$ of B_0 becomes strictly negative. According to [2], Corollary 6.12 one may contract it to a point: There exist a proper birational morphism $g : S'_0 \to Y_0$ with $\mathscr{O}_{Y_0} = g_*(\mathscr{O}_{S'_0})$ sending the integral curve B'_0 to a closed point, and being an open embedding on the complement. Note that Y_0 is a proper 2-dimensional algebraic space over k, which is usually not a scheme. We also write $A_0 \subset S'_0$ for the curve A_0, considered as a closed subscheme of S'_0. The projection $A_0 \to Y_0$ is a finite morphism, which is not injective because the two points $x_0, y_0 \in A_0 \cap B'_0$ are mapped to a common image.

We now make an analogous construction in the inverse system. Consider the normalization $S'_\lambda \to S_\lambda \times_{S_0} S'_0$ of the integral component dominating S'_0. The universal properties of normalization and fiber product ensure that the S'_λ, $\lambda \in L$ form an inverse system, and the transition maps are obviously affine. In turn, we get an integral scheme

$$S' = \varprojlim S'_\lambda,$$

coming with a birational morphism $S' \to S$. This scheme S' is totally separably closed. Clearly, the projection $S'_\lambda \to S_\lambda$ are isomorphisms over a neighborhood of the preimage of $A_0 \subset S_0$, which contains $A_\lambda \subset S'$. Therefore, we may regard A_λ and the points x_λ, y_λ also as a closed subscheme of S'_λ.

Let $S'_\lambda \to Y_\lambda$ be the Stein factorization of the composition $S'_\lambda \to S'_0 \to Y_0$. This map contracts the connected components of the preimage of $B'_0 \subset S'_0$ to closed points, and is an open embedding on the complement. Using that this preimage contains the strict transform $B'_\lambda \subset S'_\lambda$ of $B_\lambda \subset S_\lambda$, which is irreducible and thus connected, we infer that $x_\lambda, y_\lambda \in S'_\lambda$ map to a common image in Y_λ. By the universal property of Stein factorizations, the Y_λ, $\lambda \in L$ form a filtered inverse system. The transition morphisms $Y_\lambda \to Y_j$ are finite, because these schemes are finite over Y_0. In turn, we get another totally separably closed scheme $Y = \varprojlim Y_\lambda$, endowed with a birational morphism $S' \to Y$. By construction, the finite morphism $A_\lambda \to Y_\lambda$ map x_λ, y_λ to a common image. Passing to the inverse limit, we get an integral morphism

$$A = \varprojlim A_\lambda \longrightarrow \varprojlim Y_\lambda = Y,$$

which maps $x \neq y$ to a common image. By assumptions, A is an integral scheme. According to Proposition 2.3, the map $A \to Y$ must be injective, contradiction. $\square$

9. Two facts on noetherian schemes

Before we come to higher-dimensional generalizations of Theorem 8.1, we have to establish two properties pertaining to quasiprojectivity and connectedness that might be of independent interest.

Proposition 9.1. *Let R be a noetherian ground ring, X a separated scheme of finite type, and $U \subset X$ be a quasiprojective open subset. Then there is a closed subscheme $Z \subset X$ disjoint from U and containing no point $x \in X$ of codimension $\dim(\mathscr{O}_{X,x}) = 1$ such that the blowing up $f : X' \to X$ with center Z yields a quasiprojective scheme X'.*

Proof. By a result of Gross [33], proof for Theorem 1.5, there exists an open subscheme $V \subset X$ containing U and all points of codimension ≤ 1 that is quasiprojective over R. (Note that his overall assumption that the ground ring R is excellent does not enter in this result.) By Chow's Lemma (in the refined form of [13], Corollary 1.4), there is a blowing-up $X' \to X$ with center $Z \subset X$ disjoint from V so that X' becomes quasiprojective over R. □

Proposition 9.2. *Let X be a noetherian scheme satisfying Serre's Condition (S_2), and $D \subset X$ a connected Cartier divisor. Suppose that the local rings $\mathscr{O}_{X,x}$, $x \in X$ are catenary and homomorphic images of Gorenstein local rings. Then there is a sequence of irreducible components $D_0, \ldots, D_r \subset D$, possibly with repetitions and covering D such that the successive intersections $D_{i-1} \cap D_i$, $1 \leq i \leq r$ have codimension ≤ 2 in X.*

Proof. Since D is connected and noetherian, there is a a sequence of irreducible components $D_0, \ldots, D_s \subset D$, possibly with repetitions and covering D so that the successive intersections are nonempty ([17], Corollary 2.1.10). Fix an index $1 \leq i \leq r$, choose closed points $x \in D_{i-1} \cap D_i$, let $R = \mathscr{O}_{X,x}$ be the corresponding local ring on X, and $f \in R$ the regular element defining the Cartier divisor D at x.

To finish the proof, we insert between D_{i-1}, D_i a sequence of further irreducible components satisfying the codimension condition of the assertion on the local ring R. This can be achieved with Grothendieck's Connectedness Theorem ([24], Exposé XIII, Theorem 2.1) given in the form of Flenner, O'Carroll and Vogel ([15], Section 3.1) as follows:

Recall that a noetherian scheme S is called *connected in dimension d* if we have $\dim(S) > d$, and the complement $S \smallsetminus T$ is connected for all closed subsets of dimension $\dim(T) < d$. Set $n = \dim(R)$. Since the local ring R is catenary and satisfies Serre's condition (S_2), it must be connected in dimension $n-1$ (loc. cit., Corollary 3.1.13). It follows that R/fR is connected in dimension $n-2$ (loc. cit., Lemma 3.1.11, which is essentially Grothendieck's Connectedness Theorem). Consequently, there is a sequence $H_0, \ldots, H_q$ of irreducible components of $\mathrm{Spec}(R/fR)$ so that H_0, H_q are the local schemes attached to $x \in D_{i-1}, D_i$, and whose successive intersections have dimension $\geq n-2$ in $\mathrm{Spec}(R)$, whence codimension ≤ 2. The closures of the $H_0, \ldots, H_q$ inside D yield the desired sequence of further irreducible components. □

Let me state this result in a simplified form suitable for most applications:

Corollary 9.3. *Let X be a normal irreducible scheme that is of finite type over a regular noetherian ring R, and $D \subset X$ a connected Cartier divisor. Then there* *is a sequence of irreducible components $D_0, \ldots, D_r \subset D$, possibly with repetitions and covering D such that the successive intersections $D_{i-1} \cap D_i$, $1 \leq i \leq r$ have codimension ≤ 2 in X.*

Proof. For every affine open subset $U \subset X$, the ring $A = \Gamma(U, \mathscr{O}_X)$ is the homomorphic image of some polynomial ring $R[T_1, \ldots, T_n]$. Since the latter is regular, in particular Gorenstein and catenary, it follows that all local rings $\mathscr{O}_{X,x}$ are catenary, and homomorphic images of Gorenstein local rings. □

10. Contractions to points

Let k be a ground field, and X a proper scheme, assumed to be normal and irreducible. Set $n = \dim(X)$. We say that a connected closed subscheme $D \subset X$ is *contractible to a point* if there is a proper algebraic space Y and a morphism $g : X \to Y$ with $\mathscr{O}_Y = f_*(\mathscr{O}_X)$ sending D to a closed point and being an open embedding on the complement. Note that such a morphism is automatically proper, because X is proper and Y is separable. Moreover, it is unique up to unique isomorphism.

