THERE IS NO ENRIQUES SURFACE OVER THE INTEGERS

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Third revised version, 19 July 2022

Abstract. We show that there is no family of Enriques surfaces over the ring of integers. This extends non-existence results of Minkowski for families of finite étale schemes, of Tate and Ogg for families of elliptic curves, and of Fontaine and Abrashkin for families of abelian varieties and more general smooth proper schemes with certain restrictions on Hodge numbers. Our main idea is to study the local system of numerical classes of invertible sheaves. Among other things, our result also hinges on counting rational points, Lang's classification of rational elliptic surfaces in characteristic two, the theory of exceptional Enriques surfaces due to Ekedahl and Shepherd-Barron, some recent results on the base of their versal deformation, Shioda's theory of Mordell–Weil lattices, and an extensive combinatorial study for the pairwise interaction of genus-one fibrations.

Contents

Introduction

Which smooth proper morphism $X \to \mathrm{Spec}(\mathbb{Z})$ do exist? This tantalizing question seems to go back to Grothendieck ([43], page 242), but it is rooted in algebraic

2010 *Mathematics Subject Classification.* 14J28, 14J26, 14J27, 14G15, 14K30.

number theory: By Minkowski's discriminant bound, for each number field $K \neq \mathbb{Q}$ the corresponding number ring is not étale over the ring $R = \mathbb{Z}$, so the only examples in relative dimension $n = 0$ are finite sums of $\operatorname{Spec}(\mathbb{Z})$. Moreover, this reduces the general question to the case $\Gamma(X, \mathscr{O}_X) = \mathbb{Z}$, such that the structure morphism $X \to \operatorname{Spec}(\mathbb{Z})$ is a contraction of fiber type.

A result of Ogg [50], which he himself attributes to Tate, asserts that each Weierstraß equation with integral coefficients has discriminant $\Delta \neq \pm 1$. Thus there is no family of elliptic curves $E \to \operatorname{Spec}(\mathbb{Z})$. In other words, the Deligne–Mumford stack $\mathscr{M}_{1,1}$ of smooth pointed curves of genus one has fiber category $\mathscr{M}_{1,1}(\mathbb{Z}) = \varnothing$. This was generalized by Fontaine [21], who showed that there are no families $A \to \operatorname{Spec}(\mathbb{Z})$ of non-zero abelian varieties. In other words, the fiber category $\mathscr{M}_{\mathrm{Ab}}(\mathbb{Z})$ for the stack of abelian varieties contains only trivial objects. This was independently established by Abrashkin [3], who also settled the case of abelian surfaces and threefolds earlier ([1], [2]).

The theory of relative Picard schemes [23] then shows that there is no family of smooth curves $C \to \operatorname{Spec}(\mathbb{Z})$ of genus $g \geq 2$. In other words, the Deligne–Mumford stack $\mathscr{M}_g$ of smooth curves of genus g has fiber category $\mathscr{M}_g(\mathbb{Z}) = \varnothing$. Since the ring of integers has trivial Picard and Brauer groups, the only example in relative dimension $n = 1$ are finite sums of the projective line $\mathbb{P}^1_{\mathbb{Z}}$. Abrashkin [4] and Fontaine [22] also established that for each smooth proper $X \to \operatorname{Spec}(\mathbb{Z})$, the Hodge numbers of the complex fiber $V = X_{\mathbb{C}}$ are very restricted, namely $H^j(V, \Omega^i_{V/\mathbb{C}}) = 0$ for $i + j \leq 3$ and $i \neq j$. Note that this ensures that the stack $\mathscr{M}_{\mathrm{HK}}$ of hyperkähler varieties has fiber category $\mathscr{M}_{\mathrm{HK}}(\mathbb{Z}) = \varnothing$.

The only examples for smooth proper schemes over the integers that easily come to mind are the projective space $\mathbb{P}^n$ and schemes stemming from this, for example sums, products, blowing ups with respect to linear centers, or the Hilbert scheme $X = \operatorname{Hilb}^n_{\mathbb{P}^2/\mathbb{Z}}$. Similar examples arise from G/P, where G is a Chevalley group and P is a parabolic subgroup, and from torus embeddings $\operatorname{Temb}_\Delta$, stemming from a fan of regular cones $\sigma \in \Delta$. I am not aware of any other construction. It would be interesting to understand the case of elliptic surfaces over $\mathbb{P}^1$.

In light of the Enriques classification of algebraic surfaces, it is natural to consider families $Y \to \operatorname{Spec}(\mathbb{Z})$ of smooth proper surfaces with $c_1 = 0$. These are the abelian surfaces, bielliptic surfaces, K3 surfaces and Enriques surfaces. They can be distinguished by their second Betti number, which takes the values $b_2 = 6, 2, 22, 10$. The former two are ruled out by the Hodge numbers $h^{0,1} \geq 1$ of the complex fiber. Similarly, K3 surfaces do not occur, by $h^{2,0} = 1$. So only the Enriques surfaces remain, which over the complex numbers indeed have $h^{i,j} = 0$ for $i + j \leq 3$ and $i \neq j$.

At first glance, the family of K3 coverings $X \to Y$ seems to show that there is no family of Enriques surfaces. There is, however, a crucial loophole: At prime $p = 2$, the K3 cover can degenerate into a *K3-like covering*, and then the fiber $X \otimes \mathbb{F}_2$ becomes singular. In light of this, it actually seems possible that such families of Enriques surfaces exist. The main result of this paper asserts that it does not happen:

Theorem. (see Thm. 5.1) *The stack $\mathscr{M}_{\mathrm{Enr}}$ of Enriques surfaces has fiber category $\mathscr{M}_{\mathrm{Enr}}(\mathbb{Z}) = \varnothing$. In other words, there is no smooth proper family of Enriques surfaces over the ring $R = \mathbb{Z}$.*

Note that the finiteness of the fiber categories $\mathscr{M}_{\mathrm{Enr}}(R)$ for rings of integers R in number fields was recently established by Takamatsu [63]. A similar finiteness holds for abelian varieties by Faltings [19] and Zarhin [69], and also for K3 surfaces by André [5]. Results of this type are often refereed to as the *Shafarevich Conjecture* [59]

The proof of our non-existence result is long and indirect, and is given in the final Section 15. It is actually a consequence of the following:

Theorem. (see Thm. 15.1) *There is no Enriques surface Y over $k = \mathbb{F}_2$ that is non-exceptional and has constant Picard scheme $\mathrm{Pic}_{Y/k} = (\mathbb{Z}^{\oplus 10} \oplus \mathbb{Z}/2\mathbb{Z})_k$.*

The *exceptional Enriques surfaces* where introduced by Ekedahl and Shepherd-Barron [17]. They indeed can be discarded from our considerations, as I recently showed in [57] that they do not admit a lifting to the ring $W_2(\mathbb{F}_2) = \mathbb{Z}/4\mathbb{Z}$. The above result will be deduced from an explicit classification of *geometrically rational elliptic surfaces* $\phi : J \to \mathbb{P}^1$ over $k = \mathbb{F}_2$ that have constant Picard scheme, and satisfy certain additional technical conditions stemming from the theory of Enriques surfaces:

Theorem. (see Thm. 10.4 and Thm. 10.5) *Up to isomorphisms, there are exactly eleven Weierstraß equations $y^2+a_1xy+\ldots = x^3+a_2x^2+\ldots$ with coefficients $a_i \in \mathbb{F}_2[t]$ that define a geometrically rational elliptic surface $\phi : J \to \mathbb{P}^1$ with constant Picard scheme $\mathrm{Pic}_{J/\mathbb{F}_2}$ having at most one rational point $a \in \mathbb{P}^1$ where J_a is semistable or supersingular.*

For our Enriques surfaces Y over the prime field $k = \mathbb{F}_2$ with constant Picard scheme, this narrows down the possible configurations of fibers over the three rational points $t = 0, 1, \infty$ in genus-one fibrations $\varphi : Y \to \mathbb{P}^1$, by passing to the jacobian fibration $\phi : J \to \mathbb{P}^1$ and using a result of Liu, Lorenzini and Raynaud [42].

It turns out that there are fifteen possible configurations, where the additional four cases come from quasielliptic fibrations. In all but one situation there is precisely one "non-reduced" Kodaira symbol, from the list $\mathrm{II}^*, \mathrm{III}^*, \mathrm{IV}^*, \mathrm{I}_4^*, \mathrm{I}_2^*, \mathrm{I}_1^*$. We then make an extensive combinatorial analysis for the possible interaction of pairs of genus-one fibrations $\varphi, \psi : Y \to \mathbb{P}^1$ whose intersection number takes the minimal value $\varphi^{-1}(\infty) \cdot \psi^{-1}(\infty) = 4$, in the spirit of Cossec and Dolgachev [11], Chapter III, §5. This only becomes feasible after discarding the exceptional Enriques surfaces, which roughly speaking have "too few" fibrations and "too large" degenerate fibers.

The paper is organized as follows: In Section 1 we treat the group of numerical classes of invertible sheaves as a local system $\mathrm{Num}_{X/k}$, that is, a sheaf on the étale site of the ground field k. Here the main result is that if it is constant, then every projective contraction of the base-change to k^{alg} descends to a contraction of X. Geometric consequences for smooth surfaces $X = S$ are examined in more detail in Section 2. Then we analyze curves of canonical type in Section 3. They occur in genus-one fibrations, and we stress arithmetical aspects. In Section 4 we review the theory of Enriques surfaces over algebraically closed fields, including the notion of exceptional Enriques surfaces. Section 5 contains a detailed study of families of Enriques surfaces, in particular over the ring of integers. In Section 6 we collect basic facts on geometrically rational elliptic surfaces, again from an arithmetical point of view. In Section 7 we turn to counting points over finite fields with the Weil Conjectures, and give some formulas that apply to Enriques surfaces and geometrically rational elliptic surfaces whose local system $\text{Num}_{X/k}$ is constant. Section 8 contains an observation on the behavior of reduced fibers in passing from an elliptic fibration to its jacobian fibration. Section 9 contains a main technical result: If Y is an Enriques surface over $k = \mathbb{F}_2$ with constant Picard scheme, then the resulting geometrically rational surfaces J, arising as jacobian for elliptic fibrations, also has constant Picard scheme. Using this, we classify in Section 10 all Weierstraß equations with coefficients from $\mathbb{F}_2[t]$ that possibly could describe such J. In Section 11, we make a list of the possible fiber configurations over the rational points $t = 0, 1, \infty$ that could arise for the Enriques surface Y, where we also take into account quasielliptic fibrations. This list is narrowed down to two configurations in Sections 12 and 13. In Section 14 we work again over arbitrary algebraically closed ground fields k, and show that Enriques surfaces all whose genus-one fibrations are subject to certain severe combinatorial restrictions do not exist. In the final Section 15 we combine all these observations and deduce our main results.

Acknowledgement. The research was supported by the Deutsche Forschungsgemeinschaft by the grant *SCHR 671/6-1 Enriques-Mannigfaltigkeiten*. It was also conducted in the framework of the research training group *GRK 2240: Algebro-geometric Methods in Algebra, Arithmetic and Topology*. I wish to thank Will Sawin for useful remarks, and Matthias Schütt for helpful comments, in particular for pointing out [61] and [58], which correct two entries in the classification of Mordell–Weil lattices for rational elliptic surfaces. I would also like to thank the referees for very careful and thorough reading and many valuable suggestions, which helped to improve the paper and remove mistakes. In particular, for pointing out a defective argument in the first version concerning two-sections in what are now Propositions 12.2 and 12.3.

1. The local system of numerical classes

Throughout, k denotes a ground field of characteristic $p \geq 0$. Let X be a proper scheme, $\text{Pic}_{X/k}$ the Picard scheme, and $\text{Pic}^\tau_{X/k}$ be the open subgroup scheme parameterizing numerically trivial invertible sheaves (see for example [10], Exposé XII, Corollary 1.5 and Exposé XIII, Theorem 4.7, compare also [41], Section 2). The latter is *algebraic*, which means that the structure morphism is of finite type and separated, and the resulting quotient

$$\text{Num}_{X/k} = (\text{Pic}_{X/k})/(\text{Pic}^\tau_{X/k})$$

is an étale group scheme. As such, it corresponds to the Galois representation on the stalk

$$\text{Num}_{X/k}(k^{\text{alg}}) = \text{Num}_{X/k}(k^{\text{sep}}) = \mathbb{Z}^{\oplus\rho},$$

where $k^{\text{sep}} \subset k^{\text{alg}}$ are chosen separable and algebraic closures, and $\rho \geq 0$ denotes the Picard number for the base-change $\bar{X} = X \otimes k^{\text{alg}}$. The case $\rho = 0$ is also allowed, it may occur for schemes without ample sheaves ([54], Section 3). One may call

$\mathrm{Num}_{X/k}$ the *local system of numerical classes*. Note that all these group schemes exist, even if X fails to be reduced, irreducible, or equidimensional.

We say that $\mathrm{Num}_{X/k}$ is a *constant local system* if the corresponding Galois representation $\mathrm{Gal}(k^{\mathrm{sep}}/k) \to \mathrm{GL}_\rho(\mathbb{Z})$ is trivial. In other words, the group scheme $\mathrm{Num}_{X/k}$ is isomorphic to $(\mathbb{Z}^{\oplus\rho})_k = \bigcup \mathrm{Spec}(k)$, where the disjoint union runs over all elements $l \in \mathbb{Z}^{\oplus\rho}$.

In this section, we collect some noteworthy consequences of this condition. It can be solely characterized in terms of invertible sheaves as follows:

Lemma 1.1. *The group scheme* $\mathrm{Num}_{X/k}$ *is constant if and only if there is an integer* $d \geq 0$ *with the following property: For every invertible sheaf* $\mathscr{L}$ *on* $\bar{X}$ *there is a numerically trivial invertible sheaf* $\mathscr{N}$ *on* $\bar{X}$ *such that* $\mathscr{L}^{\otimes d} \otimes \mathscr{N}$ *is the pullback of some invertible sheaf on* X*.*

Proof. The condition is sufficient: It ensures that the Galois action on the stalk $\mathrm{Num}_{X/k}(k^{\mathrm{alg}}) = \mathbb{Z}^{\oplus\rho}$ becomes trivial on some subgroup of finite index. Hence it is trivial after tensoring with $\mathbb{Q}$, and thus on the torsion-free group $\mathbb{Z}^{\oplus\rho}$.

The condition is also necessary: Each rational point $l \in \mathrm{Num}_{X/k}$ can be seen as a representable torsor $T \subset \mathrm{Pic}_{X/k}$ for the algebraic group scheme $P = \mathrm{Pic}^\tau_{X/k}$. Fix a closed point $a \in T$. The torsor becomes trivial after base-changing to the finite extension field $E = \kappa(a)$. So we may regard the isomorphism class of T as an element in $l \in H^1(k, P)$, where the cohomology is taken with respect to the finite flat topology.

Such classes have finite order, which follows from general principles: Set $X' = X \otimes E$ and $P' = \mathrm{Pic}_{X'/E}$, and view them as k-schemes. First note that for each k-algebra A, the projection $\mathrm{pr} : X' \otimes A \to X \otimes A$ is locally free of rank $n = [E : k]$, so besides the pull-back $\mathrm{pr}^* : \mathrm{Pic}(X \otimes A) \to \mathrm{Pic}(X' \otimes A)$ we also have a *norm map* N in the other direction, where $N \circ \mathrm{pr}^*$ is multiplication by n (for details see [25], Section 6.5). Now consider the diagram

$$\begin{array}{ccccccc} & & & & H^1(k,P) & & \\ & & \overset{N}{\nearrow} & \swarrow^{\mathrm{pr}^*} & \downarrow{\scriptstyle \mathrm{pr}^*} & & \\ 0 \longrightarrow & H^1(k, \mathrm{pr}_* P') & & \longrightarrow & H^1(E, P') & \longrightarrow & H^0(k, R^1 \mathrm{pr}_* P'). \end{array}$$

Here the lower sequence is exact, coming from the Leray–Serre spectral sequence. Note that the term on the right is zero, because the coefficient sheaf $R^1 \mathrm{pr}_* P'$ vanishes, by the arguments for Lemma 1.4 below. Our class $l \in H^1(k, P)$ becomes trivial in $H^1(E, P')$, hence $n \cdot l = N(\mathrm{pr}^*(l)) = 0$.

Summing up, there is some $n \geq 1$ such that nl comes from a rational point $l' \in \mathrm{Pic}_{X/k}$. We next show that some multiple of l' comes from an invertible sheaf. Set $S = \mathrm{Spec}(k)$. The Leray–Serre spectral sequence for the structure morphism $h : X \to S$ yields an exact sequence

$$H^1(X, \mathbb{G}_m) \longrightarrow H^0(S, R^1 h_*(\mathbb{G}_m)) \longrightarrow H^2(S, h_*(\mathbb{G}_{m,X})).$$

The arrow on the left coincides with the canonical map $\mathrm{Pic}(X) \to \mathrm{Pic}_{X/k}(k)$. The term on the right is a torsion group by Lemma 1.4 below. Consequently there is some integer $n' \geq 1$ so that the point $n'l'$ comes from some invertible sheaf $\mathscr{L}$ on X.

Applying the above reasoning for the members of some generating set $l_1, \ldots, l_r \in \mathrm{Num}_{X/k}(k)$, we conclude that the composite map $\mathrm{Pic}(X) \to \mathrm{Num}_{X/k}(k^{\mathrm{alg}})$ has finite cokernel, and the assertion follows. $\square$

Attached to any noetherian scheme Z is the *dual graph* $\Gamma = \Gamma(Z)$. Its finitely many vertices $v_i \in \Gamma$ correspond to the irreducible components $Z_i \subset Z$, and two vertices $v_i \neq v_j$ are joined by an edge if $Z_i \cap Z_j \neq \varnothing$. Any morphism $Z' \to Z$ that sends irreducible components to irreducible components, in particular flat maps, induces a map $\Gamma(Z') \to \Gamma(Z)$ between vertex sets that sends edges to edges. Such maps are termed *graph morphisms*. They are called *graph isomorphism* if the maps on vertex and edge sets are bijective.

Proposition 1.2. *Suppose X is one-dimensional. Then the local system $\mathrm{Num}_{X/k}$ is constant if and only if $\Gamma(\bar{X}) \to \Gamma(X)$ is a graph isomorphism.*

Proof. Let $C_1, \ldots, C_r \subset X$ be the irreducible components, endowed with reduced scheme structure. First note that an invertible sheaf $\mathscr{L}$ is numerically trivial if and only if $(\mathscr{L} \cdot C_i) = 0$ for $1 \leq i \leq r$.

We now show that our condition is sufficient: If $\Gamma(\bar{X}) \to \Gamma(X)$ is a graph isomorphism, the stalk $\mathrm{Num}_{X/k}(k^{\mathrm{alg}})$ has rank $\rho = r$. For each $1 \leq i \leq r$, choose a closed point $a_i \in X$ not contained in $\mathrm{Ass}(\mathscr{O}_X)$ and the union of the C_j, $j \neq i$, and some effective Cartier divisor $D_i \subset X$ supported by a_i. Recall that $\mathrm{Ass}(\mathscr{O}_X)$ is the set of all points where the local ring $\mathscr{O}_{X,\zeta}$ admits a copy of the residue field $\kappa(\zeta)$ as a submodule. Here it comprises the generic points, together with closed points that give embedded components. The invertible sheaves $\mathscr{L}_i = \mathscr{O}_X(D_i)$ define elements $l_i \in \mathrm{Num}_{X/k}(k^{\mathrm{alg}})$ that generate a subgroup of finite index. In turn, the Galois representation must be trivial.

The condition is also necessary: Let $E_1, \ldots, E_s \subset \bar{X}$ be the irreducible components. Fix some $1 \leq i_0 \leq s$, and choose some effective Cartier divisor on X supported by E_{i_0} but disjoint from the other components. Using Lemma 1.1, we find some invertible sheaf $\mathscr{L}$ on X with $(\mathscr{L}_{\bar{X}} \cdot E_i) \geq 0$, with inequality if and only if $i = i_0$. It follows that $(\mathscr{L} \cdot C_j) \geq 0$, with inequality for precisely one index $j = j_0$, such that $C_{j_0} \otimes k^{\mathrm{alg}} = E_{i_0}$ as closed sets. Consequently $\Gamma(\bar{X}) \to \Gamma(X)$ is bijective, whence a graph isomorphism. $\square$

Let us use the following terminology: A *contraction* of X is a proper scheme Z together with a morphism $f : X \to Z$ with $\mathscr{O}_Z = f_*(\mathscr{O}_X)$.

Theorem 1.3. *Suppose the local system $\mathrm{Num}_{X/k}$ is constant. Then every contraction $\bar{X} \to \bar{Y}$ to some projective scheme $\bar{Y}$ is isomorphic to the base-change of a contraction $X \to Y$.*

Proof. First note that each invertible sheaf $\mathscr{L}$ on X yields the graded ring

$$R(X, \mathscr{L}) = \bigoplus_{t=0}^{\infty} H^0(X, \mathscr{L}^{\otimes t}),$$

and the resulting homogeneous spectrum $P(X, \mathscr{L}) = \mathrm{Proj}\, R(X, \mathscr{L})$. If $\mathscr{L}$ is semiample, then $Z = P(X, \mathscr{L})$ is projective and yields a contraction, and each contraction with Z projective is of this form. The sheaf $\mathscr{L}$ is unique up to preimages of semiample sheaves on Z.

According to [28], Theorem 8.8.2 there is a finite extension $k \subset k'$ such that the given contraction $\bar{X} \to \bar{Y}$ is the base-change of some contraction $f' : X' \to Y'$ with Y' projective, where $X' = X \otimes k'$. Choose some semiample invertible sheaf $\mathscr{L}'$ on X' defining the contraction, as discussed in the preceding paragraph. Enlarging k', we may assume that it is a splitting field. Then it is the composition of a Galois extension by some purely inseparable extension, and it suffices to treat these cases individually.

Suppose first that k' is Galois, with Galois group $G = \mathrm{Gal}(k'/k)$. For each $\sigma \in G$, the pullback $\mathscr{L}'_\sigma = \sigma^*(\mathscr{L}')$ is semiample, and also numerically equivalent to $\mathscr{L}'$ because $\mathrm{Num}_{X/k}$ is constant. Replacing $\mathscr{L}'$ by the tensor product $\bigotimes_\sigma \mathscr{L}'_\sigma$, we may assume that the k'-valued point $\mathrm{Pic}_{X/k}(k')$ corresponding to $\mathscr{L}'$ is Galois-invariant, hence defines a rational point $l \in \mathrm{Pic}_{X/k}$. Passing to a multiple, it comes from an invertible sheaf $\mathscr{L}$ on X, and the assertion follows.

It remains to treat the case that k' is purely inseparable. In this setting we can prove a stronger statement: Any contraction $X' \to Y'$ to a *proper* scheme Y' is the base-change of some contraction $X \to Y$. To see this, recall that our scheme $X = (|X|, \mathscr{O}_X)$ comprises an underlying topological space and a structure sheaf, and that the given contraction is a pair (f', φ') where $f : |X'| \to |Y'|$ is a continuous map and $\varphi' : \mathscr{O}_{Y'} \to \mathscr{O}_{X'}$ is an f'-homomorphism ([24], Chapter 0, Section 3.5.1). Since $\mathrm{pr} : X' \to X$ is a homeomorphism, we are forced to set $|Y| = |Y'|$ and $f = f'$ and $\mathscr{O}_Y = f_*(\mathscr{O}_X)$. Define $Y = (|Y|, \mathscr{O}_Y)$ and write $\varphi : \mathscr{O}_Y \to \mathscr{O}_X$ for the canonical f-homomorphism. This gives a morphism of ringed spaces $(f, \varphi) : X \to Y$.

It remains to verify that this is the desired contraction. Obviously, the topological space $|Y|$ is quasicompact, and $f_*(\mathscr{O}_X) = \mathscr{O}_Y$. The rest of the problem is essentially local: Let $V' \subset Y'$ be an affine open subscheme, $U' = f'^{-1}(V)$ the preimage, and $U \subset X$ the corresponding open subscheme. The condition $f'_*(\mathscr{O}_{X'}) = \mathscr{O}_{Y'}$ ensures that $V' = \mathrm{Spec}\,\Gamma(U', \mathscr{O}_{X'})$. In other words, $U' \to V'$ is the *affine hull*, and this morphism is proper. Consider the affine hull $V = \mathrm{Spec}\,\Gamma(U, \mathscr{O}_X)$. Then $U' \to V'$ is the base-change of $U \to V$, according to [26], Proposition 1.4.15. Using that $V' \to V$ is a homeomorphism, we get an identification $(|V|, \mathscr{O}_Y|_{|V|}) = V$, compatible with the morphisms from U. It follows that the ringed space $Y = (|Y|, \mathscr{O}_Y)$ is a scheme and that $f : X \to Y$ is a morphisms of schemes. Moreover, we infer $Y' = Y \otimes k'$ and $f' = f \otimes k'$. Applying [27], Proposition 2.7.1 to the structure morphism $Y \to \mathrm{Spec}(k)$, we see that Y is proper. ∎

Note that the part in the previous proof that deals with purely inseparable extensions $k \subset k'$ works without any assumptions on $\mathrm{Num}_{X/k}$. The proof of Lemma 1.1 relies on the following observation:

Lemma 1.4. *Write $S = \mathrm{Spec}(k)$, and let $h : X \to S$ be the structure morphism. Then we have an identification $H^2(S, h_*(\mathbb{G}_{m,X})) = \mathrm{Br}(X^{\mathrm{aff}})$, and this is a torsion group.*

Proof. For any k-algebra A, we have $\Gamma(X \otimes A, \mathbb{G}_a) = \Gamma(X^{\mathrm{aff}} \otimes A, \mathbb{G}_a)$, because X is quasicompact and separated, and $k \to A$ is flat. It follows that the multiplicative groups on X and X^{aff} have the same direct image on S, and it suffices to treat the case that $X = \mathrm{Spec}(R)$ is finite. One easily checks that the restriction map

$H^2(X, \mathbb{G}_m) \to H^2(X_{\text{red}}, \mathbb{G}_m)$ is bijective. The term on the right is the sum of Brauer groups for fields, which is torsion.

Consider the Leray–Serre spectral sequence $H^i(S, R^j h_*(\mathbb{G}_m)) \Rightarrow H^{i+j}(X, \mathbb{G}_m)$. We show that the edge map $H^2(X, \mathbb{G}_m) \to H^2(S, h_*(\mathbb{G}_m))$ is bijective by checking that $R^i h_*(\mathbb{G}_m) = 0$ for $i = 1, 2$. Let A be a finite k-algebra. Then $X \otimes A$ is semilocal, hence its Picard group vanishes, and we immediately see the vanishing for $i = 1$. Now suppose we have a class $\alpha \in H^2(X \otimes A, \mathbb{G}_m)$. Let k^{sep} be some separable closure. The base-change $A \otimes_k k^{\text{sep}}$ is the product of finitely many strictly henselian local Artin rings, so the pullback of α vanishes. Then there is already a finite extension $k \subset k'$ on which the pullback vanishes. This shows $R^2 h_*(\mathbb{G}_m) = 0$. □

2. Contractions of surfaces

We now examine in dimension two the results of the preceding section. Let k be a ground field of characteristic $p \geq 0$, and S be a smooth proper surface with $h^0(\mathscr{O}_S) = 1$. Let $C \subset X$ be a curve. Decompose it into irreducible components $C = m_1 C_1 + \ldots + m_r C_r$ with multiplicities $m_i \geq 1$, and let

$$N = N(C) = (C_i \cdot C_j)_{1 \leq i,j \leq r} \in \text{Mat}_r(\mathbb{Z})$$

be the resulting intersection matrix. We say that the curve C is *negative-definite* or *negative-semidefinite* if the intersection matrix N has the respective property. One may use the intersection numbers to endow $\Gamma = \Gamma(C)$ with the structure of an *edge-labeled graph*, where the labels attached to the edges are the intersection numbers $(C_i \cdot C_j) > 0$ for $i \neq j$

Now let $f : S \to Z$ be a contraction. Then either Z is a geometrically normal surface and the morphism is birational, or Z is a smooth curve and the morphism is of fiber type. Let $\bar{a} : \text{Spec}(\Omega) \to Z$ be a geometric point, for some algebraically closed field Ω, with image point $a \in Z$. Write $S_{\bar{a}} = X \times_Z \text{Spec}(\Omega) = f^{-1}(\bar{a})$ and $S_a = S \times_Z \text{Spec}\,\kappa(a) = f^{-1}(a)$ for the resulting geometric and schematic fibers, respectively. The goal of this section is to establish the following:

Theorem 2.1. *Suppose the geometric fiber $S_{\bar{a}}$ is one-dimensional. If the group scheme* $\text{Num}_{S/k}$ *is constant, the following holds:*

(i) *The map $\Gamma(S_{\bar{a}}) \to \Gamma(S_a)$ is a graph isomorphism.*
(ii) *The above graph isomorphism respects edge labels, provided that the reduced scheme $(S_a)_{\text{red}}$ is geometrically reduced over the residue field $\kappa(a)$.*
(iii) *If the geometric fiber $S_{\bar{a}}$ is reducible, the residue field extension $k \subset \kappa(a)$ is purely inseparable.*

Proof. Assertion (ii) is an immediate consequence from (i) and the definitions, and left to the reader. To verify (i) and (iii) we write $E = \kappa(a)$. Without restriction, it suffices to treat the case that $\Omega = E^{\text{alg}}$. The geometric fiber $S_{\bar{a}} = S_a \otimes_E \Omega \subset S_a \otimes_k \Omega$ is a curve on the base-change $S \otimes_k \Omega$. This curve is connected, by the condition $\mathscr{O}_Z = f_*(\mathscr{O}_S)$. Let $C_i \subset S_{\bar{a}}$, $1 \leq i \leq r$ be the irreducible components, and write $N = (C_i \cdot C_j)_{1 \leq i,j \leq r}$ for the resulting intersection matrix.

We first consider the case that $f : X \to Z$ is birational. Then N is negative-definite, in particular the canonical map $\bigoplus \mathbb{Z} C_i \to \text{Num}_{S/k}(\Omega)$ is injective. It follows that the Galois action on the set $\{C_1, \ldots, C_r\}$ is trivial. In turn, $\Gamma(S_{\bar{a}}) \to \Gamma(S_a)$ is a graph isomorphism. Now suppose that $r \geq 2$. Seeking a contradiction, we assume that $k \subset E$ is not purely inseparable. Then $S_a \otimes_k \Omega$ is disconnected. On the other hand, the fiber S_a is connected, by the condition $\mathscr{O}_Z = f_*(\mathscr{O}_S)$. It follows that the Galois action on the irreducible components $E_i \subset S_a \otimes_k \Omega$ is non-trivial. As above, the canonical map $\bigoplus \mathbb{Z}E_i \to \operatorname{Num}_{S/k}(\Omega)$ is injective, so the Galois action on the set $\{E_1, \ldots, E_s\}$ must be trivial, contradiction.

It remains to treat the case that $f : X \to Z$ is of fiber type. Then the matrix N is negative-semidefinite. For $r = 1$ our graphs have but one vertex, and the map $\Gamma(S_{\bar{a}}) \to \Gamma(S_a)$ is obviously a graph isomorphism. Now suppose $r \geq 2$. Then the curves $C_1 + \ldots + C_{r-1}$ and $C_2 + \ldots + C_r$ are negative-definite. Using the previous paragraph, we infer that $\Gamma(S_{\bar{a}}) \to \Gamma(S_a)$ is again a graph isomorphism. Likewise, one shows that $k \subset E$ is purely inseparable. $\square$

3. Curves of canonical type

Let k be a ground field of characteristic $p \geq 0$, and S be a smooth proper surface with $h^0(\mathscr{O}_S) = 1$. In this section we apply Theorem 2.1 to fibers in genus-one fibrations, and collect additional facts for the situation that $k = \mathbb{F}_q$ is finite.

We start with some useful general terminology: Let $C \subset S$ be a curve, and $C = m_0C_0 + \ldots + m_rC_r$ be the decomposition into irreducible components, together with their multiplicities. We say that C is of *fiber type* if $(C \cdot C_i) = 0$ for all indices $0 \leq i \leq r$. Obviously, this condition transfers to multiples and connected components of the curve. If C is connected and $\gcd(m_0, \ldots, m_r) = 1$, we say that C is an *indecomposable curve of fiber type*. Then the intersection matrix $N = (C_i \cdot C_j)_{1 \leq i,j \leq r}$ is negative-semidefinite, and the radical of the intersection form on the lattice $L = \bigoplus_{i=0}^r \mathbb{Z}C_i$ is generated by $m_0C_0 + \ldots + m_rC_r$.

Given a connected curve of fiber type $C = \sum m_iC_i$, we call $m = \gcd(m_0, \ldots, m_r)$ its *multiplicity*. We say that C is *simple* if $m = 1$, and *multiple* otherwise. Setting $n_i = m_i/m$, we call $C_{\text{ind}} = \sum n_iC_i$ the *underlying indecomposable curve of fiber type*.

Suppose now that we have a contraction $f : S \to B$ of fiber type onto some curve B. For each geometric point $\bar{a} : \operatorname{Spec}(\Omega) \to B$, the geometric fiber $S_{\bar{a}} = f^{-1}(\bar{a})$ is a connected curve of fiber type on the base-change $S \otimes_k \Omega$. It is useful to write $f^{-1}_{\text{ind}}(\bar{a})$ for the underlying indecomposable curve of fiber type, and $f^{-1}_{\text{red}}(\bar{a})$ for its reduction. An analogous notation will be used for the schematic fiber $S_a = f^{-1}(a)$ over the image point $a \in B$.

We are mainly interested in *curves of canonical type*. By definition, this means $(C \cdot C_i) = (K_S \cdot C_i) = 0$ for all indices $0 \leq i \leq r$. The above locutions for curves of fiber type are used in an analogous way for curves of canonical types.

Suppose now that $f : S \to B$ is a genus-one fibration that is relatively minimal. Then every geometric fiber $f^{-1}(\bar{a})$ is a connected curve of canonical type. The underlying indecomposable curves of canonical type $f^{-1}_{\text{ind}}(\bar{a})$ were classified by Néron [46] and Kodaira [36]. Throughout, we denote the *Kodaira symbols*

$$\text{I}_m,\ \text{I}_n^*,\ \text{II},\ \text{III},\ \text{IV},\ \text{IV}^*,\ \text{III}^*,\ \text{II}^* \tag{1}$$

to designate the geometric fibers. The symbols I_m, $m \geq 1$ are also called *semistable* or *multiplicative*, whereas $\text{I}_n^*, \text{II}, \ldots, \text{II}^*$, are referred to as *unstable* or *additive*. Here $n \geq 0$. As customary, the symbol I_0 indicates that $f^{-1}_{\text{ind}}(\bar{a})$ is an elliptic curve; we then distinguish the cases that the smooth curve is *ordinary* or *supersingular*. By abuse of terminology we say that a geometric fiber is *reducible or singular* if the scheme $f_{\mathrm{ind}}^{-1}(\bar{a})$ has the respective property. If this curve is singular with f elliptic, or reducible with f quasielliptic, or multiple, we say that the geometric fiber is *degenerate*.

The classification of geometric fibers $f^{-1}(\bar{a})$ was extended to schematic fibers $f^{-1}(a)$ by Szydlo [62], where the situation becomes considerably more involved. If all irreducible components of $S_a = f^{-1}(a)$ are birational to $\mathbb{P}^1$, the map $\Gamma(S_{\bar{a}}) \to \Gamma(S_a)$ is a graph isomorphism respecting edge labels, and we also use the Kodaira symbols in (1) to designate $f_{\mathrm{ind}}^{-1}(a) = m_1C_1 + \ldots + m_rC_r$. We need, however, the following modifications for $r \leq 2$, which play an important role in our later applications:

For $r = 1$ the curve $C = C_1$ has $h^0(\mathscr{O}_C) = h^1(\mathscr{O}_C) = 1$, and the normalization $\nu : \mathbb{P}^1 \to C$ is described by a *conductor square*

$$\begin{array}{ccc} \mathrm{Spec}(R) & \longrightarrow & \mathbb{P}^1 \\ \downarrow & & \downarrow \nu \\ \mathrm{Spec}(k) & \longrightarrow & C, \end{array}$$

which is both cartesian and cocartesian, see [20], Appendix A for more details. Geometrically speaking, the closed subscheme $\mathrm{Spec}(R) \subset \mathbb{P}^1$ is contracted to a rational point $\mathrm{Spec}(k) \subset C$. Here R is a finite k-algebra of length two. If $R = k \times k$ or $R = k[\epsilon]$ with $\epsilon^2 = 0$ then C is the rational nodal curve or the rational cuspidal curve and we use the standard Kodaira symbol I_1 and II, respectively. If $R = E$ is a field, then C is a twisted form, and we use *twisted Kodaira symbols* $\tilde{\mathrm{I}}_1$ and $\widetilde{\mathrm{II}}$ instead. Note that in the twisted situation, a non-rational point on $\mathbb{P}^1$ gets identified with a rational point on C, with consequences for the size of $C(k)$.

A similar situation appears for $r = 2$: Then $C = C_1 + C_2$ sits in the conductor square

$$\begin{array}{ccc} \mathrm{Spec}(R \times R) & \longrightarrow & \mathbb{P}^1 \amalg \mathbb{P}^1 \\ \downarrow & & \downarrow \nu \\ \mathrm{Spec}(R) & \longrightarrow & C, \end{array}$$

where R is a k-algebra of length two. For $R = k \times k$ or $R = k[\epsilon]$, where $\epsilon^2 = 0$, we again use the standard Kodaira symbols I_2 and III, respectively. If $R = E$ is a field extension, we have twisted forms and use $\tilde{\mathrm{I}}_2$ and $\widetilde{\mathrm{III}}$ instead. Again, the twisted situation has consequence for the size of $C(k)$.

Note that over perfect ground fields, and in particular over finite ground fields, for $r \leq 2$ the cases $\widetilde{\mathrm{II}}$ and $\widetilde{\mathrm{III}}$ do not occur, and we only have $\tilde{\mathrm{I}}_1$ and $\tilde{\mathrm{I}}_2$ as twisted Kodaira symbols. One then says that C is a *non-split semistable fiber*. For $r \geq 3$ there are further possible twists, but then $\Gamma(S_{\bar{a}}) \to \Gamma(S_a)$ is not a graph isomorphism. We record the following consequence of Theorem 2.1:

Proposition 3.1. *Suppose the ground field $k = \mathbb{F}_q$ is finite and that the group scheme* $\mathrm{Num}_{S/k}$ *is constant. Then $\Gamma(S_{\bar{a}}) \to \Gamma(S_a)$ is a graph isomorphism respecting edge labels, and the implications (i)⇒(ii)⇔(iii) hold among the following three conditions:*

(i) *The geometric fiber* $S_{\bar{a}}$ *is reducible.*
(ii) *The closed point* $a \in B$ *is a rational point.*
(iii) *The irreducible components of the schematic fiber* S_a *are birational to* $\mathbb{P}^1$.

Proof. The assertion on $\Gamma(S_{\bar{a}}) \to \Gamma(S_a)$ immediately follows from Theorem 2.1, which also ensures that $k = \kappa(a)$ provided that S_a is reducible. This already gives (i)⇒(ii). The implication (ii)⇐(iii) is trivial. For the converse, let $C \subset S_a$ be an irreducible component, and $C' \to C$ be its normalization. Then $C \otimes \Omega$ is reduced, with normalization $C' \otimes \Omega$, because the field extension $k \subset \Omega$ must be separable. The curve $C \otimes \Omega$ is birational to $\mathbb{P}^1_\Omega$, because $\Gamma(S_{\bar{a}}) \to \Gamma(S_a)$ is a graph isomorphism. Hence the same holds for $C' \otimes \Omega$. In other words, C' is a one-dimensional Brauer–Severi variety. By Wedderburn's Theorem, we must have $C' \simeq \mathbb{P}^1$. □

Proposition 3.2. *Assumptions as in Proposition 3.1. Let* $C = f^{-1}(a)$ *be a schematic fiber over a rational point* $a \in \mathbb{P}^1$*, and* $r \geq 1$ *be the number of irreducible components. Suppose the reduced scheme* C_{red} *is singular. Then the number of rational points in the fiber is given by the following table:*

Kodaira symbol	I_r	$\tilde{\mathrm{I}}_1$	$\tilde{\mathrm{I}}_2$	unstable
$\operatorname{Card} C(\mathbb{F}_q)$	rq	$q+2$	$2q+2$	$rq+1$

For the field $k = \mathbb{F}_2$ *with two elements, the Kodaira symbol* I_0^* *is impossible.*

Proof. The computation of $n \geq 0$ follows directly by comparing $C = f_{\text{red}}^{-1}(a)$ with its normalization $C' = \mathbb{P}^1 \cup \ldots \cup \mathbb{P}^1$, and is left to the reader. If the Kodaira symbol is I_0^*, then in $C = C_0 + C_1 + \ldots + C_4$ the terminal components $C_1, \ldots, C_4$ intersect the central component $C_0 = \mathbb{P}^1$ in four rational points. This implies $4 \leq \operatorname{Card} \mathbb{P}^1(\mathbb{F}_q) = q + 1$, which means $q \neq 2$. □

We now examine the situation that $k = \mathbb{F}_2$ and the scheme $f_{\text{ind}}^{-1}(a)$ is smooth. This genus-one curve contains a rational point ([38], Theorem 2), hence can be regarded as an elliptic curve $E = f_{\text{ind}}^{-1}(a)$. Let us recall from [33], Chapter 3, §6 that up to isomorphism, there are five elliptic curves $E_1, \ldots, E_5$ over the prime field $k = \mathbb{F}_2$. The groups $E_i(k)$ are cyclic, and all $1 \leq n \leq 5$ occur as orders. Deviating from loc. cit., we choose indices in a canonical way according to the order of the group of rational points, and record:

Proposition 3.3. *Up to isomorphisms, the elliptic curves* E_i *over the field* $k = \mathbb{F}_2$ *with* $\operatorname{Card} E_i(\mathbb{F}_2) = i$ *are given by the following table:*

elliptic curve	*Weierstraß equation*	*invariant*
E_1	$y^2 + y = x^3 + x^2 + 1$	*supersingular with* $j = 0$
E_3	$y^2 + y = x^3$	
E_5	$y^2 + y = x^3 + x^2$	
E_2	$y^2 + xy = x^3 + x^2 + x$	*ordinary with* $j = 1$
E_4	$y^2 + xy = x^3 + x$	

4. Enriques surfaces and exceptional Enriques surfaces

In this section we collect the relevant facts on Enriques surfaces over algebraically closed fields. For general information, we refer to the monograph of Cossec and Dolgachev [11]. Here we are particularly interested in the so-called exceptional Enriques surfaces, which were introduced and studied by Ekedahl and Shepherd-Barron [17].