Let $C \subsetneqq X$ be a connected closed subscheme. The goal of this section is to construct a projective birational morphism $f : X'' \to X$ so that some connected Cartier divisor $D'' \subset X''$ with $f(D'') = C$ becomes contractible to a point, generalizing the procedure in dimension $n = 2$ used in the proof for Proposition 8.1. The desired morphism f will be a composition $f = g \circ h$ of two birational morphisms.

The first morphism $g : X' \to X$ will replace the connected closed subscheme $C \subset X$ by some connected Cartier divisor $D' \subset X'$ that is a *projective* scheme. It will be itself a composition of three projective birational morphisms

$$X \xleftarrow{g_1} X_1 \xleftarrow{g_2} X_2 \xleftarrow{g_3} X_3 = X'.$$

Here $g_1 : X_1 \to X$ is the normalized blowing-up with center $C \subset Y$. Such a map has connected fibers by Zariski's Main Theorem. Let $D_1 = g_1^{-1}(C)$. According to Proposition 9.1, there is a closed subscheme $Z \subset D_1$ containing no point $d \in D_1$ of codimension $\dim(\mathscr{O}_{D_1,d}) \leq 1$ so that the blowing-up $\tilde{D}_1 \to D_1$ becomes projective. Let $g_2 : X_2 \to X_1$ be the normalized blowing-up with the same center regarded as a closed subscheme $Z \subset X$. Viewing $\tilde{D}_1$ as a closed subscheme on the blowing-up $\tilde{X}_1 \to X_1$ with center Z, we denote by $D_2 \subset X_2$ its preimage on the normalized blowing-up, which is a Weil divisor. Finally, let $g_3 : X = X_3 \to X_2$ be the normalized blowing-up with center $D_2 \subset X_2$, and $D' = D_3$ be the preimage of D_2. Then $D' \subset X'$ is a Cartier divisor by the universal property of blowing-ups. The morphism $X' \to X$ has connected fibers, by Zariski's Main Theorem, and induces a surjection $D' \to C$. Since C is connected, it follows that D' is connected. The proper scheme D' is projective, because $\tilde{D}$ is projective, and finite maps and blowing-ups are projective morphisms. Setting $g = g_1 \circ g_2 \circ g_3 : X' \to X$, we record:

Proposition 10.1. *The scheme X' is proper and normal, the morphism $g : X' \to X$ is projective and birational, the closed subscheme $D' \subset X'$ is a connected Cartier divisor that is a projective scheme, and $g(D') = C$.*

In a second step, we now construct the proper birational morphism $h : X'' \to X'$, so that X'' contains the desired contractible effective Cartier divisor. Choose an ample Cartier divisor $A \subset D'$ containing no generic point from the intersection of two irreducible component of D', and so that the invertible sheaf $\mathscr{O}_{D'}(A - D')$ is ample. Let $\varphi : \tilde{X}' \to X'$ be the blowing-up with center $A \subset X'$. By the universal property of blowing-ups, there is a partial section $\sigma : D' \to \tilde{X}'$. Let $\nu : X'' \to \tilde{X}'$ be the normalization map, such that the composition

$$h : X'' \xrightarrow{\nu} \tilde{X}' \xrightarrow{\varphi} X'$$

is the normalized blowing-up with center A. Set $D'' = \nu^{-1}(\sigma(D')) \subset X''$, and let $f = g \circ h : X'' \to X$ be the composite morphism.

Theorem 10.2. *The scheme X'' is proper and normal, the morphism $f : X'' \to X$ is projective and birational, and $D'' \subset X''$ is a connected Cartier divisor that is a projective scheme with $f(D'') = C$. Moreover, the closed subscheme $D'' \subset X''$ is contractible to a point.*

Proof. By construction, X'' is normal and proper, f is projective and birational, and $f(D'') = C$. According to [52], Lemma 4.4, the subschemes

$$\sigma(D'), \varphi^{-1}(A), \varphi^{-1}(D') \subset \tilde{X}'$$

are Cartier, with $\varphi^{-1}(D) = \sigma(D') + \varphi^{-1}(A)$ in the group of Cartier divisors on $\tilde{X}'$. Pulling back along the finite surjective morphism between integral schemes $\nu : X'' \to \tilde{X}'$, we get Cartier divisors $D'' = \nu^{-1}(\sigma(D')), E = h^{-1}(A), \hat{D} = h^{-1}(D')$ on X'' satisfying $\hat{D} = D'' + E$. In turn, $\mathscr{O}_{D''}(-D'') \simeq \mathscr{O}_{D''}(E - \hat{D})$. The latter is isomorphic to the preimage with respect to the normalization map $\nu : X'' \to \tilde{X}'$ of

$$\mathscr{O}_{\sigma(D')}(-1) \otimes \varphi^*\mathscr{O}_{D'}(-D').$$

The first tensor factor becomes under the canonical identification $\sigma(D') \to D'$ the invertible sheaf $\mathscr{O}_D(A)$. Summing up, $\mathscr{O}_{D''}(-D'')$ is isomorphic to the pull back of the ample invertible sheaf $\mathscr{O}_{D'}(A - D')$ under a finite morphism, whence is ample. By Artin's Contractibility Criterion [2], Corollary 6.12, the connected components of D'' are contractible to points.

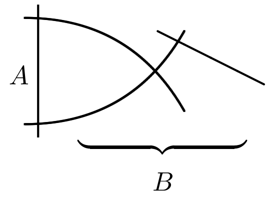
It remains to verify that D'' is connected. For this it suffices to check that the dense open subscheme $D'' \smallsetminus E$ is connected, which is isomorphic to $D' \smallsetminus A$. By Proposition 9.2, there is a sequence of irreducible components $D'_1, \ldots, D'_r \subset D'$, possibly with repetitions and covering D'' so that the successive intersections $D'_{i-1} \cap D'_i$ have codimension ≤ 2 in X', whence codimension ≤ 1 in D'_{i-1} and D'_i. By the choice of $A \subset D'$, the complements $(D'_{i-1} \cap D'_i) \smallsetminus A$ are nonempty, and it follows that $D' \smallsetminus A$ is connected. □

11. Cyclic systems

Let X be a scheme. Let us call a pair of closed subschemes $A, B \subset X$ a *cyclic system* if the following holds:

(i) The space A is irreducible.
(ii) The space B is connected.
(iii) The intersection $A \cap B$ is disconnected.

This ad hoc definition will be useful when dealing with totally separably closed schemes, and is somewhat more flexible than the "polygons" used by [3], compare also [59]. Note that this notion is entirely topological in nature, and that the conditions ensures that A, B are nonempty and $A \not\subset B$. One should bear in mind a picture like the following:



In this section, we establish some elementary but technical permanence properties for cyclic systems, which will be used later.

Proposition 11.1. *Let $A, B \subset X$ be a cyclic system, and $f : X' \to X$ closed surjection with connected fibers. Let $A' \subset X'$ be a closed irreducible subset with $f(A') = A$ and $B' = f^{-1}(B)$. Then the pair $A', B' \subset X'$ is a cyclic system.*

Proof. By assumption, A' is irreducible. Since f is a closed, surjective and continuous, the space B carries the quotient topology with respect to $f : B' \to B$. Thus B' is connected, because B is connected and f has connected fibers ([17], Proposition 2.1.14). The set-theoretic "projection formula" gives

$$f(A' \cap B') = f(A' \cap f^{-1}(B)) = f(A') \cap B = A \cap B,$$

whence $A' \cap B'$ is disconnected. □

Proposition 11.2. *Let $A, B \subset X$ be a cyclic system, and $f : X' \to X$ be a morphism. Let $U \subset X$ be an open subset over which f becomes an isomorphism, and let $A', B' \subset X'$ be the closures of $f^{-1}(A \cap U), f^{-1}(B \cap U)$, respectively. Suppose the following:*

(i) *$B \cap U$ is connected.*
(ii) *U intersects at least two connected components of $A \cap B$.*

Then the pair $A', B' \subset X'$ is a cyclic system.

Proof. Since A is irreducible, so is the nonempty open subset $A \cap U$, and thus the closure A'. Since $B \cap U$ is connected by Condition (i), the same holds for its closure B'. It remains to check that $A' \cap B'$ is disconnected. Since f is continuous and $A \subset X$ is closed, we have

$$f(A') = f(\overline{f^{-1}(A \cap U)}) \subset \overline{A \cap U} \subset A,$$

and similarly $f(B') \subset B$. Hence $f(A' \cap B') \subset A \cap B$. Seeking a contradiction, we assume that $A' \cap B'$ is connected. Choose points $u, v \in A \cap B$ from two different connected components with $u, v \in U$. Regarding u, v as elements from X' via $U = f^{-1}(U)$, we see that they are contained in the connected subset $A' \cap B' \subset X'$, whence the points $u, v \in A \cap B$ are contained in a connected subset of $f(A' \cap B')$, contradiction. □

Proposition 11.3. *Let $X = \varprojlim X_\lambda$ be a filtered inverse system of quasicompact quasiseparated schemes with integral transition maps, and $A, B \subset X$ be a cyclic system. Then their images $A_\lambda, B_\lambda \subset X_\lambda$ form cyclic systems for a cofinal subset of indices λ.*

Proof. The subsets $A_\lambda, B_\lambda \subset X_\lambda$ are irreducible respective connected, because they are images of such spaces. They are closed, because the projections $X \to X_\lambda$ are closed maps. It remains to check that $A_\lambda \cap B_\lambda$ are disconnected for a cofinal subset of indices. Since inverse limits commute with fiber products, we have

$$\varprojlim(A_\lambda) \cap \varprojlim(B_\lambda) = \varprojlim(A_\lambda \cap B_\lambda).$$

The canonical inclusion $A \subset \varprojlim(A_\lambda)$ of closed subsets in X is bijective, since the underlying set of the letter is the inverse limit of the underlying sets of the X_λ. Similarly we have $B = \varprojlim(B_\lambda)$ as closed subsets. Thus $A \cap B = \varprojlim(A_\lambda \cap B_\lambda)$. By assumption, $A \cap B$ is disconnected and the $A_\lambda \cap B_\lambda$ are quasicompact. It follows from Proposition 14.2 that $A_\lambda \cap B_\lambda$ must be disconnected for a cofinal subset of indices λ. □

12. Algebraic schemes

We now come to the main result of this paper, which generalizes Theorem 8.1 from surfaces to arbitrary dimensions. It takes the following form:

Theorem 12.1. *Let k be a ground field and X_0 a integral scheme that is separated, connected and of finite type. Then its total separable closure $X = \mathrm{TSC}(X_0)$ contains no cyclic system. In particular, for any pair $A, B \subset X$ of irreducible closed subsets, the intersection $A \cap B$ remains irreducible.*

Proof. Seeking a contradiction, we assume that there is a cyclic system $A, B \subset X$. Passing to normalization and compactification, we easily reduce to the situation that X_0 is normal and proper. As usual, write the total separable closure as an inverse limit of finite X_0-schemes X_λ, $\lambda \in L$ that are connected normal, and let $A_\lambda, B_\lambda \subset X_\lambda$ be the images of $A, B \subset X$, respectively. Then we have

$$X = \varprojlim X_\lambda, \quad A = \varprojlim A_\lambda \quad \text{and} \quad B = \varprojlim B_\lambda,$$

and all transition morphisms are finite and surjective. Clearly, the A_λ are irreducible and the B_λ are connected. In light of Proposition 11.3, we may replace L by a cofinal subset and assume that the $A_\lambda, B_\lambda \subset X_\lambda$ form cyclic systems for all $\lambda \in L$, including $\lambda = 0$.

To start with, we deal with a rather *special case*, namely that the connected subset $B_0 \subset X_0$ is *contractible to a point.* Let $X_0 \to Y_0$ be the contraction. Then the composite map $A_0 \to Y_0$ has a disconnected fiber, because the connected components of the disconnected subset $A_0 \cap B_0$ are mapped to a common image point. Next, consider the induced maps $X_\lambda \to Y_\lambda$ contracting the connected components of the preimage $X_\lambda \times_{X_0} B_0 \subset X_\lambda$ to points. Again the composite map $A_\lambda \to Y_\lambda$ has a disconnected fiber. This is because the irreducible subset A_λ is not contained in the preimage $X_\lambda \times_{X_0} B_0$, the connected subset B_λ is contained in $X_\lambda \times_{X_0} B_0$, thus contained in a connected component of the latter, whence maps to a point in the scheme Y_λ.

To proceed, consider the Stein factorization $A_\lambda \to A'_\lambda \to Y_\lambda$. Then $A'_\lambda \to Y_\lambda$ is finite, but not injective. Passing to the inverse limits $A' = \varprojlim A'_\lambda$ and $Y = \varprojlim Y_\lambda$, we obtain a totally separably closed scheme Y, an integral scheme A', and an integral morphism $A' \to Y$. The latter must be radical, by Proposition 2.3. In turn, the maps $A'_\lambda \to Y_\lambda$ are radical for sufficiently large indices $\lambda \in L$, according to [22], Theorem 8.10.5. In particular, these maps are injective, contradiction.

We now come to the *general case.* We shall proceed in six steps, some of which are repeated. In each step, a given proper normal connected scheme X_0 whose absolute separable closure $X = \mathrm{TSC}(X_0)$ contains a cyclic system $A, B \subset X$ will be replaced by another such scheme with slightly modified properties, until we finally reach the special case discussed in the preceding paragraphs. As we saw, the latter is impossible. To start with, choose points $u, v \in A \cap B$ coming from different connected components. Let $u_\lambda, v_\lambda \in A_\lambda \cap B_\lambda$ be their images, respectively.

Step 1: *We first reduce to the case that $u_0, v_0 \in A_0 \cap B_0$ lie in different connected components.* Suppose that for all indices $\lambda \in L$, the points $u_\lambda, v_\lambda \in A_\lambda \cap B_\lambda$ are contained in the same connected component $C_\lambda \subset A_\lambda \cap B_\lambda$. Then the C_λ, $\lambda \in L$ form an inverse system with finite transition maps, and their inverse limit $C = \varprojlim C_\lambda$ is connected ([22], Proposition 8.4.1, compare also Appendix A). Using $u, v \in C \subset A \cap B$, we reach a contradiction. Hence there exist an index $\lambda \in L$ such that $u_\lambda, v_\lambda \in A_\lambda \cap B_\lambda$ stem from different connected components. Replacing X_0 by such an X_λ we thus may assume that $u_0, v_0 \in A_0 \cap B_0$ lie in different connected components.