Throughout, we fix an algebraically closed ground field k of characteristic $p \geq 0$. An *Enriques surface* is a smooth proper scheme Y with

$$h^0(\mathscr{O}_Y) = 1 \quad \text{and} \quad c_1 = 0 \quad \text{and} \quad b_2 = 10.$$

The first condition means that Y is connected, and the second condition signifies that the dualizing sheaf ω_Y is numerically trivial. In the Enriques classification of algebraic surfaces, Bombieri and Mumford showed in [8], §3 that the Picard scheme $\operatorname{Pic}^\tau_{Y/k}$ of numerically trivial sheaves is a finite group scheme of order two, whose group of rational points is generated by the dualizing sheaf ω_Y. Moreover, $\operatorname{Num}(Y)$ is a free abelian group of rank $\rho = 10$, the Betti numbers satisfies $b_1 = b_3 = 0$ and $b_2 = \rho$. Furthermore, $h^1(\mathscr{O}_Y) = h^2(\mathscr{O}_Y)$, and this number is at most one. The intersection form on $\operatorname{Num}(Y)$ has signature $(1, 9)$ by the Hodge Index Theorem, must be even by Riemann–Roch, and is actually unimodular ([34], Corollary 7.3.7). By the classification of indefinite unimodular forms ([44], Chapter II, Theorem 5.3), we have $\operatorname{Num}(Y) \simeq U \oplus E_8$, where the first summand has Gram matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and the second summand is the negative-definite E_8-lattice. Each curve of canonical type $C \subset Y$ induces a genus-one fibration $\varphi : Y \to \mathbb{P}^1$, and this leads to the following fact ([7], Theorem 3 and Proposition 11, combined with [39], Theorem 2.2):

Proposition 4.1. *There is at least one genus-one fibration $\varphi : Y \to \mathbb{P}^1$. Any such fibration has a multiple fiber mF of multiplicity $m = 2$, and there is an integral curve $C \subset Y$ with $C \cdot \varphi^{-1}(\infty) = 2$. Such a two-section can be chosen to be a (-2)-curve, or a half-fiber in some other genus-one fibration $\psi : Y \to \mathbb{P}^1$.*

Here a curve $F \subset Y$ is called a *half-fiber* if $2F$ is a fiber in some genus-one fibration $\varphi : Y \to \mathbb{P}^1$ over some rational point. Another important fact ([7], Proposition 11):

Proposition 4.2. *The following four conditions are equivalent:*

- (i) *The dualizing sheaf ω_Y has order two in the Picard group.*
- (ii) *The cohomology group $H^1(Y, \mathscr{O}_Y)$ vanishes.*
- (iii) *There is a genus-one fibration $\varphi : Y \to \mathbb{P}^1$ without wild fibers.*
- (iv) *There is a genus-one fibration with two multiple fibers $2F_1$ and $2F_2$.*

Under these conditions, (iii) and (iv) are valid for all genus-one fibrations, and the dualizing sheaf is given by $\omega_Y \simeq \mathscr{O}_Y(F_1 - F_2)$. Moreover, the above conditions are automatic for $p \neq 2$.

Lemma 4.3. *Let $\varphi : Y \to \mathbb{P}^1$ be a genus-one fibration with two multiple fibers $2F_1 \neq 2F_2$ in characteristic $p = 2$. Then each F_i is either an ordinary elliptic curve or unstable.*

Proof. Write $C = F_i$. The multiple fibers must be tame, hence the sheaf $\mathscr{N} = \mathscr{O}_C(C)$ has order $m = 2$ in $\operatorname{Pic}(C)$. This Picard group has no element of order two if

C is a supersingular elliptic curve. If C is semistable, then $\operatorname{Pic}^0(C) = \mathbb{G}_m(k) = k^\times$ likewise has no elements of order two. □

Suppose now that we are in characteristic $p = 2$. Up to isomorphism, there are three group schemes of order two, namely μ_2 and $\mathbb{Z}/2\mathbb{Z}$ and α_2. In turn, there are three types of Enriques surfaces Y, which are called *ordinary*, *classical* or *supersingular*, according to the following table:

group scheme $P = \operatorname{Pic}^\tau_{Y/k}$	μ_2	$\mathbb{Z}/2\mathbb{Z}$	α_2
Cartier dual $G = \underline{\operatorname{Hom}}(P, \mathbb{G}_m)$	$\mathbb{Z}/2\mathbb{Z}$	μ_2	α_2
fundamental group $\pi_1(Y, \Omega)$	$\mathbb{Z}/2\mathbb{Z}$	trivial	trivial
dualizing sheaf ω_Y	$\mathscr{O}_Y$	$\mathscr{O}_Y(F_1 - F_2)$	$\mathscr{O}_Y$
designation of Y	ordinary	classical	supersingular

In case that $P = \operatorname{Pic}^\tau_{Y/k}$ is unipotent, in other words, the Cartier dual $G = \underline{\operatorname{Hom}}(P, \mathbb{G}_m)$ is local, one says that Y is *simply-connected*. The canonical inclusion of P into the Picard scheme corresponds to a non-trivial G-torsor $\epsilon : X \to Y$, compare the discussion in [56], Section 4. If the Enriques surface Y is simply-connected then $\epsilon : X \to Y$ is a universal homeomorphism, and X is called the *K3-like covering*. Indeed, the scheme X is an integral proper surface that is Cohen–Macaulay with $\omega_X = \mathscr{O}_X$ and $h^0(\mathscr{O}_X) = h^2(\mathscr{O}_X) = 1$ and $h^1(\mathscr{O}_X) = 0$. However, the singular locus $\operatorname{Sing}(X)$ is non-empty.

The simply-connected Enriques surface can be divided into two subclasses depending on properties of the normalization $\nu : X' \to X$. The induced projection $X' \to Y$ is purely inseparable and flat of degree two, hence the ramification locus for ν is an effective Cartier divisor $R' \subset X'$, giving $\omega_{X'} = \mathscr{O}_{X'}(-R')$. Ekedahl and Shepherd-Barron [17] observed that R' is the preimage of some effective Cartier divisor $C \subset Y$ referred to as the *conductrix*, and $2C$ is called the *biconductrix*. They called an Enriques surface *exceptional* if it is simply-connected and the biconductrix has $h^1(\mathscr{O}_{2C}) \neq 0$. From this amazing definition, they obtained a complete classification of exceptional Enriques surfaces. Moreover, Salomonsson [53] described birational models by explicit equations. The following property of *non-exceptional Enriques surfaces* will play an important role later:

Proposition 4.4. *Suppose $p = 2$, and that the Enriques surface Y is non-exceptional. Then the following holds:*

(i) *There is no quasielliptic fibration $\varphi : Y \to \mathbb{P}^1$ having a simple fiber with Kodaira symbol* III* *or* II*.
(ii) *There is no genus-one fibration $\varphi : Y \to \mathbb{P}^1$ having a multiple fiber with Kodaira symbol* IV*, III* *or* II* *that admits a (-2)-curve as two-section.*
(iii) *For every genus-one fibration $\varphi : Y \to \mathbb{P}^1$ there is another genus-one fibration $\psi : Y \to \mathbb{P}^1$ with intersection number $\varphi^{-1}(\infty) \cdot \psi^{-1}(\infty) = 4$.*

Proof. The first two assertions are [17], Theorem A. The third result is [11], Theorem 3.4.1. □

5. Families of Enriques surfaces

In this section we examine arithmetic properties of Enriques surfaces Y over non-closed ground fields. In fact, we treat them as a special case of the following general notion: A *family of Enriques surfaces* over an arbitrary ground ring R is an algebraic space $\mathfrak{Y}$, together with a morphism $f : \mathfrak{Y} \to \operatorname{Spec}(R)$ that is flat, proper and of finite presentation such that for each geometric point $\operatorname{Spec}(\Omega) \to \operatorname{Spec}(R)$, the base-change Y is an Enriques surfaces over $k = \Omega$. Let $\mathscr{M}_{\text{Enr}}$ be the resulting stack in groupoids whose fiber categories $\mathscr{M}_{\text{Enr}}(R)$ consists of the families of Enriques surfaces $f : \mathfrak{Y} \to \operatorname{Spec}(R)$. This stack lies over the site $(\text{Aff}/\mathbb{Z})$ of all affine schemes, endowed with the fppf topology. We already can formulate the main result of this paper:

Theorem 5.1. *The fiber category $\mathscr{M}_{\text{Enr}}(\mathbb{Z})$ is empty. In other words, there is no family of Enriques surfaces over the ring $R = \mathbb{Z}$.*

The proof is long and indirect, and will finally be achieved in Section 15. The following result ([57], Theorem 7.2) tells us that we do not have to worry about exceptional Enriques surface:

Theorem 5.2. *There is no family of Enriques surfaces over the ring $R = \mathbb{Z}/4\mathbb{Z}$ whose geometric fiber is exceptional.*

We now collect further preliminary facts. Throughout, $f : \mathfrak{Y} \to \operatorname{Spec}(R)$ denotes a family of Enriques surfaces over an arbitrary ground ring R. To simplify notation, we set $S = \operatorname{Spec}(R)$. According to [6], Theorem 7.3 the fppf sheafification of the naive Picard functor $A \mapsto \operatorname{Pic}(\mathfrak{Y} \otimes_R A)$ is representable by a group object in the category of algebraic spaces over R. The following observation, which immediately follows from [18], Corollary 4.3, will be important:

Proposition 5.3. *The algebraic space $\operatorname{Pic}_{\mathfrak{Y}/R}$ is a scheme. The sheaf of subgroups $\operatorname{Pic}^\tau_{\mathfrak{Y}/R}$ is representable by an open embedding, and its structure morphism is locally free of rank two. Moreover, the quotient sheaf $\operatorname{Num}_{\mathfrak{Y}/R}$ is representable by a local system of free abelian groups of rank $\rho = 10$.*

Each triple (L, a, b) where L is an invertible R-module and $a \in L$, $b \in L^{\otimes -1}$ are elements with $a \otimes b = 2_R$ yields a finite flat group scheme $G^L_{a,b}$, characterized by

$$G^L_{a,b}(A) = \{f \in L \otimes_R A \mid f^{\otimes 2} = a \otimes f\},$$

with group law $f_1 \star f_2 = f_1 + f_2 + b \otimes f_1 \otimes f_2$. According to [66], Theorem 2 each group scheme that is locally free of rank two is isomorphic to some $G^L_{a,b}$. Cartier duality corresponds to the involution $(L, a, b) \mapsto (L^{\otimes -1}, b, a)$. The special case $L = R$, $a = 1$, $b = 2$ yields the constant group scheme $(\mathbb{Z}/2\mathbb{Z})_R$, whereas $a = 2$, $b = 1$ defines the multiplicative group scheme $\mu_{2,R}$.

Write $P = \operatorname{Pic}^\tau_{\mathfrak{Y}/R}$, and $G = \underline{\operatorname{Hom}}(P, \mathbb{G}_{m,R})$ for the Cartier dual. According to [52], Proposition 6.2.1, the inclusion $P \subset \operatorname{Pic}_{\mathfrak{Y}/R}$ corresponds to a global section of $R^1 f_*(G_{\mathfrak{Y}})$, which we denote by a formal symbol $[\mathfrak{X}]$. This notation may be explained as follows: The Leray–Serre spectral sequence for the structure morphism $f : \mathfrak{Y} \to S = \operatorname{Spec}(R)$ yields an exact sequence

$$(2) \quad 0 \to H^1(S, G) \to H^1(\mathfrak{Y}, G_{\mathfrak{Y}}) \to H^0(S, R^1 f_*(G_{\mathfrak{Y}})) \xrightarrow{d} H^2(S, G) \to H^2(\mathfrak{Y}, G_{\mathfrak{Y}}).$$

If $[\mathfrak{X}] \in H^0(S, R^1f_*(G_\mathfrak{Y}))$ maps to zero under the differential d, the section comes from a $G_\mathfrak{Y}$-torsor $\epsilon : \mathfrak{X} \to \mathfrak{Y}$, which we call a *family of canonical coverings*.

Proposition 5.4. *A family of canonical coverings $\epsilon : \mathfrak{X} \to \mathfrak{Y}$ exists under any of the following three conditions:*

(i) *The pullback map $H^2(S, G) \to H^2(\mathfrak{Y}, G_\mathfrak{Y})$ is injective.*
(ii) *The structure morphism $f : \mathfrak{Y} \to S$ admits a section.*
(iii) *The group scheme $P \to S$ is étale and* $\operatorname{Pic}(S) = 0$.

If a family of canonical coverings exists, it is unique up to isomorphism and twisting by pullbacks of G-torsors over S.

Proof. The first assertion and the statement about uniqueness immediately follow from the five-term exact sequence (2). If the structure morphism admits a section, the map $H^2(S, G) \to H^2(\mathfrak{Y}, G_\mathfrak{Y})$ has a retraction and is thus injective, which gives (ii).

Finally, suppose that $P \to S$ is étale. Suppose first that R is noetherian. Consider the invertible sheaf $\mathscr{L} = \Omega^2_{\mathfrak{Y}/R}$. For each $a \in S$, the restriction $\mathscr{L}|Y$ to the fiber $Y = f^{-1}(a)$ is the dualizing sheaf, which has order two in $\operatorname{Pic}(Y)$. Moreover, we have $h^i(\mathscr{L}^{\otimes 2}|Y) = h^i(\mathscr{O}_Y) = 0$ for all $i \geq 1$. In turn, the formation of the direct image $\mathscr{N} = f_*(\mathscr{L}^{\otimes 2})$ commutes with base-change, and we conclude that $\mathscr{N}$ is invertible. With Nakayama, we infer that $f^*(\mathscr{N}) \to \mathscr{L}^{\otimes 2}$ is bijective. Using $\operatorname{Pic}(R) = 0$ we obtain a trivialization $\varphi : \mathscr{O}_\mathfrak{Y} \to \mathscr{L}^{\otimes 2}$. We endow the locally free sheaf $\mathscr{A} = \mathscr{O}_\mathfrak{Y} \oplus \mathscr{L}^{\otimes -1}$ with an algebra structure, by declaring $(a, b)\cdot(a', b') = (aa' + \varphi^{-1}(bb'), ab' + a'b)$. Then the relative spectrum $\mathfrak{X} = \operatorname{Spec}(\mathscr{A})$ is the desired canonical covering.

It remains to treat general rings R. Since $f : \mathfrak{X} \to \operatorname{Spec}(R)$ is of finite presentation, it comes from a family of Enriques surfaces over some noetherian subring R'. We then can argue as in the previous paragraph, after enlarging the noetherian ring R' to make $\mathscr{N}$ trivial. □

We say that our family of Enriques surfaces has *constant local system* if the local system $\operatorname{Num}_{\mathfrak{Y}/R}$ is isomorphic to $(\mathbb{Z}^{\oplus 10})_R$. Likewise, we say that it has *constant Picard scheme* if the group space $\operatorname{Pic}_{\mathfrak{Y}/R}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^{\oplus 10})_R$. Moreover, we say that it has *split Picard scheme* if the short exact sequence of group spaces

$$0 \longrightarrow \operatorname{Pic}^\tau_{\mathfrak{Y}/R} \longrightarrow \operatorname{Pic}_{\mathfrak{Y}/R} \longrightarrow \operatorname{Num}_{\mathfrak{Y}/R} \longrightarrow 0 \tag{3}$$

splits. Our interest in these notions stems from the following fact:

Proposition 5.5. *If the base ring is $R = \mathbb{Z}$, then our family of Enriques surfaces $f : \mathfrak{Y} \to \operatorname{Spec}(R)$ has constant Picard scheme.*

Proof. Choose a geometric point $\operatorname{Spec}(\Omega) \to S$. The local system $\operatorname{Num}_{\mathfrak{Y}/R}$ comes from a representation of the algebraic fundamental group $\pi_1(S, \Omega)$ on the stalk $\operatorname{Num}_{Y/k}(\Omega) = \mathbb{Z}^{\oplus 10}$. According to Minkowski's theorem (see for example [47], Theorem 2.18), the scheme $S = \operatorname{Spec}(\mathbb{Z})$ is simply-connected, so our family of Enriques surfaces has constant local system.

For the ring $R = \mathbb{Z}$, up to isomorphism the possible triples describing the group scheme $P = \operatorname{Pic}^\tau_{\mathfrak{Y}/R}$ are either $(R, 1, 2)$ or $(R, 2, 1)$. In other words, we either have $P = (\mathbb{Z}/2\mathbb{Z})_R$ or $P = \mu_{2,R}$. Seeking a contradiction, we assume the latter holds. The Picard group $\operatorname{Pic}(R)$ vanishes, because the ring $R = \mathbb{Z}$ is a principal ideal domain.

Furthermore, we have $\mathrm{Br}(R)=0$. Indeed, the Hasse principle (see for example [48], Theorem 8.1.17) gives a short exact sequence

$$0 \longrightarrow \mathrm{Br}(\mathbb{Q}) \longrightarrow \bigoplus \mathrm{Br}(\mathbb{Q}_{\mathfrak{p}}) \stackrel{\delta}{\longrightarrow} \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$

where the sum runs over all places $\mathfrak{p}$ including the Archimedean one. Since we have $\mathrm{Br}(\mathbb{Z}_p)=\mathrm{Br}(\mathbb{F}_p)=0$ for all primes $p>0$ by Wedderburn, one sees that $\mathrm{Br}(\mathbb{Z})$ is contained in the Brauer group $\mathrm{Br}(\mathbb{R})\cap\mathrm{Ker}(\delta)=0$.

Over the localization $R'=R[1/2]$, the group scheme $\mu_{2,R}$ becomes étale, hence there is a canonical covering $\mathfrak{X}\otimes R'\to\mathfrak{Y}\otimes R'$, by Proposition 5.4. Now consider the generic fiber $X=\mathfrak{X}\otimes\mathbb{Q}$, which is a K3 surface. In particular, the Hodge number $h^{2,0}(X)=h^0(\Omega^2_X)$ is non-zero. By construction, X has good reduction over the localization $\mathbb{Z}_{(p)}$ for all primes $p\geq 3$. We now consider the ring of Witt vectors $A=W(\mathbb{F}_2^{\mathrm{sep}})$, which is the maximal unramified extension of the local ring $\mathbb{Z}_2$.

Recall that Fontaine showed that for each proper smooth scheme over $\mathbb{Q}$ that has good reduction over $W(\mathbb{F}_p^{\mathrm{sep}})$ for all primes $p>0$, the Hodge numbers $h^{i,j}$ must vanish for $i\neq j$ and $i+j\leq 3$ ([22], Theorem 1 on page 44, and first remark afterwards). Essentially the same result was independently obtained by Abrashkin, who showed that for each smooth proper scheme over $\mathbb{Z}$, the Hodge numbers $h^{i,j}$ of the complex fiber vanish for $i\neq j$ and $i+j\leq 3$ ([4], §7, Section 6, Theorem on page 516. Note that the formulation given there contains an obvious misprint, confer the Theorem on page 514).

Since our K3 surface $X=\mathfrak{X}\otimes\mathbb{Q}$ has Hodge number $h^{2,0}=1$ and good reduction over $W(\mathbb{F}_p^{\mathrm{sep}})$ for all $p\neq 2$, it follows that the base-change X_F to the field of fractions $F=\mathrm{Frac}(A)$ must have bad reduction over A.

One the other hand, using Hensel's Lemma, any $\mathbb{F}_2^{\mathrm{alg}}$-valued point for $\mathfrak{Y}\to S$ can be extended to an A-valued point. Again with Proposition 5.4 we conclude that a family of canonical covering $\mathfrak{X}\otimes A\to\mathfrak{Y}\otimes A$ exists, whence X_F has good reduction over A, contradiction.

Summing up, the Picard scheme sits in a short exact sequence

$$0 \longrightarrow (\mathbb{Z}/2\mathbb{Z})_R \longrightarrow \mathrm{Pic}_{\mathfrak{Y}/R} \stackrel{\mathrm{pr}}{\longrightarrow} (\mathbb{Z}^{\oplus 10})_R \longrightarrow 0. \tag{4}$$

For each global section l on the right, the preimage $T=\mathrm{pr}^{-1}(l)$ is a representable torsor for the group scheme $H=(\mathbb{Z}/2\mathbb{Z})_R$. In particular, the structure morphism $T\to\mathrm{Spec}(R)$ is finite étale, of degree two. Since $\pi_1(S,\Omega)=0$, the total space T is the disjoint union of two copies of $S=\mathrm{Spec}(\mathbb{Z})$. From this we infer that the short exact sequence (4) splits. In turn, our family of Enriques surfaces has constant Picard scheme. □

When combined with suitable assumptions on the Picard group or the Picard group of the base ring R, our constancy condition on the Picard scheme have remarkable consequence for the fibers of the morphism $f:\mathfrak{Y}\to\mathrm{Spec}(R)$. In what follows, let $R\to k$ be a homomorphism to a field, choose an algebraic closure $\bar{k}=k^{\mathrm{alg}}$, and write $Y=\mathfrak{Y}\otimes_R k$ and $\bar{Y}=\mathfrak{Y}\otimes_R\bar{k}$ for the resulting Enriques surfaces.

Theorem 5.6. *Suppose that the family of Enriques surfaces* $f:\mathfrak{Y}\to\mathrm{Spec}(R)$ *has constant Picard scheme. Assume furthermore that* $\mathrm{Pic}(R)$ *and* $\mathrm{Br}(R)$ *vanish. Then there is at least one canonical covering* $\epsilon:\mathfrak{X}\to\mathfrak{Y}$*, and the set of isomorphism* *classes of such coverings is a principal homogeneous space for the group* $R^\times/R^{\times 2}$. *Furthermore, the induced Enriques surfaces* Y *and* $\bar{Y}$ *enjoy the following properties:*

(i) *The dualizing sheaf* ω_Y *has order two in the Picard group.*
(ii) *Every class* $l \in \mathrm{Num}_{Y/k}(\bar{k})$ *comes from an invertible sheaf* $\mathscr{L}$ *on* Y*, which is unique up to twisting by* ω_Y.
(iii) *Every* (-2)*-curve* $\bar{E} \subset \bar{Y}$ *is the base-change of a* (-2)*-curve* $E \subset Y$*, which is isomorphic to the projective line* $\mathbb{P}^1_k$.
(iv) *Every genus-one fibration* $\bar{Y} \to \mathbb{P}^1_{\bar{k}}$ *is the base-change of a genus-one fibration* $Y \to \mathbb{P}^1_k$*. The latter has exactly two multiple fibers, both of which are tame and lie over* k*-rational points.*
(v) *There is at least one genus-one fibration* $\varphi : Y \to \mathbb{P}^1$.