Step 2: *Next we reduce to the case that $u_0, v_0 \in A_0 \cap B_0$ are not contained in a common irreducible component of B_0.* Let $f : X_0' \to X_0$ be the normalized blowing-up with center $A_0 \cap B_0 \subset X_0$, such that the strict transforms of A_0, B_0 on X_0' become disjoint.

Let X_λ' be the normalization of $X_\lambda \times_{X_0} X_0'$ inside the function field $k(X_\lambda) = k(X_\lambda')$. Then the X_λ', $\lambda \in L$ form an inverse system of proper normal connected schemes whose transition morphisms are finite and dominant. Let X' be their inverse limit. By construction, we have projective birational morphisms $X_\lambda' \to X_\lambda$ and a resulting birational morphism $X' \to X$, which is a proper surjective map with connected fibers, according to [22], Proposition 8.10.5 (xii), (vi) and (vii). Note that all these X-morphisms become isomorphisms when base-changed to $X_0 \smallsetminus (A_0 \cap B_0)$.

Let $A' \subset X'$ be the strict transform of $A \subset X$, and $B' \subset X'$ the reduced scheme-theoretic preimage of $B \subset X$. These form a cyclic system on X', by Proposition 11.1. Consider their images $A_\lambda', B_\lambda' \subset X_\lambda'$, which are closed subsets. The A_λ' are irreducible and the B_λ' are connected. Clearly, the A_λ' can be viewed as the strict transforms of $A_\lambda \subset X_\lambda$. Moreover, the inclusion $B_\lambda' \subset X_\lambda' \times_{X_\lambda} B_\lambda$ is an equality of sets. This is because the morphisms $B \to B_\lambda$ and $X' \to X$ are surjective, by [22], Theorem 8.10.5 (vi). Again with Proposition 11.1, we infer that all $A_\lambda, B_\lambda \subset X_\lambda$ form a cyclic system.

Let $\varphi : X' \to X$ be the canonical morphism. The set-theoretical Projection Formula gives $\varphi(A' \cap B') = A \cap B$. Choose points $u', v' \in A' \cap B'$ mapping to u and v, respectively, and consider their images $u_0', v_0' \in A_0' \cap B_0'$. The latter are not contained in the same connected component, because they map to $u_0, v_0 \in A_0 \cap B_0$. Moreover, we claim that the $u_0', v_0' \in A_0' \cap B_0'$ are not contained in a common irreducible component of B_0': Suppose they would lie in a common irreducible component $W_0' \subset B_0'$. Since the strict transforms of A_0, B_0 on X_0' are disjoint, the image $f(W_0') \subset X_0$ under the blowing-up $f : X_0' \to X_0$ must lie in the center $A_0 \cap B_0$. Then $u_0, v_0 \in f(W_0) \subset A_0 \cap B_0$ are contained in some connected subset, contradiction.

Summing up, after replacing X_0 by X_0', we may additionally assume that $u_0, v_0 \in B_0$ are not contained in a common irreducible component of B_0.

Step 3: *The next goal is to turn the closed subscheme $B_0 \subset X_0$ into a Cartier divisor.* Let $X_0' \to X_0$ be the normalized blowing-up with center $B_0 \subset X_0$, such that its preimage on X'_0 becomes a Cartier divisor. Define $X'_\lambda, X', A', B', A'_\lambda, B'_\lambda$ as in Step 2. Then $A', B' \subset X'$ and the $A'_\lambda, B'_\lambda \subset X'_\lambda$ form cyclic systems.

Again let $\varphi : X' \to X$ be the canonical morphism. Then $\varphi(A' \cap B') = A \cap B$ by the projection formula. Let $u', v' \in A' \cap B'$ be in the preimage of the points $u, v \in A \cap B$. Since their images $u_0, v_0 \in A_0 \cap B_0$ do not lie in a common connected component and are not contained in a common irreducible component of B, the same necessarily holds for their images $u'_0, v'_0 \in A'_0 \cap B'_0$.

Replacing X_0 by X'_0, we thus may additionally assume that $B_0 \subset X_0$ is the support of an effective Cartier divisor.

Step 4: *We now reduce to the case that all $B_\lambda \subset X_\lambda$ are supports of connected effective Cartier divisors.* The preimages $X_\lambda \times_{X_0} B_0 \subset X_\lambda$ of the Cartier divisor $B_0 \subset X_0$ remain Cartier divisors, because $X_\lambda \to X$ is a dominant morphism between normal schemes. Let $C_\lambda \subset X_\lambda \times_{X_0} B_0$ be the connected component containing the connected subscheme $B_\lambda \subset X_\lambda \times_{X_0} B_0$. Clearly, the $C_\lambda \subset X_\lambda$ are connected Cartier divisors, which form an inverse system such that $C = \varprojlim C_\lambda$ contains $B = \varprojlim B_\lambda$. Note that the transition maps $C_j \to C_\lambda$, $i \leq j$ and the projections $C \to C_\lambda$ are not necessarily surjective. Nevertheless, we conclude with [22], Proposition 8.4.1 that C is connected. Clearly, the points $u, v \in A \cap C$ lie in different connected components, and are not contained in a common irreducible component of C, because this hold for their images $u_0, v_0 \in A_0 \cap B_0$. Thus $A, C \subset X$ form a cyclic system.

Replacing B by C and B_λ by C_λ, we thus may additionally assume that all $B_\lambda \subset X_\lambda$ are supports of connected Cartier divisors. Note that at this stage we have sacrificed our initial assumption that the the projections $B \to B_\lambda$ and the transition maps $B_j \to B_\lambda$ are surjective.

Step 5: *Here we reduce to the case that the proper scheme B_0 is projective.* Since $u_0, v_0 \in B_0$ are not contained in a common irreducible component, they admit a common affine open neighborhood inside B_0: To see this, let $B'_0 \subset B_0$ be the finite union of the irreducible components that do not contain u_0, and let $B''_0 \subset B_0$ be the finite union of the irreducible components that do not contain v_0. Then $B_0 = B'_0 \cup B''_0$, so any affine open neighborhoods of $u_0 \in B_0 \smallsetminus B'_0$ and $v_0 \in B_0 \smallsetminus B''_0$ are disjoint, and their union is the desired common affine open neighborhood.

It follows that there is a closed subscheme $Z_0 \subset B_0$ containing neither u_0, v_0 nor any point $b \in B_0$ of codimension $\dim(\mathscr{O}_{B_0,b}) = 1$ so that the blowing-up $\tilde{B}_0 \to B_0$ with center Z_0 yields a projective scheme $\tilde{B}_0$. This holds by Proposition 9.1. Note that $Z_0 \subset B_0$ has codimension ≥ 2.

Let $X'_0 \to X_0$ be the normalized blowing-up with the same center Z_0, regarded as a closed subscheme $Z_0 \subset X_0$. Define X'_λ and $X' = \varprojlim X'_\lambda$ as in Step 2. Let $U \subset X$ be the complement of the closed subscheme $X \times_{X_0} Z_0 \subset X$. This open subscheme intersects at least two connected components of $A \cap B$, because $u, v \in U$. Clearly, the canonical morphism $f : X' \to X$ becomes an isomorphism over U. Furthermore, the intersection $B \cap U$ is connected. To see this, set $Z_\lambda = X_\lambda \times_{X_0} Z_0$ and $U_\lambda = X_\lambda \smallsetminus Z_\lambda$, such that $B \cap U = \varprojlim (B_\lambda \cap U_\lambda)$. In light of [22], Proposition 8.4.1, it suffices to check that $B_\lambda \cap U_\lambda$ is connected for each $\lambda \in L$. Indeed, by the Going-Down Theorem applied to the finite morphism $B_\lambda \to B_0$, all $Z_\lambda \subset B_\lambda$ have codimension ≥ 2. The $B_\lambda \subset X_\lambda$ are supports of connected Cartier divisors. In light of Corollary 9.3, the set $B_\lambda \cap U_\lambda = B_\lambda \smallsetminus Z_\lambda$ must remain connected.

According to Proposition 11.2, the closures $A', B' \subset X'$ of $A \cap U, B \cap U$ form a cyclic system. Clearly, the image of the composite map $B' \to X_0$ is $B_0 \subset X_0$. Replacing X by X', we may assume that B_0 is a projective scheme. Note that with this step we have sacrificed the property that the $B_\lambda \subset X_\lambda$ are Cartier divisors.

Step 6: *We now repeat Step 4, and achieve again that $B_\lambda \subset X_\lambda$ are supports of connected Cartier divisors.* Moreover, the B_λ stay projective, because B_0 is projective and the $X_\lambda \to X_0$ are finite.

Step 7: *In this last step we achieve that $B_0 \subset X_0$ becomes contractible to a point, thus reaching a contradiction.* Choose an ample Cartier divisor $H_0 \subset B_0$ containing no generic point from the intersection of two irreducible components of B_0, and no generic point from the intersection $A_0 \cap B_0$, and no point from $\{u_0, v_0\}$. Furthermore, we demand that the invertible $\mathscr{O}_{B_0}$-module $\mathscr{O}_{B_0}(H_0 - B_0)$ is ample. Let $X'_0 \to X_0$ be the normalized blowing-up with center $H_0 \subset X_0$, and $A'_0, B'_0 \subset X'_0$ the strict transform of $A_0, B_0 \subset X_0$, respectively. As in the proof for Theorem 10.2, the subscheme $B'_0 \subset X'_0$ is connected and contractible to a point. Define X'_λ and $X' = \varprojlim X'_\lambda$ as in Step 2.

Let $U \subset X$ be the complement of the closed subscheme $X \times_{X_0} H_0$. This open subscheme intersects at least two connected components of $A \cap B$, because $u, v \in U \cap V$. Clearly, the canonical morphism $f : X' \to X$ becomes an isomorphism over U. Furthermore, the intersection $B \cap U$ is connected. To see this, set $H_\lambda = X_\lambda \times_{X_0} H_0$ and $U_\lambda = X_\lambda \smallsetminus H_\lambda$, such that $B \cap U = \varprojlim(B_\lambda \cap U_\lambda)$. In light of [22], Proposition 8.4.1, it suffices to check that $B_\lambda \cap U_\lambda = B_\lambda \smallsetminus H_\lambda$ is connected for each $\lambda \in L$. Since $B_\lambda \subset X_\lambda$ is the support of a connected Cartier divisor, there is a sequence of irreducible components $C_1, \ldots, C_n$ covering B_λ so that each successive intersection $C_{j-1} \cap C_j$ has codimension ≤ 2 in X_λ. Let $f : X_\lambda \to X_0$ be the canonical morphisms, and consider the inclusion

$$f(C_{j-1} \cap C_j) \subset f(C_{j-1}) \cap f(C_j).$$

Since f is finite and dominant, the left hand side has codimension ≤ 2 in X_0, and the $f(C_{j-1}), f(C_j)$ are irreducible components from B_0. By the choice of the ample Cartier divisor $H_0 \subset X_0$, the image $f(C_{j-1} \cap C_j)$ is not entirely contained in H_0. We conclude that $(C_{j-1} \cap C_j) \smallsetminus H_\lambda$ is nonempty, thus $B_\lambda \smallsetminus H_\lambda$ remains connected.

Let $A', B' \subset X'$ be the closure of $A \cap U, B \cap U$. The former comprise a cyclic system, in light of Proposition 11.2. Clearly, the image of $B' \to X_0$ equals B'_0. Replacing X_0 by X'_0, we thus have achieved the situation that $B_0 \subset X_0$ is contractible to a point. We have seen in the beginning of this proof that such a cyclic system $A, B \subset X$ does not exist. □

13. Nisnevich Čech cohomology

We now apply our results to prove the following:

Theorem 13.1. *Let k be a ground field, and X be a quasicompact and separated k-scheme. Then $\check{H}^p(X_{\mathrm{Nis}}, \underline{H}^q(F)) = 0$ for all integers $p \geq 0$, $q \geq 1$ and all abelian Nisnevich sheaves F on X. In particular, the canonical maps*

$$\check{H}^p(X_{\mathrm{Nis}}, F) \longrightarrow H^p(X_{\mathrm{Nis}}, F)$$

are bijective for all $p \geq 0$.

Proof. Recall that $\check{H}^p(X_{\text{Nis}}, \underline{H}^q(F)) = \varinjlim_U \check{H}^p(U_{\text{Nis}}, \underline{H}^q(F))$, where the direct limit runs over the system of all completely decomposed étale surjections $U \to X$. For an explanation of the transition maps, we refer to [32], Chapter II, Section 5.7. Since X is quasicompact and étale morphisms are open, one easily sees that the the subsystem of all étale surjections $U \to X$ with U affine form a cofinal subsystem.

Let $[\alpha] \in \check{H}^p(X_{\text{Nis}}, \underline{H}^q(F))$ be a Čech class with $q \geq 1$. Represent the class by some cocycle $\alpha \in H^q(U^p, F)$, where $U \to X$ is a completely decomposed étale surjection, and U^p denotes the p-fold selfproduct in the category of X-schemes. This selfproduct remains quasicompact, because X is quasiseparated. Since $q \geq 1$, there is a completely decomposed étale surjection $W \to U^p$ with W affine so that $\alpha|W = 0$. It suffices to show that there is a refinement $U' \to U$ so that the structure morphisms $U'^p \to U^p$ factors over $W \to U^p$, because then the class of α in the direct limit of the Čech complex vanishes. From this point on we merely work with the schemes W, U, X and may forget about the sheaf F and the cocycle α.

This allows us to reduce to the case of k-schemes of finite type: Write $X = \varprojlim X_\lambda$ as an inverse limit with affine transition maps of k-schemes X_λ that are of finite type. Passing to a cofinal subset of indices, we may assume that there is a smallest index $\lambda = 0$, so that X_0 is separated ([62], Appendix C, Proposition 7). Since U is quasicompact, our morphisms $U \to X$ is of finite presentation, so we may assume that $U = U_0 \times_{X_0} X_0$ for some X_0-scheme U_0 of finite presentation. Similarly, we can assume $W = W_0 \times_{U_0^p} U^p$ for some U_0^p-scheme W_0 of finite presentation. We may assume that $U_0 \to X_0$ and $W_0 \to U_0^p$ are étale ([23], Proposition 17.7.8) and surjective ([22], Proposition 8.10.5). According to Lemma 13.2 below, we can also impose that these morphisms are completely decomposed. Summing up, it suffices to produce a refinement $U_0' \to U_0$ so that $U_0'^p \to U_0^p$ factors over W_0. In other words, we may assume that X is of finite type over the ring $\mathbb{Z}$. In particular, X has only finitely many irreducible components, the normalization of the corresponding integral schemes are finite over X, and X is noetherian.