Proof. By assumption we have $\mathrm{Pic}^\tau_{\mathfrak{Y}/R} = (\mathbb{Z}/2\mathbb{Z})_R$. Write $S = \mathrm{Spec}(R)$. The Kummer sequence $0 \to \mu_2 \to \mathbb{G}_m \stackrel{2}{\to} \mathbb{G}_m \to 0$ gives an exact sequence

$$\mathrm{Pic}(S) \xrightarrow{2} \mathrm{Pic}(S) \longrightarrow H^2(S, \mu_2) \longrightarrow \mathrm{Br}(S) \xrightarrow{2} \mathrm{Br}(S).$$

Our assumptions ensure that the term in the middle vanishes. According to Proposition 5.4, there is at least one canonical covering $\epsilon : \mathfrak{X} \to \mathfrak{Y}$. Moreover, the set of isomorphism classes is a principal homogeneous space for the group $H^1(S, \mu_2)$. Again by the Kummer sequence we obtain an identification $H^1(S, \mu_2) = R^\times/R^{\times 2}$. This establishes the statements on the family of canonical coverings.

From $\mathrm{Pic}^\tau_{\mathfrak{Y}/R} = (\mathbb{Z}/2\mathbb{Z})_R$ we immediately get (i). To see (ii), we may regard the element l as a section of $R^1 f_*(\mathbb{G}_{m,\mathfrak{Y}})$. The Leray–Serre spectral sequence for the structure morphism $f : \mathfrak{Y} \to S$ yields

$$H^1(\mathfrak{Y}, \mathbb{G}_{m,\mathfrak{Y}}) \longrightarrow H^0(S, R^1 f_*(\mathbb{G}_{m,\mathfrak{Y}})) \longrightarrow H^2(S, \mathbb{G}_{m,S}).$$

The arrow on the right is the zero map by assumption, hence the section l comes from an invertible sheaf on $\mathfrak{Y}$. Base-changing along $R \to k$, we find the desired invertible sheaf $\mathscr{L}$ on Y.

We now check (iii). Let $\bar{E} \subset \bar{Y}$ be a (-2)-curve, and consider the invertible sheaf $\bar{\mathscr{L}} = \mathscr{O}_{\bar{Y}}(\bar{E})$. We just saw that it is the base-change of some invertible sheaf $\mathscr{L}$ on Y. Using $h^0(\bar{\mathscr{L}}) = 1$ we infer that there is a unique curve $E \subset Y$ inducing $\bar{E} \subset \bar{Y}$, and therefore $E^2 = -2$. This curve is a twisted form of the projective line. Since its class in $\mathrm{Num}(Y)$ must be primitive in light of the self-intersection number, and the intersection form is unimodular, there is an invertible sheaf $\mathscr{N}$ with $(\mathscr{N} \cdot E) = 1$, hence $E \simeq \mathbb{P}^1$.

It remains to verify (iv). Let $\bar{Y} \to \mathbb{P}^1_{\bar{k}}$ be a genus-one fibration. According to Theorem 1.3, it is the base-change of a fibration $Y \to B$, where B is a twisted form of the projective line. Consider the invertible sheaf $\bar{\mathscr{L}} = \mathscr{O}_{\bar{Y}}(\bar{F})$, where $\bar{F} \subset \bar{Y}$ is a half-fiber. It arises from some invertible sheaf $\mathscr{L}$ on Y. Using $h^0(\bar{\mathscr{L}}) = 1$ we infer that there is a unique curve $F \subset Y$ inducing $\bar{F} \subset \bar{Y}$. This curve is a half-fiber for $Y \to B$. If $\bar{F}$ is reducible, it contains a (-2)-curve. By the previous paragraph, this gives an inclusion $\mathbb{P}^1 \subset F$. In particular, Y contains a rational point, hence $B = \mathbb{P}^1$. Now suppose that $\bar{F}$ is irreducible. Then F is an integral curve with $h^0(\mathscr{O}_F) = h^1(\mathscr{O}_F) = 1$. Since $\mathrm{Num}(Y)$ is unimodular, there is an invertible sheaf $\mathscr{N}$ on Y with $(\mathscr{N} \cdot F) = 1$. With Riemann–Roch we conclude that there is an effective Cartier divisor of degree one on F. Again there is a rational point, and $B = \mathbb{P}^1$.

Summing up, we have shown that each half-fiber on $\bar{Y}$ induces a half-fiber on Y that lies over a rational point. Since there are exactly two half-fibers on $\bar{Y}$, both of which are tame, the same holds for Y. □

6. Geometrically rational elliptic surfaces

In this section we discuss the relation between Enriques surfaces and geometrically rational surfaces. The results are well-known, but I could not find suitable references in the required generality. Let k be an arbitrary ground field of arbitrary characteristic $p \geq 0$. A proper surface J is called a *geometrically rational surface* if it is geometrically integral and the base-change to k^{alg} is birational to the projective plane. Clearly we have $h^0(\mathscr{O}_J) = 1$.

Suppose J is such a surface that is endowed with a genus-one fibration $\phi : J \to \mathbb{P}^1$. We assume that the fibration is jacobian and relatively minimal, and that the total space is smooth. Fix a section $E \subset J$, and let $J \to Z$ be the contraction of all vertical curves disjoint from the zero-section $E \subset J$. The normal proper surface Z is called the *Weierstraß model*. Write $\mathbb{P}^1 = \operatorname{Spec} k[t] \cup \operatorname{Spec} k[t^{-1}]$ for some indeterminate t.

Proposition 6.1. *There are polynomials $a_i \in k[t]$ of degree $\deg(a_i) \leq i$ such that*

$$
\begin{aligned}
&y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,\\
&y^2 + a_1'xy + a_3'y = x^3 + a_2'x^2 + a_4'x + a_6' \qquad \textit{with } a_i' = a_i/t^i
\end{aligned}
\tag{5}
$$

are Weierstraß equations for Z over the affine open sets $\mathbb{P}^1 \smallsetminus \{\infty\} = \operatorname{Spec} k[t]$ and $\mathbb{P}^1 \smallsetminus \{0\} = \operatorname{Spec} k[t^{-1}]$, respectively.

Proof. To simplify notation we write $\mathscr{O}_X(n)$ for the pullbacks of the invertible sheaves $\mathscr{O}_{\mathbb{P}^1}(n)$. The Canonical Bundle Formula [7] gives $\omega_J = \mathscr{O}_X(d)$ for some integer d. Base-changing to the algebraic closure k^{alg} we can apply [11], Proposition 5.6.1 and conclude $d = -1$. The Adjunction formula for $E \subset J$ gives the self-intersection number $(E \cdot E) = -1$, and we have $\omega_{J/\mathbb{P}^1} = \mathscr{O}_X(1)$.

As explained in [14], Section 1 the sheaves $\mathscr{E}_i = f_*\mathscr{O}_X(iE)$ are locally free, with $\operatorname{rank}(\mathscr{E}_i) = i$ for $i \geq 1$, and $\mathscr{E}_0 = \mathscr{O}_{\mathbb{P}^1}$. The canonical inclusions of $\mathscr{E}_i$ define an increasing filtration on $\mathscr{E} = \mathscr{E}_3$ with $\operatorname{gr}^*(\mathscr{E}) = \bigoplus_{i=0,2,3} \mathscr{O}_{\mathbb{P}^1}(-i)$. The invertible sheaf $\mathscr{L} = \mathscr{O}_X(3E)$ is relatively very ample, and yields a closed embedding $Z \subset \mathbb{P}(\mathscr{E})$. On the projective line, we actually may choose splittings $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-2) \oplus \mathscr{O}_{\mathbb{P}^1}(-3)$.

Write $\mathbb{P}^1 = \operatorname{Proj} k[t_0, t_1]$, such that $t = t_0/t_1$, and consider the global section $\omega = t_1$ of the invertible sheaf $\mathscr{O}_{\mathbb{P}^1}(1) = \Omega^1_{J/\mathbb{P}^1}|E$. Over the affine open set $U = \operatorname{Spec} k[t]$, we choose local sections $x = t_0^{-2}$ and $y = t_0^{-3}$ in the second and third summand of $\mathscr{E}$. As explained in loc. cit., there are polynomials $a_i \in k[t]$ such that the first equation in (5) holds in the sheaf of graded rings $\operatorname{Sym}^\bullet(\mathscr{E})$ over U. The situation over $U' = \operatorname{Spec} k[t^{-1}]$ is similar: Here we choose $\omega' = t_0$ and $x' = t_1^{-2}$ and $y' = t_1^{-3}$. This gives polynomials $a_i' \in k[t^{-1}]$ with $y'^2 + a_1'x'y' + \ldots = x'^3 + \ldots$. Now we use that the coefficients are unique, once the local sections in the summands of $\mathscr{E}$ are chosen. Multiplying the equation over U with $t^{-6} = (t_0/t_1)^6$ and comparing coefficients with the equation over U' we get $a_i' = a_i/t^i$. This gives the second equation in (5), and ensures $\deg(a_i) \leq i$. □

Note that at least one of the coefficients a_i in (5) is non-constant, and at least one is not divisible by t^i, because otherwise the geometrically rational smooth surface

J would be isomorphic to the product of $\mathbb{P}^1$ and some elliptic curve. Furthermore, this property of the coefficients remains true after any degree-preserving change of coordinates.

Now let Y be an Enriques surface, and suppose there is a genus-one fibration $\varphi : Y \to \mathbb{P}^1$. Let $\eta \in \mathbb{P}^1$ be the generic point, with function field $K = k(t)$. If the fibration is elliptic, $J_\eta = \operatorname{Pic}^0_{Y_\eta/K}$ is an elliptic curve. If the fibration is quasielliptic, $\operatorname{Pic}^0_{Y_\eta/K}$ is a twisted form of the additive group $\mathbb{G}_{a,K}$, and we write J_η for its unique regular compactification. In both cases, J_η is a regular curve with $h^0(\mathscr{O}_{J_\eta}) = h^1(\mathscr{O}_{J_\eta}) = 1$, and we write $\phi : J \to \mathbb{P}^1$ for the resulting relatively minimal genus-one fibration. It comes with a zero-section, and is thus a jacobian fibration.

Proposition 6.2. *The regular surface J is geometrically rational.*

Proof. Let $U \subset Y$ be the open set where the coherent sheaf $\Omega^1_{Y/\mathbb{P}^1}$ is invertible. It is dense, because the generic fiber Y_η is geometrically reduced, hence the image $V = \varphi(U)$ is a dense open set. For each point $b \in V$, the fibration $\varphi : Y \to \mathbb{P}^1$ acquires a section after base-changing to the strict henselization $\mathscr{O}_{\mathbb{P}^1,b} \subset R$, and we get $Y \times_{\mathbb{P}^1} \operatorname{Spec}(R) \simeq J \times_{\mathbb{P}^1} \operatorname{Spec}(R)$. It follows that the scheme J is smooth on the open set $\psi^{-1}(V)$, and we infer that the base-change $\bar{J}$ remains integral.

Moreover, the jacobian for the base-change $\bar{Y}$ coincides with the base-change of the jacobian $\bar{J}$, at least over some dense open set in $\mathbb{P}^1_{k^{\mathrm{alg}}}$, so we may assume from the start that k is algebraically closed. The assertion now follows from [11], Theorem 5.7.2. $\square$

In turn, the genus-one fibration $\varphi : Y \to \mathbb{P}^1$ yields polynomials $a_i \in k[t]$ of degree $\deg(a_i) \leq i$ such that the two Weierstraß equations in (5) describe the Weierstraß model $Z \to \mathbb{P}^1$ of the jacobian fibration $\phi : Y \to \mathbb{P}^1$. We shall see that for $k = \mathbb{F}_2$, the possibilities for these polynomials impose strong restrictions on the Kodaira symbols occurring for φ.

7. Counting points over finite fields

In this section we count the number of rational points on Enriques surfaces with constant local system over finite fields. The observation has nothing in particular to do with Enriques surfaces, so we first work in a general setting. Let $k = \mathbb{F}_q$ be a finite ground field, for some prime power $q = p^\nu$, and X be a smooth proper scheme of dimension $n \geq 0$, with $h^0(\mathscr{O}_X) = 1$.

Set $N_s = \operatorname{Card} X(\mathbb{F}_{q^s})$ for the integers $s \geq 1$, such that N_1 is the number of rational points. The *Hasse–Weil zeta function* is

$$Z(X,t) = \prod_a \frac{1}{1 - t^{\deg \kappa(a)}} = \exp\left(\sum_{s=1}^\infty \frac{N_s}{s} t^s\right), \tag{6}$$

where the product runs over all closed points $a \in X$. This formal power series actually belongs to the subring $\mathbb{Q}(t) \subset \mathbb{Q}[[t]]$, by Dwork's Rationality Theorem [16].

Choose some algebraic closure $\bar{k} = k^{\mathrm{alg}}$, write $\bar{X} = X \otimes \bar{k}$, and fix a prime $\ell \neq p$. For each $i \geq 0$ and each integer j we can form the ℓ-adic cohomology groups

$$H^i(\bar{X}, \mathbb{Q}_\ell(j)) = \left(\varprojlim_r H^i(\bar{X}, \mu_{\ell^r}^{\otimes j})\right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell. \tag{7}$$

Here $j \in \mathbb{Z}$ refers to the *Tate twist* in the coefficient sheaves. Note that $H^i(\bar{X}, \mathbb{Q}_\ell(j))$ arises from $H^i(\bar{X}, \mathbb{Q}_\ell)$ by tensoring with the one-dimensional vector space $\mathbb{Q}_\ell(j) = (\varprojlim_r \mu_{l^r}(\bar{k})) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, which comprises only information on the action of $\mathrm{Gal}(\bar{k}/k)$ on the l-primary roots of unity in $\bar{k}$. In particular, the $\mathbb{Q}_\ell$-dimensions $b_i \geq 0$ of the ℓ-adic cohomology $H^i(\bar{X}, \mathbb{Q}_\ell(j))$ do not depend on the Tate twist, and are called the *Betti numbers* of X.

The power $F_X^\nu : X \to X$ of the absolute Frobenius is a k-morphism, and induces by base-change a $\bar{k}$-morphism $\Phi = F_X^\nu \otimes \mathrm{id}_{\bar{k}}$ on $\bar{X}$. Let us write $f_{i,j} = \Phi^*$ for the induced $\mathbb{Q}_\ell$-linear endomorphism on the cohomology group $H^i(\bar{X}, \mathbb{Q}_\ell(j))$. Of particular interest are the $f_i = f_{i,0}$: The zeta function takes the form

$$Z(X,t) = \prod \det(1 - tf_i)^{(-1)^{i+1}},$$

by the Grothendieck–Lefschetz Trace Formula [31]. Here the polynomial factors can be seen as "reciprocal" characteristic polynomials $\det(1 - f_i t) = t^{b_i} \det(t^{-1} - f_i)$ of the endomorphisms f_i. The characteristic polynomials of $f_{i,j}$ have coefficients from $\mathbb{Q}$, and the eigenvalues $\alpha \in \mathbb{Q}^{\mathrm{alg}}$ are algebraic integers, of length $|\alpha| = q^{i/2-j}$ in all complex embeddings, according to the Riemann Hypothesis ([13], Corollary 3.3.9, see [12], Theorem 1.6 for the projective case).

The following observations explain the effect of the Tate twist: First of all, we have a canonical identification $H^i(\bar{X}, \mathbb{Q}_\ell(j)) = H^i(\bar{X}, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell(j)$. Moreover, the composition $(F_X^\nu \otimes \mathrm{id}_{\bar{k}}) \circ (\mathrm{id}_X \otimes F_{\bar{k}}^\nu)$ is a power of the absolute Frobenius of $\bar{X}$, which acts trivially on étale cohomology with respect to any coefficient sheaf. Finally, the Frobenius power $F_{\bar{k}}^\nu$ acts by multiplication with $q = p^\nu$ on $\mathbb{Q}_\ell(1)$. Combining these three observations, we see that $f_{i,j} = f_i \otimes q^{-j}$, which reveals that the eigenvalues of f_i are obtained from the eigenvalues of $f_{i,j}$ by multiplication with q^j.

The Tate twists become relevant in connection with algebraic cycles: Let $\mathrm{CH}^j(\bar{X})$ be the *Chow group* of j-codimensional cycles modulo linear equivalence, which comes with a *cycle class map* $\mathrm{CH}^j(\bar{X}) \to H^{2j}(\bar{X}, \mathbb{Q}_\ell(j))$. We likewise can form Chow groups and ℓ-adic cohomology on the scheme X, before base-changing to the algebraic closure, and have a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^j(X) & \longrightarrow & H^{2j}(X, \mathbb{Q}_\ell(j)) \\ \downarrow & & \downarrow \\ \mathrm{CH}^j(\bar{X}) & \longrightarrow & H^{2j}(\bar{X}, \mathbb{Q}_\ell(j)) \end{array} \tag{8}$$

of cycle class maps. For more details on cycle class maps see [15].

Proposition 7.1. *Suppose for all integers $j \geq 0$ we have $H^{2j+1}(\bar{X}, \mathbb{Q}_\ell) = 0$, and that the composite map $\mathrm{CH}^j(X) \otimes \mathbb{Q}_\ell \to H^{2j}(\bar{X}, \mathbb{Q}_\ell(j))$ is surjective. Then the zeta function is*

$$Z(X,t) = \prod_{j=0}^{n} \left(\frac{1}{1 - q^j t} \right)^{b_{2j}}. \tag{9}$$

In particular, the number of rational points is $\mathrm{Card}\, X(\mathbb{F}_q) = \sum_{j=0}^n b_{2j} q^j$*, and the set of rational points is non-empty.*

Proof. Suppose that the endomorphism f_{2j} is the homothety with eigenvalue q^j. Then

$$t^{b_{2j}} \det(t^{-1} - f_{2j}) = t^{b_{2j}}(t^{-1} - q^j)^{b_{2j}} = (1 - q^j t)^{b_{2j}},$$

and the assertion on the zeta function follows. Expanding (6) and (9) as formal power series in t and comparing linear terms, we get the statement on the number of rational points. Since $b_0 = 1$ the set $X(\mathbb{F}_q)$ must be non-empty.

It remains to verify the assertion on the endomorphism f_{2j}. Set $r = b_{2j}$ and choose prime cycles $Z_1, \ldots, Z_r \subset X$ whose cycle classes in $H^{2j}(\bar{X}, \mathbb{Q}_\ell(j))$ form a vector space basis. Obviously, the action of $\mathrm{id}_X \otimes F_{\bar{k}}$ on the base changes $\bar{Z}_1, \ldots, \bar{Z}_r \subset \bar{X}$ is trivial, so the same holds for the action on $H^{2j}(\bar{X}, \mathbb{Q}_\ell(j))$. The latter action is inverse to the action of $\Phi = F_X^\nu \otimes \mathrm{id}_{\bar{k}}$, as observed above. In other words, $f_{2j,j}$ is the identity map. We also observed above that $f_{2j,j} = f_{2j} \otimes q^{-j}$, thus f_{2j} is the homothety with eigenvalue q^j. □

The conditions obviously hold if $b_{2j+1} = 0$ and $b_{2j} = 1$ for all $j \geq 0$. One may see the above result as a generalization of Wedderburn's Theorem, which states that every Brauer–Severi variety X contains a rational point, and is thus isomorphic to projective space $\mathbb{P}^n$. Let us now specialize to surfaces. Recall that $\rho = \operatorname{rank} \operatorname{Num}(\bar{X})$ denotes the Picard number.

Corollary 7.2. *Suppose that X is a surface with $b_1 = 0$ and $b_2 = \rho$. If the local system $\operatorname{Num}_{X/k}$ is constant, we have $\operatorname{Card} X(\mathbb{F}_p) = 1 + b_2 q + q^2$.*

Proof. We have $b_0 = 1$, and with Poincaré Duality get $b_4 = 1$ and $b_1 = 0$. The cycle class map $\mathrm{CH}^1(X) \to H^2(\bar{X}, \mathbb{Q}_\ell(1))$ factors over $\operatorname{Num}(X)$, and by the assumptions the inclusion $\operatorname{Num}(X) \otimes \mathbb{Q}_\ell \subset H^2(\bar{X}, \mathbb{Q}_\ell(1))$ is an equality. Thus we may apply the Proposition and the formula on $\operatorname{Card} X(\mathbb{F}_p)$ follows. □

Corollary 7.3. *Suppose that X is either an Enriques surface, or a geometrically rational surface endowed with a relatively minimal genus-one fibration $X \to \mathbb{P}^1$. If the local system $\operatorname{Num}_{X/k}$ is constant, we have $\operatorname{Card} X(\mathbb{F}_q) = 1 + 10q + q^2$.*

Proof. In both cases, we have $b_1 = 0$. In light of the previous corollary, we only have to show that $\bar{X}$ has Betti number $b_2 = 10$. For Enriques surfaces, this holds by definition. In the other case, the induced fibration over $\bar{k}$ remains a relatively minimal genus one-fibration. We have $K_{\bar{X}}^2 = 0$ by [11], Proposition 5.6.1. Hence there is a morphism $\bar{X} \to \mathbb{P}^2_{\bar{k}}$ that can be factored as a sequence of nine blowing ups, so we also have $b_2 = 10$. □

We shall be particularly concerned with the case $q = p = 2$. The above formula then gives $\operatorname{Card} X(\mathbb{F}_2) = 25$.