Next, we make use of the weak Cartan–Artin property. Indeed, according to Artin's induction argument in [3], Theorem 4.1, applied to the Nisnevich topology rather than the étale topology, we see that it suffices to check that X satisfies the weak Cartan–Artin property. Let X_0 be the normalization of some irreducible component of X, viewed as an integral scheme. According to Propositions 6.4 and 6.6, it is permissible to replace X by X_0. Changing notation, we write $X = \operatorname{TSC}(X_0)$ for the total separable closure of the integral scheme X_0. In light of Theorem 7.1, it is enough to verify the weak Cartan–Artin property for X.

Here, our problem reduces to a simple geometric statement: According to Theorem 7.6, it suffices to check that for each pair of irreducible closed subsets $A, B \subset X$, the intersection $A \cap B$ remains irreducible. And indeed, this holds by Theorem 12.1. Note that only in this very last step, we have used that X is separated, and the existence of a ground field k. □

In the preceding proof, we have used the following facts:

Lemma 13.2. *Suppose that $X = \varprojlim X_\lambda$ is a filtered inverse system of quasicompact schemes with affine transition maps, and $U \to X$ is a completely decomposed étale surjection, with U quasicompact. Then there is an index α and a completely decomposed étale surjection $U_\alpha \to X_\alpha$ with $U = X \times_{X_\alpha} U_\alpha$.*

Proof. The morphism $U \to X$ is of finite presentation, because it is étale and its domain is quasicompact. According to [22], Theorem 8.8.2 there is an X_α-scheme U_α of finite presentation with $U = X \times_{X_\alpha} U_\alpha$ for some index α. We may assume that $U_\alpha \to X_\alpha$ is surjective ([22], Theorem 8.10.5) and étale ([23], Proposition 17.7.8).

For each index $\lambda \geq \alpha$, let $F_\lambda \subset X_\lambda$ be the subset of all points $s \in X_\lambda$ so that the étale surjection $U_\lambda \otimes \kappa(s) \to \operatorname{Spec} \kappa(s)$ is completely decomposed, that is, admits a section. According to Lemma 13.3 below, this subset is ind-constructible. In light of [23], Corollary 8.3.4, it remains to show that the inclusion $\bigcup_\lambda u_\lambda^{-1}(F_\lambda) \subset X$ is an equality, where $u_\lambda : X \to X_\lambda$ denote the projections.

Fix a point $x \in X$, and choose a point $u \in U$ above so that the inclusion $k(x) \subset \kappa(u)$ is an equality. Let $x_\lambda \in X_\lambda$ and $u_\lambda \in U_\lambda$ be the respective image points. Then $U_x = \varprojlim (U_\lambda)_{x_\lambda}$, because inverse limits commute with fiber products. Furthermore, the section of $U_x \to \operatorname{Spec} \kappa(x)$ corresponding to u comes from a section of $(U_\lambda)_{x_\lambda} \to \operatorname{Spec} \kappa(x_\lambda)$, for some index λ, by [22], Theorem 8.8.2. $\square$

Recall that a subset $F \subset X$ is called *ind-constructible* if each point $x \in X$ admits an open neighborhood $x \in U$ so that $F \cap U$ is the union of constructible subsets of U (compare [17], Chapter I, Section 7.2 and Chapter 0, Section 2.2).

Lemma 13.3. *Let $f : U \to X$ be a étale surjection. Then the subset $F \subset X$ of points $x \in X$ for which $U_x \to \operatorname{Spec} \kappa(x)$ admits a section is ind-constructible.*

Proof. The problem is local, so we may assume that $X = \operatorname{Spec}(R)$ is affine. Fix a point $x \in X$ so that $U_x \to \operatorname{Spec} \kappa(x)$ admits a section, and let $A \subset X$ be its closure. The section extends to a section for $U_A \to A$ over some dense open subset $C \subset A$. Shrinking C if necessary, we may assume that $C = A \cap \operatorname{Spec}(R_g)$ for some $g \in R$. Then $C \subset X$ is constructible and contains x. $\square$

14. Appendix A: Connected components of schemes

Let X be a topological space and $a \in X$ a point. The corresponding *connected component* $C = C_a$ is the union of all connected subsets containing a. This is then the largest connected subset containing a, and it is closed but not necessarily open. In contrast, the *quasicomponent* $Q = Q_a$ is defined as the intersection of all open-and-closed neighborhoods of a, which is also closed but not necessarily open. Clearly, we have $C \subset Q$. Equality holds if X has only finitely many quasicomponents, or if X is compact or locally connected, but in general there is a strict inclusion.

The simplest example with $C \subsetneqq Q$ seems to be the following: Let U be an infinite discrete space, $Y_i = U \cup \{a_i\}$ be two copies of the Alexandroff compactification, and $X = Y_1 \cup Y_2$ the space obtained by gluing along the open subset U, which is quasicompact but not Hausdorff. Then all connected components are singletons, but the discrete space $Q = \{a_1, a_2\}$ is a quasicomponent.

Assume that X is a locally ringed space. Write $R = \Gamma(X, \mathscr{O}_X)$ for the ring of global sections and $\mathfrak{p} \subset R$ for the prime ideal of all global sections vanishing at our point $a \in X$. Obviously, the open-and-closed neighborhoods $U \subset X$ correspond to the idempotent elements $\epsilon \in R \smallsetminus \mathfrak{p}$, via $U = X_\epsilon = \{x \in X \mid \epsilon(x) = 1\}$. Let $L \subset R \smallsetminus \mathfrak{p}$ the multiplicative subsystem of all idempotent elements. We put an order relation by declaring $\epsilon \leq \epsilon'$ if $\epsilon' \mid \epsilon$. Then $\epsilon \mapsto X_\epsilon$ is a filtered inverse system of subspaces, and the quasicomponent is

$$Q = \bigcap_{\epsilon \in L} X_\epsilon = \varprojlim_{\epsilon \in L} X_\epsilon.$$

Now suppose that X is a scheme. Then each inclusion morphism $\iota_\epsilon : X_\epsilon \to X$ is affine, hence $\mathscr{A}_\iota = \iota_{\epsilon *}(\mathscr{O}_{X_\epsilon})$ are quasicoherent, and the same holds for $\mathscr{A} = \varinjlim \mathscr{A}_\epsilon$. In turn, we have $Q = \varprojlim_{\epsilon \in L} X_\epsilon = \operatorname{Spec}(\mathscr{A})$, which puts a scheme structure on the quasicomponent. The quasicoherent ideal sheaf for the closed subscheme $Q \subset X$ is $\mathscr{I} = \bigcup_{\epsilon \in L}(1-\epsilon)\mathscr{O}_X$. The following observation is due to Ferrand in the affine case, and Lazard in the general case (see [44], Proposition 6.1 and Corollary 8.5):

Lemma 14.1. *Let X be a quasicompact and quasiseparated scheme and $a \in X$ be a point. Then we have an equality $C = Q$ between the connected component and the quasicomponent containing $a \in X$.*

If $X = \operatorname{Spec}(R)$ is affine, and $a \in X$ is the point corresponding to a prime ideal $\mathfrak{p} \subset R$, then the corresponding $C = Q$ is defined by the ideal $\mathfrak{a} = \bigcup Re$, where e runs through the idempotent elements in $\mathfrak{p}$. Clearly, we have $R/\mathfrak{a} = \varinjlim_e A/eA$. The latter description was already given by Artin in [3], page 292.

Let X_0 be a scheme, and $X = \varprojlim_{\lambda \in L} X_\lambda$ be an inverse system of affine X_0-schemes. The argument for the previous Lemma essentially depends on [22], Proposition 8.4.1 (ii). There, however, it is incorrectly claimed that a sum decomposition of X comes from a sum decomposition of X_λ for sufficiently large $\lambda \in L$, provided that X_0 is merely quasicompact. This only becomes true only under the additional hypothesis of quasiseparatedness:

Proposition 14.2. *Let X_0 be a quasicompact and quasiseparated scheme, and X_λ an filtered inverse system of affine X_0-schemes, and $X = \varprojlim X_\lambda$. If $X = X' \cup X''$ is a decomposition into disjoint open subsets, that there is some $\lambda \in L$ and a decomposition $X_\lambda = X'_\lambda \cup X''_\lambda$ into disjoint open subset so that $X', X'' \subset X$ are the respective preimages.*

Proof. This is a special case of [22], Theorem 8.3.11, but I would like to give an alternative proof relying on absolute noetherian approximation rather than ind- and pro-constructible sets: Choose a $\mathbb{Z}$-scheme of finite type S so that there is an affine morphism $X_0 \to S$ ([62], Appendix C, Theorem 9). Replacing X_0 by S, we may assume that X_0 is noetherian. Let $\mathscr{A}_\lambda$ be the quasicoherent $\mathscr{O}_{X_0}$-algebras corresponding to the X_λ, such that X is the relative spectrum of $\mathscr{A} = \varinjlim(\mathscr{A}_\lambda)$. Since the topological space of X_0 is noetherian, the canonical map

$$\varinjlim H^0(X_0, \mathscr{A}_\mathscr{L}) \longrightarrow H^0(X_0, \varinjlim \mathscr{A}_\lambda)$$

is bijective, see [34], Chapter III, Proposition 2.9. Thus each idempotent $e \in \Gamma(X_0, \mathscr{A})$ comes from an element $e_\lambda \in \Gamma(X_0, \mathscr{A}_\lambda)$. Passing to a larger index, we may assume that $e_\lambda^2 = e_\lambda$, such that our element e_λ becomes idempotent. The statement follows from the aforementioned correspondence between idempotent global sections and open-and-closed subsets. $\square$

Here is a counterexamples with X_0 not quasiseparated: Let U be an infinite discrete set, and $X = Y_1 \cup Y_2$ be the gluing of two copies $Y = Y_1 = Y_2$ of the Alexandroff compactification along the open subset $U \subset Y_i$ mentioned above. We may regard the Alexandroff compactification as a profinite space, by choosing a total order on U: Let L be the set of all finite subsets of U, ordered by inclusion. Given $\lambda \in L$, we write $F_\lambda \subset U$ for the corresponding finite subset. For every $\lambda \leq \mu$, let $F_\mu \to F_\lambda$ by the retraction of the canonical inclusion $F_\lambda \subset F_\mu$ sending the complementary points to the largest element $f_\mu \in F_\mu$. Then one easily sees that $Y = \varprojlim F_\lambda$, where the point at infinity becomes the tuple of largest elements $a = (f_\lambda)_{\lambda \in L}$.

To proceed, fix a ground field k. Then we may endow the profinite space Y with the structure of an affine k-scheme, by regarding F_λ as the spectrum of the k-algebra $R_\lambda = \mathrm{Hom}_{\mathrm{Set}}(F_\lambda, k)$. In turn, we regard the gluing $X = Y_1 \cup Y_2$ as a k-scheme. This scheme is quasicompact, but not quasiseparated, because the intersection $Y_1 \cap Y_2 = U$ is infinite and discrete. Consider the subset $V_\lambda = X \smallsetminus F_\lambda$, which is open-and-closed. Moreover, the inclusion morphism $V_\lambda \to X$ is affine, and $\varprojlim_{\lambda \in L} V_\lambda = Q = \{a_1, a_2\}$ is the disconnected quasicomponent. The sum decomposition of the quasicomponent Q, however, does not come from a sum decomposition of any V_λ.

Here is a counterexample with X_0 not quasicompact: Let X_0 be the disjoint union of two infinite chains $C' = \bigcup C'_n$ and $C'' = \bigcup C''_n$, $n \in \mathbb{Z}$ of copies of the projective line over the ground field k, together with further copies B_n of the projective line joining the intersection $C'_{n-1} \cap C'_n$ with $C''_{n-1} \cap C''_n$. Let L be the collection of all finite subset of $\mathbb{Z}$, ordered by inclusion. Consider the inverse system of closed subsets $X_\lambda \subset X$ consisting of $C' \cup C''$ and the union $\bigcup_{n \notin \lambda} B_n$. Then all X_λ are connected, but $X = \varprojlim(X_\lambda) = \bigcap X_\lambda = C' \amalg C''$ becomes disconnected.

15. Appendix B: Čech and sheaf cohomology

Here we briefly recall H. Cartan's Criterion for equality for Čech and sheaf cohomology, which is described in Godement's monograph [32], Chapter II, Theorem 5.9.2. Throughout we work in the context of topoi and sites. The general reference is [4]. Suppose $\mathscr{E}$ is a topos. Choose a site $\mathscr{C}$ and an equivalence $\mathrm{Sh}(\mathscr{C}) = \mathscr{E}$, and make the choice so that the category $\mathscr{C}$ has a terminal object $X \in \mathscr{C}$ and all fiber products, and so that the Grothendieck topology is given in terms of a pretopology of coverings. Suppose we additionally have a subcategory $\mathscr{B} \subset \mathscr{C}$ such that every object $U \in \mathscr{C}$ admits a covering $(U_\alpha \to U)_\alpha$ with $U_\alpha \in \mathscr{B}$, and that $\mathscr{B}$ has all fiber products. Then the subcategory inherits a pretopology, and the restriction functor of sheaves on $\mathscr{C}$ to sheaves on $\mathscr{B}$ is an equivalence.

For each abelian sheaf F on $\mathscr{C}$ and each object $U \in \mathscr{C}$, we write $H^p(U, F)$ for the cohomology groups in the sense of derived functors, and

$$\check{H}^p(U, F) = \varinjlim \check{H}^p(\mathfrak{V}, \check{H}^q(F)),$$

for the Čech cohomology groups, where the direct limit runs over all coverings $\mathfrak{V} = (U_\alpha \to U)_\alpha$, and the transition maps are defined as in [32], Chapter II, Section 5.7. Note that me may restrict to those coverings with all $U_\alpha \in \mathscr{B}$.

Let F be an abelian sheaf on the site $\mathscr{C}$. We denote by $\underline{H}^q(F)$ the presheaf $U \mapsto H^q(U, F)$. The functors $F \mapsto \underline{H}^p(F)$, $p \geq 0$ form a δ-functor from the category of abelian sheaves to the category of abelian presheaves. This δ-functor is universal, because it vanishes on injective objects for $p \geq 1$.