8. Multiple fibers and isogenies

In this section we relate multiple fibers in elliptic fibrations with simple elliptic fibers via isogenies. The general set-up is as follows: Suppose R is an discrete valuation ring, with field of fractions $F = \operatorname{Frac}(R)$ and residue field $k = R/\mathfrak{m}_R$. For the sake of exposition, we assume that the local ring R is excellent, which here means that the field extension $\operatorname{Frac}(R) \subset \operatorname{Frac}(\hat{R})$ is separable, compare the discussion in [55], Section 4. Let E_F be an elliptic curve over F and X_F be a principal homogeneous space, representing a cohomology class in $H^1(F, E_F)$ with respect to the flat topology. Suppose that E_F has good reduction over R, that is, arises as generic fiber for some family of elliptic curves $E \to \mathrm{Spec}(R)$. Furthermore, we assume that the principal homogeneous space is the generic fiber of some relatively minimal regular model $X \to \mathrm{Spec}(R)$, and that the underlying reduced subscheme $X_{k,\mathrm{red}}$ of the closed fiber X_k is an elliptic curve. We then observe:

Proposition 8.1. *Under the above assumptions, the two elliptic curves E_k and $X_{k,\mathrm{red}}$ over the residue field k are isogeneous.*

Proof. Without restriction, we may assume that the discrete valuation ring R is henselian. Let $m \geq 1$ be the multiplicity of the closed fiber for $X \to \mathrm{Spec}(R)$, such that $X_k = mX_{k,\mathrm{red}}$. By assumption, the reduction $X_{k,\mathrm{red}}$ is an elliptic curve, in particular contains a k-rational point $d \in X_{k,\mathrm{red}}$. This is an effective Cartier divisor, and can be extended to an effective Cartier divisor $D_k \subset X_k$ on the fiber. In light of [29], Theorem 18.5.11 we may extend further, and regard it as the closed fiber of some effective Cartier divisor $D \subset X$. The scheme D is regular, because the intersection of Cartier divisors $D \cap X_{k,\mathrm{red}} = \mathrm{Spec}\,\kappa(d)$ is zero-dimensional and regular. Writing $D = \mathrm{Spec}(R')$, we get a finite extension of discrete valuation rings $R \subset R'$. The residue field extension has degree one, hence $k' = R'/\mathfrak{m}_{R'}$ is a copy of $k = R/\mathfrak{m}_R$. By construction, the base-change $X_{R'}$ acquires a section; by abuse of notation, we use the same symbol for the effective Cartier divisor $D \subset X$ and the resulting section $D \subset X_{R'}$. Note, however, that the scheme $X_{R'}$ usually becomes non-normal, and contains D as a Weil divisor rather than a Cartier divisor. Let $X' \to X_{R'}$ be the normalization. By the valuative criterion for properness, the inclusion $D \subset X_{R'}$ lifts to an inclusion $D \subset X'$.

To proceed, consider the base-change $E' = E_{R'}$, which is a family of elliptic curves over R'. The section $D \subset X'$ yields a rational map $X' \dashrightarrow E'$, which is defined on the generic fiber. Choose proper birational maps $X' \leftarrow Y \to E'$ that restrict to identities on the generic fiber, where Y is a normal scheme. According to [52], Theorem 8.2.1 the morphism $Y \to \mathrm{Spec}(R')$ is cohomologically flat, because it admits a section. In turn, the closed fiber $C = Y \otimes_{R'} k'$ has cohomological invariants $h^0(\mathscr{O}_C) = h^1(\mathscr{O}_C) = 1$.

Let $C_1, \ldots, C_r \subset C$ be those irreducible components that are strict transforms of irreducible components in $X'_{k'}$, endowed with reduced scheme structure. Then each C_i dominates the elliptic curve $X_{k,\mathrm{red}}$, hence is a curve with cohomological invariants $h^0(\mathscr{O}_{C_i}) \geq 1$ and $h^1(\mathscr{O}_{C_i}) \geq 1$, by Lemma 8.4 below. We now claim that $r = 1$. Seeking a contradiction, suppose that $r \geq 2$. In the exact sequence $\mathscr{O}_C \to \bigoplus_{i=1}^r \mathscr{O}_{C_i} \to \mathscr{F} \to 0$, the term on the right is a skyscraper sheaf, whereas the other sheaves are one-dimensional. From this we infer that the map $H^1(C, \mathscr{O}_C) \to \bigoplus H^1(C_i, \mathscr{O}_{C_i})$ is surjective. Since each summand is a non-trivial vector space, we must have $r = 1$, with $h^1(\mathscr{O}_{C_1}) = 1$.

Now write $C_1' \subset Y$ for the strict transform of $E_{k'}$. This is an elliptic curve, because the birational map $C_1' \to E_{k'}$ must be an isomorphism. Arguing as in the preceding paragraph, we see $C_1 = C_1'$. Summing up, the morphism $C_1 \to E_{k'} = E_k$ is an isomorphism. Using its inverse, we obtain the desired isogeny $E_k = C_1 \to X_{k,\mathrm{red}}$. □

It is easy to construct an example, even in arbitrary dimensions: Suppose for simplicity that R contains a field of representatives, that is, a subfield that surjects onto the residue field, for example $R = k[[t]]$. Let A be an abelian variety over k, endowed with a finite subgroup $G \subset A(k)$, and suppose that there is a finite Galois extension $F \subset F'$ with Galois group $\mathrm{Gal}(F'/F) = G$ such that $R \subset R'$ has trivial residue field extension. Consider the diagonal action on $A \times \mathrm{Spec}(R')$, which is free, and the resulting quotient $X = (A \times \mathrm{Spec}(R'))/G$. Then the generic fiber is a twisted form of $A \otimes_k F$, whereas the closed fiber contains the abelian variety A/G as reduction. See the dissertation of Zimmermann for more on this construction ([67], [68]).

In light of general results of Serre, Tate and Honda on isogeny classes of abelian varieties over finite fields, the above result has the following consequences for finite fields:

Corollary 8.2. *Assumptions as in the proposition. If the residue field $k = \mathbb{F}_q$ is finite, then for each integer $r \geq 1$, the number of $\mathbb{F}_{q^r}$-valued points in E_k and X_k coincide.*

Proof. According to [65], Theorem 1 any two isogeneous elliptic curves over the finite field $\mathbb{F}_q$ have the same zeta function, in other words, the same number of $\mathbb{F}_{q^r}$-valued points, $r \geq 1$. □

Corollary 8.3. *Assumptions as in the proposition. If the residue field is $k = \mathbb{F}_2$, then the elliptic curve E_k is isomorphic to the scheme $X_{k,\mathrm{red}}$.*

Proof. We recalled in Proposition 3.3 that there are five isomorphism classes of elliptic curves over $k = \mathbb{F}_2$. In each case, the group of rational points is cyclic, and each of the groups $\mathbb{Z}/n\mathbb{Z}$, $1 \leq n \leq 5$ does occur. Hence the number of rational points already determines the isomorphism class, and the assertion follows from Corollary 8.2. □

Note that according to the results of Tate [65] and Honda [32], the isogeny classes of simple abelian varieties A over a finite field $k = \mathbb{F}_q$, $q = p^\nu$ corresponds to the conjugacy classes of q-Weil numbers $\pi \in \mathbb{C}$.

In the proof for Proposition 8.1, we have used the following observation on algebraic curves:

Lemma 8.4. *Let k be a ground field, E be an elliptic curve, C be an integral proper curve, and $f : C \to E$ be a dominant morphism. Then $h^1(\mathscr{O}_C) \geq 1$. Moreover, if $h^1(\mathscr{O}_C) = 1$ then the curve C is regular.*

Proof. Let $d \geq 1$ be the degree of the dominant morphism $f : C \to E$. The induced homomorphism $\mathrm{Pic}(E) \to \mathrm{Pic}(C)$ comes with a norm map in the reverse direction (see [25] Section 6.5), with the property $N_{C/E}(\mathscr{L}|C) = \mathscr{L}^{\otimes d}$ for all invertible sheaves $\mathscr{L}$ on E.

To check $h^1(\mathscr{O}_C) \geq 1$, we may assume that k is separably closed. Seeking a contradiction, we assume that $h^1(\mathscr{O}_C) = 0$, such that $\mathrm{Pic}^0_{C/k}$ is trivial. We conclude that the abelian group $\mathrm{Pic}^0(E) = E(k)$ is annihilated by d. By Bertini's Theorem (for example [30], Proposition 4.3 or [35], Chapter I, Theorem 6.3), we find some smooth divisor $D \subset E$ that is disjoint from the scheme $E[d]$ of d-torsion. Each point $a \in D$ of this subscheme is rational, because k is separably closed. The corresponding invertible sheaf $\mathscr{L} = \mathscr{O}_E(a - 0_E)$ has $\mathscr{L}^{\otimes d} \neq \mathscr{O}_E$, contradiction.

Now suppose that $h^1(\mathscr{O}_C) = 1$. It remains to check that C is regular. Let $C' \to C$ be the normalization. The short exact sequence $0 \to \mathscr{O}_C \to \mathscr{O}_{C'} \to \mathscr{F} \to 0$ yields a long exact cohomology sequence

$$0 \to H^0(C, \mathscr{O}_C) \to H^0(C', \mathscr{O}_{C'}) \to H^0(C, \mathscr{F}) \to H^1(C, \mathscr{O}_C) \to H^1(C', \mathscr{O}_{C'}) \to 0.$$

The term on the right does not vanish, by the preceding paragraph, hence the map on the right is bijective. Thus $H^1(C, \mathscr{O}_C)$ is a one-dimensional vector space over $k = H^0(C, \mathscr{O}_C)$. It is also a vector space over the field extension $k' = H^0(C', \mathscr{O}_{C'})$, and we infer $k = k'$. Then the above exact sequence gives $h^0(\mathscr{F}) = 0$, whence the torsion sheaf $\mathscr{F}$ vanishes. Thus the one-dimensional scheme C is normal, hence regular. □

9. Passage to jacobian fibrations

Throughout this section, Y is an Enriques surface with constant Picard scheme $\mathrm{Pic}_{Y/k}$ over the field $k = \mathbb{F}_2$. According to Theorem 5.6, there is at least one genus-one fibration $\varphi : Y \to \mathbb{P}^1$. We now assume that this fibration is elliptic, and let $\phi : J \to \mathbb{P}^1$ be the resulting jacobian fibration, where J is a geometrically rational surface. The goal of this section is to establish the following result:

Theorem 9.1. *The geometrically rational surface J has constant Picard scheme.*

This ensures that the relation between the elliptic fibration on the Enriques surface and its jacobian is stronger that one might expect. This crucial observation will allow us in the next sections to use Weierstraß equations to reduce the possibilities for J, and thus also for Y.

Choose an an algebraic closure $k \subset k^{\mathrm{alg}}$, and write $\bar{Y} = Y \otimes k^{\mathrm{alg}}$ and $\bar{J} = J \otimes k^{\mathrm{alg}}$ for the base-changes. Throughout, $\bar{a} : \mathrm{Spec}(\Omega) \to \mathbb{P}^1$ denotes a geometric point whose image point $a \in \mathbb{P}^1$ is closed.

Proposition 9.2. *For each such $\bar{a} : \mathrm{Spec}(\Omega) \to \mathbb{P}^1$, the geometric fibers $J_{\bar{a}}$ and $Y_{\bar{a}}$ have the same Kodaira symbols. If Y_a is simple, we actually have $J_a \simeq Y_a$.*

Proof. The first assertion follows from a general result of Liu, Lorenzini and Raynaud ([42], Theorem 6.6). Now suppose that Y_a is simple. In order to check $J_a \simeq Y_a$ we may base-change to $\kappa(a) = \mathbb{F}_{2^\nu}$ and assume that $a \in \mathbb{P}^1$ is rational. If the fiber contains a rational point $y \in Y_a$ in the regular locus, the resulting effective Cartier divisor gives an identification $J \otimes R \simeq Y \otimes R$, where $R = \mathscr{O}^h_{\mathbb{P}^1,a}$ is the henselization, and in particular $J_a \simeq Y_a$. It remains to verify the existence of such a rational point. For smooth fibers, this follows from Lang's result ([38], Theorem 2). Suppose now that $C = Y_a$ is singular, with irreducible components $C_1, \ldots, C_r$. Each C_i is birational to $\mathbb{P}^1$, according to Proposition 3.1. In case $r = 1$ the description before Proposition 3.2 reveals that the desired rational point exists. If $r \geq 2$, we actually have $C_i \simeq \mathbb{P}^1$, and the canonical map $\Gamma(\bar{C}) \to \Gamma(C)$ is a graph isomorphism respecting edge labels, again by Proposition 3.1. Choose a component C_i corresponding to a terminal vertex of $\Gamma(C)$. Then there is exactly one component C_j intersecting C_i, which has $(C_i \cdot C_j) \leq 2$. In turn, from the three rational points in $C_i = \mathbb{P}^1$ at least one is contained in the regular locus of C_{red}. □

We now state several facts about the fibers of $\phi : J \to \mathbb{P}^1$, which can be formulated without referring to the Enriques surface Y:

Proposition 9.3. *Let $\bar{a} : \mathrm{Spec}(\Omega) \to \mathbb{P}^1$ be a geometric point whose image point $a \in \mathbb{P}^1$ is closed. Then the following holds:*

(i) *If $a \in \mathbb{P}^1$ is non-rational, then the geometric fiber $J_{\bar{a}}$ is irreducible.*
(ii) *The Kodaira symbol of $J_{\bar{a}}$ is different from I_0^*.*
(iii) *If $J_{\bar{a}}$ is semistable, then the map $\Gamma(J_{\bar{a}}) \to \Gamma(J_a)$ is a graph isomorphism.*
(iv) *There is at most one rational point $b \in \mathbb{P}^1$ whose fiber J_b is semistable or supersingular.*
(v) *The canonical inclusion $\mathrm{MW}(J/\mathbb{P}^1) \subset \mathrm{MW}(\bar{J}/\bar{\mathbb{P}}^1)$ of Mordell–Weil groups is an equality.*

Proof. Recall from Proposition 9.2 that for all closed points $a \in \mathbb{P}^1$, the fibers J_a and Y_a have the same Kodaira symbol. Suppose now that $k \subset \kappa(a)$ is not an equality. Then $Y_{\bar{a}}$ is irreducible by Theorem 2.1, which settles (i). From Proposition 3.2 we get (ii). Suppose next that J_a is semistable. Then Y_a must be simple, by Lemma 4.3, hence $J_a \simeq Y_a$, again from Proposition 9.2. Assertion (iii) thus follows from Theorem 2.1.

By Theorem 5.6 there are two rational points $a \in \mathbb{P}^1$ where the fiber Y_a is multiple. The corresponding $\varphi_{\text{ind}}^{-1}(a)$ are neither semistable nor supersingular, again by Lemma 4.3. So from the three rational points, there is only one $b \in \mathbb{P}^1$ where $\varphi_{\text{ind}}^{-1}(b)$ and hence J_b can be semistable or supersingular, which gives (iv).

It remains to verify the last assertion (v). According to the definition of Mordell–Weil groups we have

$$\mathrm{MW}(J/\mathbb{P}^1) = J(F) = \mathrm{Pic}^0_{Y_F/F}(F) = \mathrm{Pic}^0(Y_F),$$

where F is the function field of the projective line. The latter equality holds because $\mathrm{Br}(F) = 0$, by Tsen's Theorem. Choose a two-section $R \subset Y$, and consider the surjective homomorphism

$$\mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}^0(Y_F), \quad \mathscr{L} \longmapsto \mathscr{L}(-nR)|Y_F,$$

where $2n = \deg(\mathscr{L}|Y_F)$. The kernel $T(Y/\mathbb{P}^1)$ of this map is generated by the irreducible components of the closed fibers for $\varphi : Y \to \mathbb{P}^1$, together with R. Consider the commutative diagram

$$\begin{array}{ccc} T(Y/\mathbb{P}^1) & \longrightarrow & \mathrm{Pic}(Y) \\ \downarrow & & \downarrow \\ T(\bar{Y}/\bar{\mathbb{P}}^1) & \longrightarrow & \mathrm{Pic}(\bar{Y}) \end{array}$$

induced by base-change. By assumption, the vertical map on the right is bijective. According to Theorem 2.1, the same holds for the vertical map on the left. In turn, the induced vertical map between cokernels is bijective, thus (v) holds. □

The number of irreducible components of the geometric fiber $J_{\bar{a}}$ depends only on the image point $a \in \mathbb{P}^1$; we denote this integer by $r_a \geq 1$. It coincides with the number of irreducible components in both $Y_{\bar{a}}$ and Y_a. If $a \in \mathbb{P}^1$ is rational, we furthermore set

$$n_a = \begin{cases} i & \text{if } J_a \text{ is smooth and isomorphic to } E_i; \\ 2r_a & \text{if } J_a \text{ is split semistable;} \\ 2r_a+2 & \text{if } J_a \text{ is non-split semistable;} \\ 2r_a+1 & \text{if } J_a \text{ is unstable.} \end{cases}$$

Recall from Proposition 3.3 that the E_i denote the five elliptic curves over the prime field $k = \mathbb{F}_2$, indexed by the order of the group of rational points. Also note that $n_a = |Y_a(k)|$, and thus $\sum n_a = 25$ by Corollary 7.3, where the sum runs over the three rational points $a \in \mathbb{P}^1$. The numbers $n_a \geq 1$ are defined, however, in terms of the irreducible components of the schematic and geometric fibers J_a and $J_{\bar{a}}$, and it is a priori not clear how they are related to $|J_a(k)|$.

We now forget that J comes from an elliptic Enriques surface, and only demand that it satisfies the properties observed above. Theorem 9.1 becomes a consequence of the following more general statement:

Theorem 9.4. *Let J be a geometrically rational surface, endowed with an elliptic fibration $\phi : J \to \mathbb{P}^1$ that is relatively minimal and jacobian. Suppose that conditions* (i)–(v) *of Proposition 9.3 and the equation $\sum n_a = 25$ hold. Then J has constant Picard scheme.*

The elliptic surface J is geometrically rational, so the Picard scheme $\mathrm{Pic}_{Y/k}$ is a local system of free abelian groups. Their rank is $\rho = 10$, by the minimality of the fibration. We have to show that the Galois group $G = \mathrm{Gal}(k^{\mathrm{alg}}/k)$ acts trivially on $\mathrm{Pic}_{J/k}(k^{\mathrm{alg}}) = \mathrm{Pic}(\bar{J}) = \mathbb{Z}^{\oplus 10}$. Since there is no torsion part, it suffices to find a subgroup of finite index for which this holds. The group $\mathrm{Pic}(\bar{J})$ is generated by a fiber, the vertical curves Θ disjoint from the zero-section, and the sections. The former is obviously defined over k, and the latter by condition (v). According to condition (i), each $\Theta \subset \bar{J}$ projects to a fiber J_a over some rational point $a \in \mathbb{P}^1$. Thus it suffices to prove the following:

Proposition 9.5. *Let $\bar{a} : \mathrm{Spec}(\Omega) \to \mathbb{P}^1$ be a geometric point. Assume that the image $a \in \mathbb{P}^1$ is rational and that the fiber J_a is singular. Then the canonical map $\Gamma(J_{\bar{a}}) \to \Gamma(J_a)$ is a graph isomorphism.*

Proof. The assertion is trivial if $J_{\bar{a}}$ has Kodaira symbol I_1 or II. Condition (iii) ensures that it also holds for I_n with $n \geq 2$. Now suppose that $J_{\bar{a}}$ has one of the Kodaira symbol $\mathrm{III}, \mathrm{IV}, \ldots, \mathrm{II}^*$. Our task is to verify that the Galois action of $G = \mathrm{Gal}(\mathbb{F}_2^{\mathrm{alg}}/\mathbb{F}_2)$ on the dual graph $\Gamma = \Gamma(J_{\bar{a}})$ is trivial. Since it is a tree, it suffices that check that all terminal vertices are fixed. This obviously holds for the terminal vertex $v_0 \in \Gamma$ that corresponds to the irreducible component that intersects the base-change of the zero-section $O \subset J$. If there is at most one further terminal vertex, it must be fixed as well. This already settles the cases where the Kodaira symbol is III, III^* or II^*.

If the Kodaira symbol is I_m^*, we must have $m \geq 1$ by condition (ii). Besides $v_0 \in \Gamma$, there are three other terminal vertices. One has a shorter distance to v_0 than the others, and thus must be fixed, and we have to deal with the remaining two terminal vertices. For Kodaira symbols IV and IV^* there are also two remaining terminal vertices besides $v_0 \in \Gamma$. In all these cases, we shall check that there is a section $P \subset J$ whose base-change intersects an irreducible component $\Theta \subset J_{\bar{a}}$ corresponding to one of the two remaining terminal vertices.

To achieve this, first note that $\operatorname{Num}(\bar{J})$ is unimodular, because the minimal model of $\bar{J}$ is the projective plane. Consider the orthogonal complement $W \subset \operatorname{Num}(\bar{J})$ of the fiber $J_{\bar{a}}$ and the zero-section O. These two curves have Gram matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix}\right)$, and we infer that W is unimodular. It is generated by the vertical curves disjoint from the zero-section, together with the combinations $(P - O) - (1 + (P \cdot O))J_{\bar{a}}$ for $P \in \operatorname{MW}(\bar{J}/\bar{\mathbb{P}}^1)$. The latter have self-intersection $-1 - 2(P \cdot O) - 1$, hence the lattice W is even. Furthermore, we consider the sublattice $W_{\bar{a}} \subset W$ generated by the components $\Theta_1, \ldots, \Theta_r \subset J_{\bar{a}}$ disjoint from the zero-section. Note that this is a root lattice, endowed with a canonical basis.

Suppose now that $J_{\bar{a}}$ has Kodaira symbol IV or IV*, and that all sections $P \subset J$ have a base change that passes through the component $\Theta_0 \subset J_{\bar{a}}$ corresponding to $v_0 \in \Gamma$. Then $W_{\bar{a}} \subset W$ is an orthogonal direct summand, hence unimodular. On the other hand, this root lattice has type A_2 or E_6, with discriminant $d = 3$, contradiction.