The inclusion functor $F \mapsto \underline{H}^0(F)$ from the category of abelian sheaves to the category of abelian presheaves is right adjoint to the sheafification functor. This adjointness ensures that $F \mapsto \underline{H}^0(F)$ sends injective objects to injective objects. Furthermore, $F \mapsto \check{H}^p(U, F)$, $p \geq 0$ form a universal δ-functors from the category of abelian presheaves on U to the category of abelian groups, see [61], Theorem 2.2.6. In turn, we get a Grothendieck spectral sequence

$$\check{H}^p(\mathfrak{U}, \underline{H}^q(F)) \Longrightarrow H^{p+q}(U, F), \tag{1}$$

which we call *Čech-to-sheaf-cohomology spectral sequence.*

Theorem 15.1. (H. Cartan) *The following are equivalent:*

(i) $H^q(U, F) = 0$ *for all* $U \in \mathscr{B}$ *and* $q \geq 1$.
(ii) $\check{H}^q(U, F) = 0$ *for all* $U \in \mathscr{B}$ *and* $q \geq 1$.

If these equivalent conditions are satisfied, then $\check{H}^p(X, \underline{H}^q(F)) = 0$ *for all* $p \geq 0$ *and* $q \geq 1$*, and the edge maps* $\check{H}^p(X, F) \to H^p(X, F)$ *are bijective for all* $p \geq 0$.

Proof. Suppose (i) holds. Then $\underline{H}^q(F) = 0$ for all $q \geq 1$. This ensures that $\check{H}^p(V, \underline{H}^q(F)) = 0$ for all $V \in \mathscr{C}$. In turn, the edge map $\check{H}^p(V, F) \to H^p(U, F)$ in the Čech-to-sheaf-cohomology spectral sequence (1) is bijective. For $V = U \in \mathscr{B}$, this gives (ii), while for $V = X$ we get the amendment of the statement.

Conversely, suppose that (ii) holds. We inductively show that $H^p(U, F) = 0$, or equivalently that the edge maps $\check{H}^p(U, F) \to H^p(U, F)$ are bijective, for all $p \geq 1$ and all $U \in \mathscr{B}$. First note that $\check{H}^0(U, \underline{H}^r(F)) = 0$ for all $r \geq 1$, because the sheafification of $\underline{H}^r(F)$ vanishes. Thus the case $p = 1$ in the induction follows from the short exact sequence

$$0 \longrightarrow \check{H}^1(U, F) \longrightarrow H^1(U, F) \longrightarrow \check{H}^0(U, \underline{H}^1(F)).$$

Now let $p \geq 1$ be arbitrary, and suppose that $H^p(U, F) = 0$ for $1, \ldots, p-1$ and all $U \in \mathscr{B}$. The argument in the previous paragraph gives $\check{H}^r(U, \underline{H}^s(F)) = 0$ for all $0 < s < p$ and $r \geq 1$, and we already noted that $\check{H}^0(U, \underline{H}^p(F)) = 0$. Consequently, the edge map $\check{H}^p(U, F) \to H^p(U, F)$ must be bijective. $\square$

16. Appendix C: Inductive dimension in general topology

The proof of the crucial Proposition 4.5 depends on results from dimension theory that are perhaps not so well-known outside general topology. Here we briefly review the relevant material. For a comprehensive treatment, see for example Pears's monograph [51], Chapter 4.

Let X be a topological space. There are several ways to obtain suitable notions of dimension that are meaningful for compact spaces. One of them is the *large inductive dimension* $\operatorname{Ind}(X) \geq -1$, which goes back to Brouwer [10]. Here one first defines $\operatorname{Ind}(X) \leq n$ as a property of topological spaces by induction as follows: The induction starts with $n = -1$, where $\operatorname{Ind}(X) \leq -1$ simply means that X is empty. Now suppose that $n \geq 0$, and that the property is already defined for $n-1$. Then $\operatorname{Ind}(X) \leq n$ means that for all closed subsets $A \subset X$ and all open neighborhoods $A \subset V$, there is a smaller open neighborhood $A \subset U \subset V$ whose boundary $\Phi = \overline{U} \smallsetminus U$ has $\operatorname{Ind}(\Phi) \leq n-1$. Now one defines $\operatorname{Ind}(X) = n$ as the least integer $n \geq -1$ for which $\operatorname{Ind}(X) \leq n$ holds, or $n = \infty$ if no such integer exists.

A variant is the *small inductive dimension* $\operatorname{ind}(X) \geq -1$, which is defined in an analogous way, but with points $a \in X$ instead of closed subsets $A \subset X$. If every point $a \in X$ is closed, an easy induction gives

$$\operatorname{ind}(X) \leq \operatorname{Ind}(X).$$

There are, however, compact spaces with strict inequality ([51], Chapter 8, §3). Moreover, little useful can be said if the space contains non-closed points. For example, a finite linearly ordered space $X = \{x_0, \ldots, x_n\}$ obviously has $\mathrm{Ind}(S) = 0$ and $\mathrm{ind}(S) = n$. Note that such spaces arise as spectra of valuation rings R with Krull dimension $\dim(R) = n$.

Recall that a space X is called *at most zero-dimensional* if its topology admits a basis consisting of open-and-closed subsets $U \subset X$. It is immediate that this is equivalent to $\mathrm{ind}(X) \leq 0$. In fact, both small and large inductive dimension can be seen as generalizations of this concept ([51], Chapter 4, Proposition 1.1 and Corollary 2.2), at least for compact spaces:

Proposition 16.1. *If X is compact, then the following three conditions are equivalent:* (i) *X is at most zero-dimensional;* (ii) $\mathrm{ind}(X) \leq 0$*;* (iii) $\mathrm{Ind}(X) \leq 0$.

There are several *sum theorems*, which express the large inductive dimension of a covering $X = \bigcup A_\lambda$ in terms of the large inductive dimension of the subspaces $A_\lambda \subset X$. The following sum theorem is very useful for us; its proof is tricky, but only relies only basic notations of set-theoretical topology ([51], Chapter 4, Proposition 4.13):

Proposition 16.2. *Suppose that X is compact, and $X = A \cup B$ is a closed covering. If the intersection $A \cap B$ is at most zero-dimensional, and the subspaces have* $\mathrm{Ind}(A), \mathrm{Ind}(B) \leq n$*, then* $\mathrm{Ind}(X) \leq n$.

From this we deduce the following fact, which, applied to $X_i = \mathrm{Max}(Y_i)$, enters in the proof of Proposition 4.5.

Proposition 16.3. *Suppose that X is compact, and that $X = X_1 \cup \ldots \cup X_n$ is a closed covering, where the X_i are at most zero-dimensional. Then X is at most zero-dimensional.*

Proof. By induction it suffices to treat the case $n = 2$. Since X_i are at most zero-dimensional, the same holds for the subspace $X_1 \cap X_2 \subset X_i$. The spaces X_i are compact, because they are closed inside the compact space X. According to Proposition 16.1, we have $\mathrm{Ind}(X_i) \leq 0$. The sum theorem above yields $\mathrm{Ind}(X) \leq 0$, and Proposition 16.1 again tells us that X is at most zero-dimensional. $\square$

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