It remains to treat the symbols I_m^* with $1 \leq m \leq 4$. Now $W_{\bar{a}}$ is a root lattice of type D_{m+4}, which again is not unimodular. Our lattices and their dual lattices are related by a commutative diagram

$$\begin{array}{ccccc} W_{\bar{a}} & \longrightarrow & W & \longrightarrow & \operatorname{Num}(\bar{J}) \\ \downarrow & & \downarrow & & \downarrow \\ W_{\bar{a}}^* & \longleftarrow & W^* & \longleftarrow & \operatorname{Num}(\bar{J})^*, \end{array} \tag{10}$$

where the vertical maps are given by $x \mapsto (y \mapsto (x \cdot y))$. Seeking a contradiction, we now assume that each section $P \subset J$ passes through the component $\Theta_0 \subset J_{\bar{a}}$ corresponding to $v_0 \in \Gamma$, or the component $\Theta_1 \subset J_{\bar{a}}$ corresponding to the terminal vertex $v_1 \in \Gamma$ that has shorter distance to v_0. The latter indeed happens, because otherwise $W_{\bar{a}} \subset W$ would be an orthogonal direct summand, hence unimodular, contradiction. Now fix some P passing through Θ_1. Under the maps in (10), the combination $(P - O) - (1 + (P \cdot O))J_{\bar{a}} \in W$ maps to the dual basis vector $\Theta_1^* \in W_{\bar{a}}^*$. Since the maps respect the intersection pairing, we conclude that $(\Theta_1^* \cdot \Theta_1^*) \in \mathbb{Q}$ actually belongs to $2\mathbb{Z}$. On the other hand, a direct computation with the root lattice $W_{\bar{a}}$ of type D_{m+4} reveals $(\Theta_1^* \cdot \Theta_1^*) = \pm 1$, contradiction. □

My original arguments for the cases $\mathrm{IV}, \mathrm{IV}^*$ and I_m^* in the above proof relied on Lang's Classification [40], which gives the possible configuration of singular fibers for $\bar{J}$, together with the Oguiso–Shioda Table [51] for Mordell–Weil lattices and the Shioda's explicit formula [60] for the height pairing $\langle P, Q \rangle \in \mathbb{Q}$. This method was already used in [56], Section 15. The above more elegant and conceptual argument was suggested by one referee.

10. Classification of certain jacobian fibrations

Throughout, we work over the ground field $k = \mathbb{F}_2$. Let Y be a geometrically rational surface, and $\phi : Y \to \mathbb{P}^1$ be an elliptic fibration that is relatively minimal and jacobian. It is then described by some Weierstraß equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

where the coefficients $a_i \in \mathbb{F}_2[t]$ are polynomials of degree $\deg(a_i) \leq i$, and the discriminant does not vanish. The goal of this section is to classify those Weierstraß equations that might arise from an Enriques surface with constant Picard scheme. It turns out that there are exactly eleven such Weierstraß equations, up to change of coordinates and automorphism of the projective line. This is quite remarkable, because all in all there are $2^{21} = 2097152$ possible Weierstraß equations.

We make the following assumptions on the surface J and the jacobian fibration $\phi : J \to \mathbb{P}^1$:

(i) The proper scheme J has constant Picard scheme $\mathrm{Pic}_{J/k}$.
(ii) There is at most one rational point $a \in \mathbb{P}^1$ whose fiber J_a is semistable or supersingular.

The first condition arises from Theorem 9.1, and the second from Proposition 9.3. With Propositions 3.1, 3.2 and 7.3 we see that the following holds as well:

(iii) The fibers $J_{\bar{a}}$ over non-rational points $a \in \mathbb{P}^1$ have Kodaira I_0, I_1 or II.
(iv) There is no fiber $J_{\bar{a}}$ with Kodaira symbol I_0^*.
(v) The number of rational points is $\mathrm{Card}\, Y(\mathbb{F}_2) = 25$.

For each rational point $a \in \mathbb{P}^1$, let $r_a \geq 1$ and $n_a \geq 1$ be the respective numbers of irreducible components and rational points in J_a. We start our analysis by observing that there is a large fiber, in the sense that there are many irreducible components:

Proposition 10.1. *There is a rational point $a \in \mathbb{P}^1$ whose fiber J_a contains at least six irreducible components.*

Proof. Seeking a contradiction, we assume that $r_a \leq 5$ for the three rational points $a \in \mathbb{P}^1$. If J_a is unstable we actually have $r_a \leq 3$, because the Kodaira symbol I_0^* is impossible, and get $n_a \leq 7$. The same estimate holds if the fiber is smooth, because it is then isomorphic to one of the elliptic curves E_i, with $1 \leq i \leq 5$. In the semistable case we get $n_a \leq 10$, and there is at most one rational point with semistable fiber. Writing $a, b, c \in \mathbb{P}^1$ for the three rational points, we get the contradiction $25 = \mathrm{Card}\, Y(\mathbb{F}_2) = n_a + n_b + n_c \leq 10 + 7 + 7 = 24$. □

We shall see that this large fiber must be unstable. The next result is a first step into this direction:

Proposition 10.2. *There is a rational point $a \in \mathbb{P}^1$ whose fiber J_a is reducible and unstable.*

Proof. Seeking a contradiction, we assume that all reducible fibers are semistable. According to Proposition 10.1, there is a reducible fiber with at least six components, say J_a. It follows that J_a has Kodaira symbol I_r for some $r \geq 6$, contributing $n = 2r \geq 12$ rational points. We also have the upper bound $r \leq 9$, because of the Picard number $\rho(J) = 10$.

For each of the two other rational points $b, c \in \mathbb{P}^1$, it follows that the fiber is either ordinary or with Kodaira symbol II. In the former case, the number of rational points is even, in the latter case it is odd. Since $\mathrm{Card}\, Y(\mathbb{F}_2) = 25$ is odd, we may assume without restriction that $J_b = E_{2i}$ for $1 \leq i \leq 2$, and J_c has Kodaira symbol II. This gives $25 = \operatorname{Card} Y(\mathbb{F}_2) = 2r + 2i + 3$. The only possible solution is $r = 9$ and $i = 2$. But by Lang's Classification [40], the configuration of singular fibers $\mathrm{II} + \mathrm{I}_9$ does not occur. ☐

The generic fiber of the jacobian fibration $\phi : J \to \mathbb{P}^1$ has a certain invariant $j(J/\mathbb{P}^1)$ in the function field $k(\mathbb{P}^1) = \mathbb{F}_2(t)$, which is also called the *functional j-invariant* and can be seen as a morphism $j : \mathbb{P}^1 \to \mathbb{P}^1$. It is now convenient to distinguish the cases that the invariant is zero or non-zero.

Proposition 10.3. *Suppose the functional j-invariant is non-zero. Then there is exactly one unstable fiber. If this fiber contains $r \leq 5$ irreducible components, the configuration of fibers over rational points must be $\mathrm{III} + E_4 + \mathrm{I}_8$.*

Proof. First note that by Proposition 10.2 there is a rational point $a \in \mathbb{P}^1$ whose fiber J_a is unstable. As observed by Lang, there is no other unstable fiber ([40], beginning of Section 1). Indeed, a fiber J_x is unstable if and only if $c_4 \in \mathbb{F}_2[T]$ vanishes at the point x ([14], Proposition 5.1), and in our situation $c_4 = a_1^4$ with $\deg(a_1) \leq 1$. Assume now that J_a contains at most five irreducible components. Since the Kodaira symbol I_0^* does not occur, J_a has at most three irreducible components. According to Proposition 10.1, there must be a semistable fiber J_b over some rational point $b \in \mathbb{P}^1$, with $6 \leq m \leq 9$ irreducible components. In light of Lang's Classification [40], Section 2 the only possible configuration of singular fibers is $\mathrm{III} + \mathrm{I}_6$ and $\mathrm{III} + \mathrm{I}_8$. Consequently, the fiber over the last rational point $c \in \mathbb{P}^1$ is ordinary, whence $J_c = E_{2n}$ with $1 \leq n \leq 2$. The respective number of rational points in the fiber is thus $n_a = 5$ and $n_b = 2m$ and $n_c = 2n$. From $25 = \operatorname{Card} Y(\mathbb{F}_2) = n_a + n_b + n_c$ we get $m + n = 10$. The only solution is $m = 8$ and $n = 2$. ☐

We now can state the first half our our classification result:

Theorem 10.4. *Suppose the functional j-invariant is non-zero. Up to coordinate changes and automorphisms of the projective line, the fibration $\phi : J \to \mathbb{P}^1$ is given by exactly one of the following eight Weierstraß equations:*

Weierstraß equation	fibers over $t = 0, 1, \infty$	j-invariant
$y^2 + txy + t^2y = x^3 + tx^2 + t^4x + t^5(1+t)$	$\mathrm{I}_1^* + E_4 + \mathrm{I}_4$	t^4
$y^2 + txy = x^3 + t^3x + t^5(1+t)$	$\mathrm{III}^* + E_4 + E_4$	$t^2/(t^2+t+1)$
$y^2 + txy = x^3 + t^3x$	$\mathrm{III}^* + E_4 + \mathrm{I}_2$	t^2
$y^2 + txy = x^3 + t^2x^2 + t^3x$	$\mathrm{III}^* + E_2 + \tilde{\mathrm{I}}_2$	
$y^2 + txy = x^3 + t^5$	$\mathrm{II}^* + E_4 + \mathrm{I}_1$	t
$y^2 + txy = x^3 + t^2x^2 + t^5$	$\mathrm{II}^* + E_2 + \tilde{\mathrm{I}}_1$	
$y^2 + txy = x^3 + tx^2 + t^4x$	$\mathrm{I}_4^* + E_2 + E_4$	1
$y^2 + txy + ty = x^3 + tx^2 + tx$	$\mathrm{III} + E_4 + \mathrm{I}_8$	t^8

For the invariant $j = t^2/(t^2+t+1)$, there is an additional singular fiber, which occurs over the point $x \in \mathbb{P}^1$ with $\kappa(x) \simeq \mathbb{F}_4$ and has Kodaira symbol I_1. In all other cases, the fibers over the non-rational points are smooth.

Proof. Let J_a be the unstable fiber, with $1 \leq r_a \leq 9$ irreducible components and $n_a = 2r_a + 1$ rational points. Note that the latter is odd. Without restriction, we may assume that a is given by $t = 0$. Write $b, c \in \mathbb{P}^1$ for the remaining two rational points, whence

$$25 = \operatorname{Card} Y(\mathbb{F}_2) = n_a + n_b + n_c. \tag{11}$$

First, assume that $r_a \geq 6$. To start with, we consider the situation that both fibers J_b and J_c are smooth. Applying an automorphism of the projective line, we may assume that the first is ordinary. Then n_b is even. By (11) the number n_c is also even, hence J_c is also ordinary. Write $J_b = E_{2i}$ and $J_c = E_{2j}$. Applying another automorphism of the projective line, we may assume that b is given by $t = 1$. Recall that $r_a \geq 1$ is the number of irreducible components in J_a. From (11) one infers $r_a + i + j = 12$. Using $i + j \leq 4$ we obtain $r_a \geq 8$. Consequently, the possible configurations of fibers over rational points are

$$\mathrm{II}^* + E_2 + E_4, \quad \mathrm{I}_4^* + E_2 + E_4, \quad \mathrm{III}^* + E_4 + E_4, \quad \mathrm{I}_3^* + E_4 + E_4.$$

In the first and last cases, Lang's Classification [40] tells us that there must be an additional singular geometric fiber, which is unique and has Kodaira symbol I_1. This uniqueness ensures that it comes from a fiber over a rational point, contradiction.

We now analyze the case that J_a has Kodaira symbol I_4^*. We are thus in Lang Case 5D from [40], and the Weierstraß equation takes the form $y^2 + txy = x^3 + td_1x^2 + t^4x$, where $t \nmid d_1$. *In this context, symbols like $d_i, e_i, \ldots$ denote polynomials in the variable t of degree $\leq i$.* Here we have the two cases $d_1 = 1$ and $d_1 = 1 + t$. The former is transformed into the latter, by the automorphism $t \mapsto t/(1 + t)$ of the projective line, followed by the change of coordinates $x = u^2x'$ and $y = u^3y'$ with $u = 1/(1+t)$. For $d_1 = 1$ the fibers, the Weierstraß equation and the ensuing j-invariant are as in the table.

Suppose now that J_a has Kodaira symbol III^*. Here we are in Lang Case 7, and the configuration of singular geometric fibers must be $\mathrm{III}^* + \mathrm{I}_1 + \mathrm{I}_1$. The Weierstraß equation is of the form $y^2 + txy + t^3c_0y = x^3 + t^2d_0x^2 + t^3e_1x + t^5f_1$ with $t \nmid e_1$. Making a change of variables we achieve $c_0 = 0$ and $e_1 = 1$. The discriminant becomes $\Delta = t^{10}(tf_1 + 1)$. Since the fiber over the unique point $x \in \mathbb{P}^1$ with residue field $\kappa(x) \simeq \mathbb{F}_4$ is singular, we must have $f_1 = 1 + t$. The case $d_0 = 1$ yields $J_b = E_2$, contradiction. Thus $d_0 = 0$. Again the Weierstraß equation and the ensuing j-invariant are as in the table.

We next consider the situation that one of the fibers over rational points is semistable. Applying an automorphism of the projective line, we may assume that this happens over $c \in \mathbb{P}^1$, and that this point is given by $t = \infty$. Recall that $r_c \geq 1$ is the number of irreducible components in J_c. The remaining fiber is ordinary, and we write $J_b = E_{2i}$ for some $1 \leq i \leq 2$, such that $n_b = 2i$. Suppose first that the semistable fiber is untwisted. Then $n_c = 2r_c$, and (11) gives $r_a + i + r_c = 12$. In particular, we have

$$r_c = 12 - i - r_a \geq 10 - r_a.$$

For $r_a = 6$ this estimate gives $r_c \geq 4$. By Lang's Classification, the configuration of fibers over rational points must be $\mathrm{I}_1^* + E_4 + \mathrm{I}_4$. For $r_a = 7$ we get $r_a \geq 3$, and now the classification ensures that the configuration must be $\mathrm{IV}^* + E_4 + \mathrm{I}_3$. In this case, however, there is a unique additional degenerate fiber, which has Kodaira symbol I_1. The uniqueness ensures that it lies over a rational point, contradiction. For $r_a = 8$ we obtain $r_c \geq 2$, and the configuration is $\mathrm{III}^* + E_4 + \mathrm{I}_2$. Finally, for $r_a = 9$ the possible configuration is $\mathrm{II}^* + E_4 + \mathrm{I}_1$. If the semistable fiber is twisted, we have $r_c \leq 2$ and $n_c = 2r_c + 2$. Now (11) becomes $r_a + i + r_c = 11$ and thus $r_c \geq 11 - i - r_a \geq 7$, and the argument is similar. Summing up, we have the following possibilities:

$$\mathrm{I}_1^* + E_4 + \mathrm{I}_4, \quad \mathrm{III}^* + E_4 + \mathrm{I}_2, \quad \mathrm{III}^* + E_2 + \tilde{\mathrm{I}}_2, \quad \mathrm{II}^* + E_4 + \mathrm{I}_1, \quad \mathrm{II}^* + E_2 + \tilde{\mathrm{I}}_1.$$

Let me give the details for the first case: Here we are in Lang Case 5A, with a Weierstraß equation of the form $y^2 + txy + t^2c_1y = x^3 + td_1x^2 + t^3e_1x + t^4c_2$, subject to the conditions $t \nmid c_1$ and $t \nmid d_1$. Making a change of coordinates, we may assume $c_1 = 1$ and $e_1 = t$. Now the discriminant becomes

$$\Delta = t^8(t^2c_2 + t^3 + td_1 + t^4 + 1 + t).$$

The second factor is a unit, hence equals one, because we assume that the semistable fiber is at $t = \infty$. Comparing coefficients yields the solutions $c_2 = t + t^2$, $d_1 = 1$ and $c_2 = 1 + t + t^2$, $d_1 = 1 + t$. For the latter solutions, the semistable fiber is twisted, as revealed by the Tate Algorithm at $t = \infty$. Hence we are in the former case, and the Weierstraß equation and the resulting j-invariant is as in the Table. The remaining cases are even simpler, and handled in the same fashion. We leave the details to the reader. Summing up, this settles the case $r_a \geq 6$.

It remains to treat the case that $r_a \leq 5$. According to Proposition 10.3, the configuration over the singular fibers is $\mathrm{III} + E_4 + \mathrm{I}_8$, and we are in Lang Case 2A, with Weierstraß equation $y^2 + txy + tc_2y = x^3 + tc_1x^2 + tc_3x + t^2c_4$, with $t \nmid c_2$ and $t \nmid c_3$ and $\Delta = t^4$. Making successive changes of coordinates $x = x' + R$ and $y = y' + T$ where the indeterminate t divides R and T, we may assume that $c_2 = c_3 = 1$. Then $1 = \Delta/t^4 = c_4t^4 + t^3 + c_1t^3 + 1$. It follows $c_1 = 1 + \alpha t$ and $c_4 = \alpha$ for some scalar $\alpha \in \mathbb{F}_2$. In case $\alpha = 1$ we have $J_b = E_2$, contradiction, hence $\alpha = 0$. □

It remains to understand the situation with trivial functional j-invariant:

Theorem 10.5. *Suppose the functional j-invariant is zero. Up to some coordinate changes and automorphisms of the projective line, the fibration $\phi : J \to \mathbb{P}^1$ is given by exactly one of the following three Weierstraß equations:*

Weierstraß equation	fibers over $t = 0, 1, \infty$
$y^2 + t^2y = x^3 + tx^2$	$\mathrm{I}_1^* + E_5 + \mathrm{IV}$
$y^2 + t^2y = x^3 + t^3x$	$\mathrm{IV}^* + E_5 + \mathrm{III}$
$y^2 + t^2y = x^3$	$\mathrm{IV}^* + E_3 + \mathrm{IV}$

In all three cases, the fibers over non-rational points $x \in \mathbb{P}^1$ are smooth.

Proof. There are neither ordinary nor semistable fibers, because the map $j : \mathbb{P}^1 \to \mathbb{P}^1$ misses the values $t = 1$ and $t = \infty$. In particular, each smooth fiber over a rational point is isomorphic to one of the elliptic curves E_1, E_3 or E_5. So there are at least two rational points $a \neq c$ on $\mathbb{P}^1$ such that J_a and J_c are unstable. By Lang's Classification, there are at most three singular geometric fibers. If present, the third singular geometric fiber must be Galois invariant, whence occurs over the remaining rational point $b \in \mathbb{P}^1$. Thus all fibers over non-rational points are smooth.

Suppose first that all fibers of $\phi : J \to \mathbb{P}^1$ are reduced, such that the Kodaira symbols for the singular fibers are II, III or IV. Then each fiber over a rational point contains at most seven rational points, which gives the contradiction $25 = \operatorname{Card} Y(\mathbb{F}_2) \leq 3 \cdot 7$. Thus there is a non-reduced rational fiber.

Recall that J_a and J_c are unstable. Let $r_a, r_c \geq 1$ be the respective number of irreducible components. After applying an automorphism of the projective line, we may assume that J_a is non-reduced. By Lang's Classification ([40], page 5829), the remaining fiber J_b must be smooth (one referee pointed out that this also follows from the fact that the valuation of the discriminant in characteristic $p = 2$ is at least four at an additive fiber, and at least eight at a non-reduced fiber). Write $J_b = E_{2i-1}$ for some $1 \leq i \leq 3$. Then $25 = \operatorname{Card} Y(\mathbb{F}_2) = (2r_a + 1) + (2i - 1) + (2r_c + 1)$, or equivalently $r_a + i + r_c = 12$. Since J_a is non-reduced and the Kodaira symbol I_0^* does not occur, we have $6 \leq r_a \leq 9$. In particular, $r_c \leq 5$, so the unstable fiber J_c is reduced, such that actually $1 \leq r_c \leq 3$. The case $r_a = 6$ implies $r_c = i = 3$. In case $r_a = 7$ we get $r_c = 3$, $i = 2$ or $r_c = 2$, $i = 3$. Going through Lang's Classification, we see that these are the only possibilities, which already give the second column of the table.

Suppose the configuration of fibers over rational points is $\mathrm{I}_1^* + E_5 + \mathrm{IV}$. Then we are in Lang Case 13A, and the Weierstraß equation takes the form $y^2 + t^2c_1y = x^3 + td_1x^2 + t^3e_1x + t^4d_2$ with $t \nmid d_1$ and $t \nmid c_1$. The discriminant is $\Delta = t^8c_1^4$, thus $c_1 = 1$. Making a change of coordinate $y' = y + sx$, we achieve $d_1 = 1$. Making a further change of coordinates, we may assume that e_1 has no linear term, and that d_2 has neither constant nor quadratic term. Now the Weierstraß equation becomes $y^2 + t^2y = x^3 + tx^2 + t^3e_0x + t^5d_0$. One checks that only $e_0 = d_0 = 0$ yields $J_b = E_5$.

Next, suppose the configuration is $\mathrm{IV}^* + E_5 + \mathrm{III}$. Now we are in Lang Case 14, with $y^2 + t^2c_1y = x^3 + t^2d_0x^2 + t^3e_1x + t^4d_2$ subject to the condition $t \nmid c_1$. The discriminant is $\Delta = t^8c_1^4$, whence $c_1 = 1$. Making a suitable change of variables $y = y' + sx'$ we achieve $d_0 = 0$. As in the preceding paragraph, we put the Weierstraß equation into the form $y^2 + t^2y = x^3 + t^3e_0x + t^5d_0'$. Only the case $e_0 = 1$ and $d_0' = 0$ yields $J_b = E_5$.

Finally, consider the configuration $\mathrm{IV}^* + E_3 + \mathrm{IV}$. We argue as in the preceding paragraph to reduce to the very same Weierstraß equation. Now only $e_0 = d_0' = 0$ yields the fiber $J_b = E_3$. □

11. Possible configurations for Enriques surfaces

Let Y be an Enriques surface over the prime field $k = \mathbb{F}_2$ with constant Picard scheme. According to Theorem 5.6, it admits at least one genus-one fibration. Let us collect the information on fibers we have gathered so far:

Proposition 11.1. *Let $\varphi : Y \to \mathbb{P}^1$ be a genus-one fibration. Up to automorphisms of the projective line, the possible configurations of fibers over the the rational points*

$t = 0, 1, \infty$ *are given by the following table:*

(12)

elliptic	*quasielliptic*
$\mathrm{II}^* + E_4 + \mathrm{I}_1$	$\mathrm{II}^* + \mathrm{II} + \mathrm{II}$
$\mathrm{II}^* + E_2 + \tilde{\mathrm{I}}_1$	$\mathrm{III}^* + \mathrm{III} + \mathrm{II}$
$\mathrm{III}^* + E_4 + \mathrm{I}_2$	
$\mathrm{III}^* + E_2 + \tilde{\mathrm{I}}_2$	
$\mathrm{III}^* + E_4 + E_4$	
$\mathrm{IV}^* + E_3 + \mathrm{IV}$	
$\mathrm{IV}^* + E_5 + \mathrm{III}$	
$\mathrm{I}_4^* + E_4 + E_2$	$\mathrm{I}_4^* + \mathrm{II} + \mathrm{II}$
$\mathrm{I}_1^* + E_5 + \mathrm{IV}$	$\mathrm{I}_2^* + \mathrm{III} + \mathrm{III}$
$\mathrm{I}_1^* + E_4 + \mathrm{I}_4$	
$\mathrm{III} + E_4 + \mathrm{I}_8$	

If moreover the Enriques surface Y is non-exceptional, the following holds:

(i) *Each fiber with Kodaira symbol* IV, IV^*, II^*, I_1^* *is multiple.*
(ii) *If there is a simple fiber with Kodaira symbol* III^*, *the fibration is elliptic, and the configuration of fibers over rational points is* $\mathrm{III}^* + E_4 + E_4$.
(iii) *For each multiple fiber* $\varphi^{-1}(a)$ *with Kodaira symbol* IV^*, III^* *or* II^*, *there is no* (-2)*-curve* $R \subset Y$ *with intersection number* $\varphi_{\mathrm{ind}}^{-1}(a) \cdot R = 1$.

Proof. The main task is to verify the table. Let $\bar{a} : \mathrm{Spec}(\Omega) \to \mathbb{P}^1$ be a geometric point, with image $a \in \mathbb{P}^1$. If this image point is non-rational, the geometric fiber $Y_{\bar{a}}$ is irreducible, according to Theorem 2.1. In any case, the map $\Gamma(Y_{\bar{a}}) \to \Gamma(Y_a)$ between dual graphs is a graph isomorphism. Moreover, the Kodaira symbol I_0^* is impossible, according to Proposition 3.2. In particular, Y_a and $Y_{\bar{a}}$ have the same number of irreducible components. Write $r_a \geq 1$ for this number. Then we have the inequality $2 + \sum(r_a - 1) \leq \rho(Y) = 10$, and thus $r_0 + r_1 + r_\infty \leq 11$.

Suppose first that the fibration is quasielliptic. Since the Picard group $\mathrm{Pic}^0(Y_F)$ for the generic fiber is annihilated by $p = 2$, we actually have $r_0 + r_1 + r_\infty = 11$. The possible Kodaira symbols and the corresponding number of irreducible components are as follows ([11], Theorems 5.7.4–5.7.6):

Kodaira symbol	II	III	I_2^*	III^*	II^*	I_4^*
number of irr. comp.	1	2	7	8	9	9

Since $9 + 1 + 1 = 8 + 2 + 1 = 7 + 2 + 2 = 11$ are the only possible solutions, the second column of table (12) follows.

Now suppose that $\varphi : Y \to \mathbb{P}^1$ is elliptic, and consider the resulting jacobian fibration $\phi : J \to \mathbb{P}^1$. Then the smooth proper surface J is geometrically rational (Proposition 6.2), with constant Picard scheme (Theorem 9.1). As above, the geometric fibers $J_{\bar{a}}$ are irreducible if $a \in \mathbb{P}^1$ is non-rational, and in any case $\Gamma(J_{\bar{a}}) \to \Gamma(J_a)$ is a graph isomorphism. According to Theorems 10.4 and 10.5, the first column of table (12) gives the possible configurations of singular fibers over the rational points for $\phi : J \to \mathbb{P}^1$. By Proposition 9.2 the fibers $Y_{\bar{a}}$ and $J_{\bar{a}}$ have the same Kodaira symbols. Moreover, we actually have $J_a \simeq Y_a$ if the fiber Y_a is simple. In particular, this holds for all semistable fibers. According to Proposition 8.3, we also have $J_a \simeq (Y_a)_{\text{red}}$ provided that Y_a is multiple with smooth reduction. In turn, the first column of table (12) indeed describes the configuration of degenerate fibers for $\varphi : Y \to \mathbb{P}^1$.

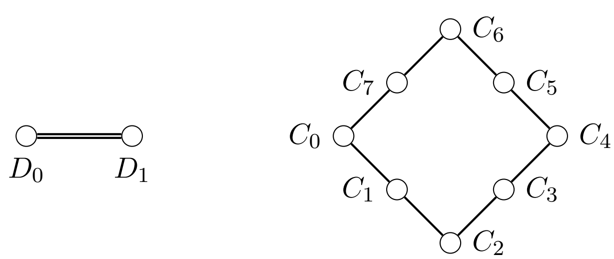
Now suppose that Y is non-exceptional. If $\varphi : Y \to \mathbb{P}^1$ is elliptic having some $\varphi^{-1}(a)$ with Kodaira symbol $\text{IV}, \text{IV}^*, \text{II}^*, \text{I}_1^*$ there must be some $\varphi_{\text{ind}}^{-1}(b)$ that is supersingular or semistable, in light of the table, so $\varphi^{-1}(a)$ is multiple by Lemma 4.3. If $\varphi : Y \to \mathbb{P}^1$ is quasielliptic, fibers with Kodaira symbol II^* are multiple according to Proposition 4.4. This establishes (i).

Next assume that $\varphi^{-1}(a)$ is simple with Kodaira symbol III^*. Again by Proposition 4.4 the fibration must be elliptic. By the table, the configuration of fibers over rational points is $\text{III}^* + E_4 + \text{I}_2$ or $\text{III}^* + E_2 + \tilde{\text{I}}_2$ or $\text{III}^* + E_4 + E_4$. The former are impossible by Lemma 4.3, which gives (ii). Finally, assertion (iii) is a consequence of Proposition 4.4. □

Starting from the above information, we rule out step by step each possible Kodaira symbol, mostly from combinatorial considerations. In this section, we will reduce from fifteen to eight possible configurations for $\varphi : Y \to \mathbb{P}^1$.

Proposition 11.2. *Assumption as in Proposition 11.1. Then there are no fibers with Kodaira symbol* I_8*.*

Proof. Seeking a contradiction, we assume that $\varphi^{-1}(\infty)$ has Kodaira symbol I_8, and $\varphi^{-1}(0)$ has symbol III. Then the configuration of reducible fibers has dual graph:



Here the double edge indicates that the scheme $D_0 \cap D_1$ has length two. The semistable fiber $\varphi^{-1}(\infty)$ is simple whereas the unstable fiber $\varphi^{-1}(0)$ is multiple. Moreover, there is a genus-one fibration $\psi : Y \to \mathbb{P}^1$ so that the multiple fibers $2F$ have $F \cdot \varphi_{\text{ind}}^{-1}(0) = 1$. After renumeration, we may assume $(F \cdot D_1) = 1$, hence D_0 is vertical with respect to ψ. Without restriction, we may assume that $\psi^{-1}(\infty)$ is simple, whereas the ψ-fibers over $a = 0, 1$ are multiple. Write $\psi_{\text{ind}}^{-1}(0) = \sum_{i=0}^r m_i \Theta_i$. After renumeration, we may assume $(D_1 \cdot \Theta_0) > 0$. From

$$1 = \varphi_{\text{ind}}^{-1}(0) \cdot \psi_{\text{ind}}^{-1}(0) = D_1 \cdot \sum_{i=0}^r m_i \Theta_i = m_0(D_1 \cdot \Theta_0) + \sum_{i=1}^r m_i (D_1 \cdot \Theta_i)$$

we see that $(D_1 \cdot \Theta_0) = m_0 = 1$, and that the curve $\Theta_1 + \ldots + \Theta_r$ is disjoint from D_1. Furthermore, D_0 does not belong to $\psi^{-1}(0)$, because $D_0 = \Theta_j$ gives the contradiction

$$1 = \varphi_{\text{ind}}^{-1}(0) \cdot \psi_{\text{ind}}^{-1}(0) = D_1 \cdot \psi_{\text{ind}}^{-1}(0) \geq D_1 \cdot m_j \Theta_j \geq 2m_j \geq 2.$$

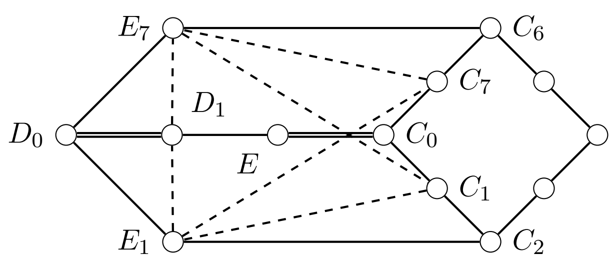
Consequently $\Theta_1 + \ldots + \Theta_r \subset \varphi^{-1}(\infty)$. From $m_0 = 1$ we infer that $\Theta_1 + \ldots + \Theta_r$ is connected, hence is a chain. So the multiple fiber $\psi_{\text{ind}}^{-1}(0)$ is either ordinary or has Kodaira symbol II, III or IV. The same analysis applies to the other multiple fiber $\psi_{\text{ind}}^{-1}(1)$. From Proposition 11.1, we see that the possible Kodaira symbols for the simple fiber $\psi^{-1}(\infty)$ are I_8 or $\mathrm{II}^*, \mathrm{III}^*, \mathrm{I}_4^*, \mathrm{I}_2^*$.

To proceed write $\psi^{-1}(\infty) = \sum_{i=0}^s n_i\Upsilon_i$. Then $s \geq 6$ and $(\Upsilon_i \cdot \Upsilon_j) \leq 1$. After renumeration $\Upsilon_0 = D_0$. From

$$2 = \varphi_{\text{ind}}^{-1}(0) \cdot \psi^{-1}(\infty) = D_1 \cdot \sum_{i=0}^{s} n_i\Upsilon_i = 2n_0 + \sum_{i=1}^{s} n_i(D_1 \cdot \Upsilon_i)$$

we infer $n_0 = 1$, and that $\Upsilon_1 + \ldots + \Upsilon_s$ is disjoint from D_1. Seeking a contradiction, we now assume that $\psi^{-1}(\infty)$ is additive. By inspection, one sees that Υ_0 has exactly one neighbor in the dual graph, say Υ_1, and that $\Upsilon = \Upsilon_2 + \ldots + \Upsilon_s$ is not a disjoint union of chains. Moreover, Υ is disjoint from $\varphi^{-1}(0)$, hence must be strictly contained in $\varphi^{-1}(\infty)$. The latter is multiplicative, and it follows that Υ is a union of chains, contradiction. We conclude that $\psi^{-1}(\infty)$ has Kodaira symbol I_8.

Summing up, the configuration of rational fibers for ψ must be $\mathrm{III} + E_4 + \mathrm{I}_8$, according to the table in Proposition 11.1. Without restriction we may assume that $\psi_{\text{ind}}^{-1}(0) = \Theta_0 + \Theta_1$ is reducible. We saw above that Θ_1 belongs to $\varphi^{-1}(\infty)$, so after renumeration $\Theta_1 = C_0$. To simplify notation set $E = \Theta_0$, such that $\psi_{\text{ind}}^{-1}(0) = E + C_0$ with $(E \cdot D_1) = 1$ and $(E \cdot C_0) = 2$. It follows that D_0 and $C_2 + \ldots + C_6$ belong to $\psi^{-1}(\infty)$. Let E_1, E_7 be the additional components. The dual graph for our curves takes the following form:



Here the dashed edges indicate potential intersections. Using

$$2 = (D_0 + D_1) \cdot \sum \psi^{-1}(\infty) = (D_0 + D_1) \cdot (E_1 + E_7) = 1 + 1 + D_1 \cdot (E_1 + E_7)$$

we conclude that D_1 is actually disjoint from $E_1 \cup E_7$. Then

$$2 = 2(D_0 + D_1) \cdot E_1 = (C_0 + \ldots + C_7) \cdot E_1 = (C_1 + C_7) \cdot E_1 + 1$$

ensures that either $(E_1 \cdot C_1) = 1$, $(E_1 \cdot C_7) = 0$ or $(E_1 \cdot C_7) = 1$, $(E_1 \cdot C_1) = 0$. In the former case we get a curve $E_1 + C_2 + C_1$ of canonical type I_3, in the letter $E_1 + C_2 + \ldots + C_7$ of canonical type I_7, contradiction. □

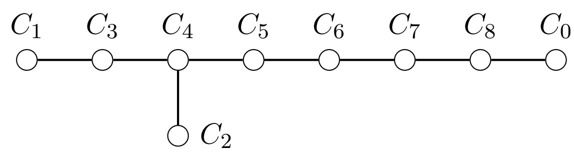
For the next observation it is crucial to exclude exceptional Enriques surfaces.

Proposition 11.3. *Assumption as in Proposition 11.1. Assume Y is non-exceptional. Then there are no fibers with Kodaira symbol* II^*, IV^* *or* IV. *Furthermore, there are no multiple fiber with Kodaira symbol* I_4^*.

Proof. According to Proposition 11.1, all fibers with Kodaira symbol II*, IV*, IV are multiple. Seeking a contradiction, we thus assume that $\varphi^{-1}(\infty)$ is a multiple with Kodaira symbol II*, IV*, IV or I$_4^*$.

By Proposition 4.4, there is another genus-one fibrations $\psi : Y \to \mathbb{P}^1$ such that $\varphi^{-1}(\infty)\cdot\psi^{-1}(\infty) = 4$. Without restriction, we may assume that $\psi^{-1}(\infty)$ is the fiber having the maximal number of irreducible components. Let $2F$ a multiple fiber for ψ, such that $F \cdot \varphi_{\text{ind}}^{-1}(\infty) = 1$.

We first treat the case that $\varphi^{-1}(\infty)$ is multiple with Kodaira symbol II*. With enumeration as in the Bourbaki tables [9], the dual graph for such a fiber is:



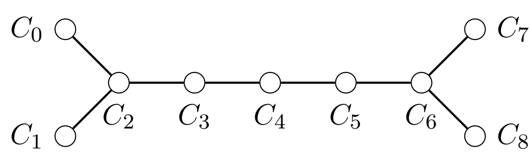
We also denote this dual graph by $T_{2,3,6}$. *In general, the symbol $T_{l,m,n}$ refers to a star-shaped graph with three terminal chains of length l, m, n, where the central vertex belongs to each of the terminal chains.*

We have $\varphi_{\text{ind}}^{-1}(\infty) = C_0 + 2C_1 + 3C_2 + \ldots + 2C_8$, and C_0 is indeed the only component with multiplicity $m = 1$. Thus $(C_0 \cdot F) = 1$. In turn, the connected curve $C_1 + \ldots + C_8$ has dual graph $T_{2,3,5}$ and is necessarily contained in the fiber $\psi^{-1}(\infty)$, which therefore must have Kodaira symbol II*. This is multiple by Proposition 11.1. Let $R \subset \psi^{-1}(\infty)$ be the additional component, with $(R \cdot C_8) = 1$. Then

$$1 = \varphi_{\text{ind}}^{-1}(\infty) \cdot \psi_{\text{ind}}^{-1}(\infty) = \varphi_{\text{ind}}^{-1}(\infty) \cdot R = (C_0 + 2C_8) \cdot R \geq 2(C_8 \cdot R) = 2,$$

contradiction. Summing up, there are no fibers with Kodaira symbol II*.

Suppose next that $\varphi^{-1}(\infty)$ is multiple with Kodaira symbol I$_4^*$. The dual graph is:

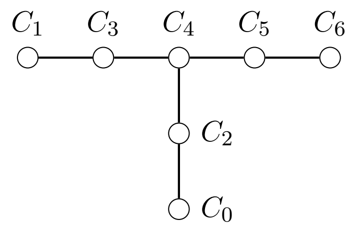


Now $\varphi_{\text{ind}}^{-1}(\infty) = C_0 + C_1 + 2(C_2 + \ldots + C_6) + C_7 + C_8$. After renumeration, we may assume $(F \cdot C_0) = 1$. In turn, the connected curve $C_1 + \ldots + C_8$ has dual graph $T_{2,2,6}$, and is necessarily contained in $\psi^{-1}(\infty)$. This must have Kodaira symbol I$_4^*$, as we already ruled out the symbol II*. Let $R \subset \psi^{-1}(\infty)$ be the additional component, with $(R \cdot C_2) = 1$. We then have

$$2 \geq \varphi_{\text{ind}}^{-1}(\infty) \cdot \psi_{\text{ind}}^{-1}(\infty) = \varphi_{\text{ind}}^{-1}(\infty) \cdot R = (C_0 + 2C_2) \cdot R = (C_0 \cdot R) + 2.$$

Consequently $C_0 \cap R = \varnothing$. In turn, $C_0 + \ldots + C_3 + R$ supports a curve of canonical type I$_0^*$, in contradiction to Proposition 11.1. So there are no multiple fibers with Kodaira symbol I$_4^*$.

Suppose next that $\varphi^{-1}(\infty)$ is multiple with Kodaira symbol IV^*. The dual graph for $\varphi^{-1}(\infty)$ is:



Now $\varphi_{\text{ind}}^{-1}(\infty) = (C_0 + C_1 + C_6) + 2(C_2 + C_3 + C_5) + 3C_4$. After renumeration we have $(F \cdot C_0) = 1$. In turn, the connected curve $C_1 + \ldots + C_6$ has dual graph $T_{2,3,3}$ and must be contained in $\psi^{-1}(\infty)$. Now recall that we already established that II^* does not appear in any genus-one fibration on Y. Consequently, the only possible Kodaira symbols for the fiber $\psi^{-1}(\infty)$ are IV^* and III^*. In the former case, the fiber is multiple by Proposition 11.1, and we write $R \subset \psi^{-1}(\infty)$ for the additional component. Then $1 = \varphi_{\text{ind}}^{-1}(\infty) \cdot \psi_{\text{ind}}^{-1}(\infty) = (C_0 + 2C_2) \cdot R \geq 2$, contradiction. So the fiber $\psi^{-1}(\infty)$ has Kodaira symbol III^*. Let $R, R' \subset \psi^{-1}(\infty)$ be the two additional components, say with $(R \cdot C_1) = (R' \cdot C_6) = 1$. Then

$$2 \geq \varphi_{\text{ind}}^{-1}(\infty) \cdot \psi_{\text{ind}}^{-1}(\infty) = (C_0 + C_1 + C_6) \cdot (R + R') = (C_0 \cdot R) + (C_0 \cdot R') + 2.$$

It follows that $\psi^{-1}(\infty)$ is simple. In light of Proposition 11.1, the configuration of fibers over the rational points for $\psi : Y \to \mathbb{P}^1$ is $\text{III}^* + E_4 + E_4$. In turn, each (-2)-curve $C \subset Y$ intersects the fiber $\psi^{-1}(\infty)$. By Proposition 11.1, we may assume that $\varphi^{-1}(0) = 2(C'_0 + \ldots + C'_{r-1})$ is multiple with Kodaira symbol III or IV with respective integers $r = 2$ or $r = 3$. Obviously $C_0, \ldots, C_6$ are disjoint from $\varphi^{-1}(0)$. In turn, $R \cup R'$ intersects each component $C'_i \subset \varphi^{-1}(0)$. Moreover, R is not contained in $\varphi^{-1}(0)$, because $R \cdot \varphi^{-1}(0) = R \cdot \varphi^{-1}(\infty) \geq 1$, and likewise for R'. So $2 = \varphi_{\text{ind}}^{-1}(0) \cdot \psi^{-1}(\infty)$ equals

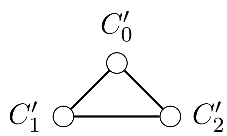
$$(C'_0 + \ldots + C'_{r-1}) \cdot \psi^{-1}(\infty) = (C'_0 + \ldots + C'_{r-1}) \cdot (R + R') \geq r \geq 2.$$

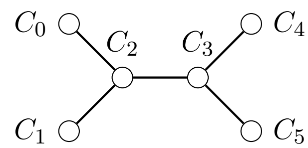
Thus we have equality at each step, hence $r = 2$, and the fiber $\varphi^{-1}(0)$ has Kodaira symbol III. If R, R' intersect the same component of $\varphi^{-1}(0)$, say C'_0, the curve $C_3 + \ldots + C_6 + R' + C'_0 + R + C_1$ is a curve of canonical type I_8, in contradiction to Proposition 11.2. So after renumeration, we may assume $(R \cdot C'_0) = (R' \cdot C'_1) = 1$ and $(R \cdot C'_1) = (R' \cdot C'_0) = 0$. In turn, $R \cap C'_0$ and $C'_1 \cap C'_0$ yield two different rational points, and similarly for C'_1. Now consider the fibers $\psi^{-1}(0)$ and $\psi^{-1}(1)$, which are linearly equivalent curves. Both are multiple, because $\psi^{-1}(\infty)$ is simple, and their reductions are smooth, as observed above. So $\psi_{\text{ind}}^{-1}(0)$ and $\psi_{\text{ind}}^{-1}(1)$ are disjoint but numerically equivalent integral curves. After renumeration, we may assume that they intersect C'_0 but not C'_1. Then

$$R \cap C'_0 \quad \text{and} \quad C'_1 \cap C'_0 \quad \text{and} \quad \psi_{\text{ind}}^{-1}(0) \cap C'_0 \quad \text{and} \quad \psi_{\text{ind}}^{-1}(1) \cap C'_0$$

define four distinct rational points on $C'_0 = \mathbb{P}^1$ contradicting $|\mathbb{P}^1(\mathbb{F}_2)| = 3$. Thus there is no fiber with Kodaira symbol IV^*.

Let us now treat that case that $\varphi^{-1}(\infty)$ is multiple with Kodaira symbol IV. By Proposition 11.1, we may apply an automorphism of $\mathbb{P}^1$ so that the configuration of fibers over the rational points $t = \infty, 0, 1$ becomes $\mathrm{IV} + \mathrm{I}_1^* + E_5$. The former two are multiple, and the latter is simple. The dual graph for $\varphi^{-1}(\infty) \cup \varphi^{-1}(0)$ is:





After renumeration, we may assume that $(F \cdot C_0) = (F \cdot C_0') = 1$. The connected curve $C_1 + \ldots + C_5$ has dual graph $T_{2,2,3}$ and must be contained in the fiber $\psi^{-1}(\infty)$. Let $m \geq 2$ be the multiplicity of $C_2 \subset \psi_{\mathrm{ind}}^{-1}(\infty)$. From $2 \geq \varphi_{\mathrm{ind}}^{-1}(\infty) \cdot \psi^{-1}(\infty) \geq C_0 \cdot mC_2 = m \geq 2$ we infer that the fiber $\psi^{-1}(\infty)$ is simple. Likewise, C_1' and C_2' are ψ-vertical, say contained in $\psi_{\mathrm{ind}}^{-1}(0)$, with respective multiplicities $m_1, m_2 \geq 1$. Now

$$2 \geq \varphi_{\mathrm{ind}}^{-1}(\infty) \cdot \psi^{-1}(0) \geq C_0' \cdot (m_1 C_1' + m_2 C_2') = m_1 + m_2 \geq 2,$$

we see that also $\psi^{-1}(0)$ is simple, contradiction. □

Combining Propositions 11.1 and 11.2 and 11.3 we get the following key reduction:

Proposition 11.4. *Suppose Y is a non-exceptional Enriques surface over $k = \mathbb{F}_2$ with constant Picard scheme. Let $\varphi : Y \to \mathbb{P}^1$ be a genus-one fibration. Up to automorphisms of the projective line, the possible configurations of fibers over the the rational points $t = 0, 1, \infty$ are given by the following table:*

elliptic	*quasielliptic*
$\mathrm{I}_4^* + E_4 + E_2$	$\mathrm{I}_4^* + \mathrm{II} + \mathrm{II}$
$\mathrm{III}^* + E_4 + \mathrm{I}_2$	$\mathrm{III}^* + \mathrm{III} + \mathrm{II}$
$\mathrm{III}^* + E_2 + \tilde{\mathrm{I}}_2$	
$\mathrm{III}^* + E_4 + E_4$	
$\mathrm{I}_1^* + E_4 + \mathrm{I}_4$	$\mathrm{I}_2^* + \mathrm{III} + \mathrm{III}$

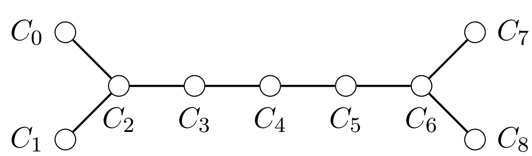
Moreover, all I_1^*-fibers are multiple, the fibers with Kodaira symbol $\mathrm{I}_4^*, \mathrm{I}_n, \tilde{\mathrm{I}}_2$ are simple, and there is another genus-one fibration $\psi : Y \to \mathbb{P}^1$ with $\varphi^{-1}(\infty) \cdot \psi^{-1}(\infty) = 4$.

12. Elimination of I_4^*-fibers

Let Y be a non-exceptional Enriques surface over the ground field $k = \mathbb{F}_2$, with constant Picard scheme. The goal of this section is to exclude the two I_4^*-cases from the table in Proposition 11.4.

Proposition 12.1. *There is no genus-one fibration $\varphi : Y \to \mathbb{P}^1$ having a fiber with Kodaira symbol I_4^*.*

Proof. Seeking a contradiction, we assume that $\varphi^{-1}(\infty)$ has Kodaira symbol I_4^*. Its dual graph is:



The fiber must be simple, according to Proposition 11.4, so the schematic fiber is $\varphi^{-1}(\infty) = C_0 + C_1 + 2(C_2 + \ldots + C_6) + C_7 + C_8$. Moreover, there is a genus-one fibration $\psi : Y \to \mathbb{P}^1$ so that the multiple fiber $2F$ have $F \cdot \varphi^{-1}(\infty) = 2$. After renumeration, we have the following possibilities: $(F \cdot C_i) = 1$ with $i \in \{2, 3, 4\}$ or $(F \cdot C_0) = (F \cdot C_1) = 1$ or $(F \cdot C_0) = (F \cdot C_8) = 1$ or $(F \cdot C_0) = 2$. Our task is to rule out each of these six possibilities. Without restriction, we assume that $\psi^{-1}(\infty)$ contains the largest number of irreducible components among all fibers.

Suppose first that $(F \cdot C_4) = 1$. Then the connected curves $C_0 + \ldots + C_3$ and $C_5 + \ldots + C_8$ both have dual graph $T_{2,2,2}$ and are contained in some fibers of ψ. According to Proposition 11.4, they both must be contained in $\psi^{-1}(\infty)$, which therefore has Kodaira symbol I_4^* and must be simple. Let $R \subset \psi^{-1}(\infty)$ the additional component, which has $(R \cdot C_i) = (R \cdot C_j) = 1$ for some $i \in \{0, 1, 3\}$ and $j \in \{5, 7, 8\}$. Let $1 \leq m_i, m_j \leq 2$ be the multiplicities of $C_i, C_j \subset \varphi^{-1}(\infty)$. From

$$4 = \varphi^{-1}(\infty) \cdot \psi^{-1}(\infty) = (m_i C_i + 2C_4 + m_j C_j) \cdot 2R = 2m_i + 4(C_4 \cdot R) + 2m_j$$

we infer that $R \cap C_4 = \varnothing$ and $m_i = m_j = 1$. After renumeration, we may assume $i = 1$ and $j = 7$. Then $C_1 + \ldots + C_7 + R$ is a curve of canonical type I_8, in contradiction to Proposition 11.4.

Suppose next that $(F \cdot C_3) = 1$. The connected curve $C_4 + \ldots + C_8$ has dual graph $T_{2,2,3}$ and must be contained in $\psi^{-1}(\infty)$. The curve $C_0 + C_2 + C_1$ is a chain and must be part of some fiber $\psi^{-1}(t)$ as well. If $t \neq \infty$, we infer from Proposition 11.4 that $\psi^{-1}(t)$ has Kodaira symbol I_4 and is simple. Let $R \subset \psi^{-1}(t)$ be the additional component, with $(R \cdot C_0) = (R \cdot C_1) = 1$. This gives

$$4 = \varphi^{-1}(\infty) \cdot \psi^{-1}(t) = (C_0 + C_1 + 2C_3) \cdot R = 1 + 1 + 2(C_3 \cdot R),$$

and hence $(C_3 \cdot R) = 1$. Thus $C_1 + C_2 + C_3 + R$ is another curve of canonical type I_4. Its intersection number with C_4 is one, so the fiber is multiple, contradiction. Summing up, $C_0 + C_1 + C_2$ and $C_4 + \ldots + C_8$ both belong to $\psi^{-1}(\infty)$, which therefore has Kodaira symbol I_4^* and must be simple, in light of Proposition 11.4. Let R be the additional component, which necessarily has $(R \cdot C_2) = (R \cdot C_4) = 1$. This gives

$$4 = \varphi^{-1}(\infty) \cdot \psi^{-1}(\infty) = (2C_2 + 2C_3 + 2C_4) \cdot 2R = 4 + 4(C_3 \cdot R) + 4 \geq 8,$$

contradiction.

Now suppose that $(F \cdot C_2) = 1$. Then the connected curve $C_3 + \ldots + C_8$ has dual graph $T_{2,2,4}$ and must be contained in $\psi^{-1}(\infty)$. Its Kodaira symbol is either II^*, I_4^*, III^* or I_2^*. The first alternative is impossible by Proposition 11.4. In the second alternative, the fiber must be simple. The additional component $R \subset \psi^{-1}(\infty)$ has $(R \cdot C_i) = 1$ for all $i \in \{0, 1, 3\}$, which gives the contradiction

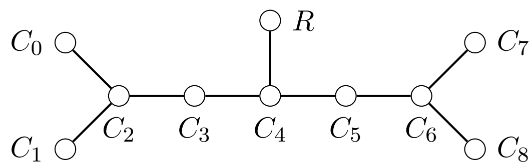
$$4 = \varphi^{-1}(\infty) \cdot \psi^{-1}(\infty) = (C_0 + C_1 + 2C_2 + 2C_3) \cdot 2R = 2 + 2 + 4(C_2 \cdot R) + 4 \geq 8.$$

In the third alternative, we deduce from Proposition 11.4 that one of the curves from $C_0 + C_1$ also belongs to $\psi^{-1}(\infty)$. Without restriction, we may assume that this is C_1. Let $R \subset \psi^{-1}(\infty)$ be the additional component, say with $(R \cdot C_7) = (R \cdot C_1) = 1$. From

$$4 \geq \varphi^{-1}(\infty) \cdot \psi^{-1}_{\mathrm{ind}}(\infty) = (C_1 + 2C_2 + C_7) \cdot 2R = 2 + 4(C_2 \cdot R) + 2$$

we get $R \cap C_2 = \varnothing$. But then $C_1 + \ldots + C_7 + R$ is a curve of canonical type I_8, in contradiction to Proposition 11.4. The only remaining possibility is that $\psi^{-1}(\infty)$

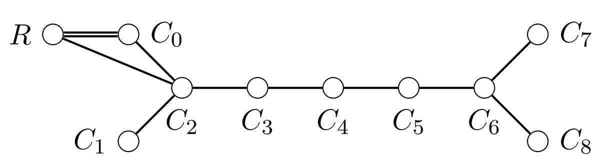
has Kodaira symbol I_2^*. So the additional component $R\subset\psi^{-1}(\infty)$ has $(R\cdot C_4)=1$. From
$$4\geq\varphi^{-1}(\infty)\cdot\psi_{\text{ind}}^{-1}(\infty)=(2C_2+2C_4)\cdot R=2(C_2\cdot R)+2$$
we deduce that $(C_2\cdot R)$ is either zero or one. In the latter case we get a curve $D=C_2+C_3+C_4+R$ of canonical type I_4. The ensuing fiber is multiple, because $(D\cdot C_5)=1$, in contradiction to Proposition 11.4. The former case yields the following dual graph, in contradiction to Proposition 12.2 below.



Next, suppose that we are in the case $(F\cdot C_0)=(F\cdot C_1)=1$. Then the connected curve $C_2+\ldots+C_8$ has dual graph $T_{2,2,5}$ and must be contained in $\psi^{-1}(\infty)$. The possible Kodaira symbols are II^* and I_4^*. The former is impossible by Proposition 11.4, so $\psi^{-1}(\infty)$ is simple with Kodaira symbol I_4^*. Let $R,R'\subset\psi^{-1}(\infty)$ be the two additional components, with $(R\cdot C_2)=(R'\cdot C_2)=1$. Then $4=\varphi^{-1}(\infty)\cdot\psi^{-1}(\infty)$ becomes
$$(C_0+C_1+2C_2)\cdot(R+R')=(C_0+C_1)\cdot(R+R')+2+2,$$
and we infer that $R\cup R'$ is disjoint from $C_0\cup C_1$. In turn, $C_0+C_1+R+R'+C_2$ supports a curve of canonical type I_0^*, in contradiction to Proposition 11.4.

Now suppose $(F\cdot C_0)=(F\cdot C_8)=1$. Then $C_1+\ldots+C_7$ is a chain and contained in $\psi^{-1}(\infty)$. The possible Kodaira symbols are I_4^* or III^*, by Proposition 11.4. The symbol I_4^* is handled as in the preceding paragraph. In the remaining case III^*, let $R\subset\psi^{-1}(\infty)$ be the additional component, which has $(R\cdot C_4)=1$. From
$$4\geq\varphi^{-1}(\infty)\cdot\psi_{\text{ind}}^{-1}(\infty)=(C_0+2C_4+C_8)\cdot 2R=2(C_0\cdot R)+4+2(C_8\cdot R)$$
we infer that R is disjoint from $C_0\cup C_8$. Now the configuration $C_0+\ldots+C_8+R$ contradicts Proposition 12.2 below.

It remains to cope with the last possibility $(F\cdot C_0)=2$, which is the most difficult. Now the connected curve $C_1+\ldots+C_8$ has dual graph $T_{2,2,6}$ and must be contained in $\psi^{-1}(\infty)$. Its Kodaira symbol has to be I_4^*, so the fiber is simple. The additional component $R\subset\psi^{-1}(\infty)$ has $(R\cdot C_2)=1$, and from
$$4=\varphi^{-1}(\infty)\cdot\psi^{-1}(\infty)=(C_0+2C_2)\cdot R=(C_0\cdot R)+2$$
we infer that the dual graph for $C_0+\ldots+C_8+R$ must be:

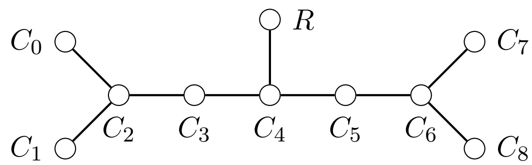


But this configuration is impossible, by Proposition 12.3 below. □

In the above arguments, we have used the following facts:

Proposition 12.2. *There is no configuration $C_0 + \ldots + C_8 + R$ of ten (-2)-curves on Y with simple normal crossings and the following dual graph:*

(13)



Proof. Seeking a contradiction, we assume that the configuration exists. The subconfiguration $C_0 + \ldots + C_8$ supports a curve of canonical type I_4^*. Let $\varphi : Y \to \mathbb{P}^1$ be the resulting genus-one fibration. Without restriction, we may assume that $\varphi^{-1}(\infty) = C_0 + C_1 + 2(C_2 + \ldots + C_6) + C_7 + C_8$. The fiber indeed must be simple by Proposition 11.4. Write $F \subset Y$ for some half-fiber, and let $S \subset \mathrm{Num}(Y)$ be the subgroup generated by $C_1, \ldots, C_8, R, F$. To proceed, we exploit properties of the resulting Gram matrix $N \in \mathrm{Mat}_{10}(\mathbb{Z})$, which can easily be checked with computer algebra. One computes in this way $\det(N) = -4$, whence $S \subset \mathrm{Num}(Y)$ if a subgroup of index two.

Nikulin's theory of discriminant forms [49] allows us to use the lattice S to gain enough control of the slightly larger $\mathrm{Num}(Y)$ and thus the geometry of Y. Set $S^* = \mathrm{Hom}(S, \mathbb{Z})$, and consider the injection $S \to S^*$ given by $x \mapsto (y \mapsto (x \cdot y))$. This linear map is described, with respect to the given basis $C_1, \ldots, F \in S$ and the dual basis $C_1^*, \ldots, F^* \in S^*$, by the matrix $N \in \mathrm{Mat}_n(\mathbb{Z})$. With respect to this dual basis, the induced $\mathbb{Q}$-valued form on S^* has Gram matrix

$$N^{-1} = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1/2 & -1/2 & 0 & 1 \\ -1 & -2 & -2 & -2 & -2 & -2 & -1 & -1 & 0 & 2 \\ -1 & -2 & -3 & -3 & -3 & -3 & -3/2 & -3/2 & 0 & 3 \\ -1 & -2 & -3 & -4 & -4 & -4 & -2 & -2 & 0 & 4 \\ -1 & -2 & -3 & -4 & -5 & -5 & -5/2 & -5/2 & 0 & 4 \\ -1 & -2 & -3 & -4 & -5 & -6 & -3 & -3 & 0 & 4 \\ -1/2 & -1 & -3/2 & -2 & -5/2 & -3 & -2 & -3/2 & 0 & 2 \\ -1/2 & -1 & -3/2 & -2 & -5/2 & -3 & -3/2 & -2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 4 & 4 & 4 & 2 & 2 & 1 & -2 \end{pmatrix} \tag{14}$$

and we have $S \subset \mathrm{Num}(Y) \subset S^*$. Note that for each column, the entries are the coordinates of the corresponding dual basis vectors with respect to the basis vectors. The invariant factors of N are computed to be $(2, 2)$, so the *discriminant group* $A_S = S^*/S$ is elementary abelian of order four, hence a 2-dimensional vector space over $\mathbb{F}_2$. As explained in loc. cit., Section 3, it comes with a quadratic form $q : A_S \to \mathbb{Q}/2\mathbb{Z}$ whose associated bilinear form $A_S \times A_S \to \mathbb{Q}/\mathbb{Z}$ is non-degenerate. Moreover, the generator for the extension $S \subset \mathrm{Num}(Y)$ corresponds to a non-zero isotropic vector in A_S, by loc. cit., Section 4. Examining the columns in (14), we see that

$$C_1^* \equiv C_3^* \equiv C_5^* \quad \text{and} \quad C_7^* \quad \text{and} \quad C_8^*$$

are non-zero modulo S, whereas the other dual basis vectors vanish modulo S. Moreover, one sees that $C_7^* + C_8^* \equiv C_1^*$ modulo S. So each two of the above form a basis of $A_S = S^*/S$, and the third one is their sum. Moreover,

$$(C_1^*)^2 = -1, \quad (C_7^*)^2 = (C_8^*)^2 = -2, \quad (C_1^* \cdot C_7^*) = (C_1^* \cdot C_8^*) = -1/2,$$

hence the isotropic vectors in the discriminant group come from C_7^* and C_8^*.

Now back to our genus-one fibration $\varphi : Y \to \mathbb{P}^1$. Consider the genus-one curve $Y_K = \varphi^{-1}(\eta)$ over the function field $K = k(\mathbb{P}^1)$. The restriction map $\operatorname{Pic}(Y) \to \operatorname{Pic}(Y_K)$ is surjective. It factors over $\operatorname{Num}(Y)$ because ω_Y can be written as the difference of half-fibers. The curve Y_K contains no rational points, the two-section R gives a splitting $\operatorname{Pic}(Y_K) = \operatorname{Pic}^0(Y_K) \oplus 2\mathbb{Z}$, and the curves $C_0, \ldots, C_8, F$ vanish in $\operatorname{Pic}(Y_K)$. This gives an identification $\operatorname{Pic}^0(Y_K) = \operatorname{Num}(Y)/S$, so this group has order two. Write $\mathscr{N}_K$ for the non-trivial invertible sheaf of degree zero on Y_K. By Riemann–Roch we have $\mathscr{N}_K(R) \simeq \mathscr{O}_{Y_K}(\tilde{R})$ for another two-section $\tilde{R} \subset Y$, and thus $\operatorname{Num}(Y) = S + \mathbb{Z}\tilde{R}$. Note that such a two-section is not unique; according to the Enriques Reducibility Theorem ([39], Theorem 4.1), we may furthermore assume that $\tilde{R}^2 = 0$ or $\tilde{R}^2 = -2$. In light of the preceding paragraph, the element $\tilde{R} \in S^*$ is either congruent to C_7^* or C_8^*, modulo S. Without loss of generality $\tilde{R} \equiv C_8^*$, which ensures $(\tilde{R} \cdot C_8) = 1$. Then $(\tilde{R} \cdot C_i) = 0$ for $2 \leq i \leq 6$, and $(\tilde{R} \cdot C_j) = 1$ for exactly one $j \in \{0, 1, 7\}$ besides $j = 8$. If $(\tilde{R} \cdot C_7) = 1$ then $\tilde{R} \equiv C_7^* + C_8^* \equiv C_1^*$, contradiction. So without loss of generality we may assume $(\tilde{R} \cdot C_0) = 1$. If $\tilde{R}^2 = -2$ then $\tilde{R} = \mathbb{P}^1$, and $C_0 + C_2 + \ldots + C_6 + C_8 + \tilde{R}$ is of canonical type I_8, which is impossible by Proposition 11.4. Thus $\tilde{R}^2 = 0$, and $\tilde{R}$ is a genus-one curve.

We can easily compute the intersection number $n = (\tilde{R} \cdot R)$: As member of S^* that is perpendicular to $C_1, \ldots, C_7$ we have $\tilde{R} = C_8^* + nR^* + F^*$. Glancing at (14) again, one sees

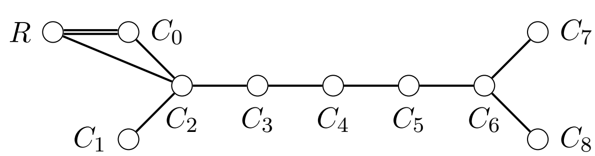
$$(C_8^*)^2 = (F^*)^2 = -2, \quad (R^*)^2 = (C_8^* \cdot R^*) = 0, \quad (C_8^* \cdot F^*) = 2, \quad (R^* \cdot F^*) = 1.$$

From $0 = \tilde{R}^2 = (C_8^* + nR^* + F^*)^2 = (-2 + 0 - 2) + 2(0 + 2 + n)$ we get $n = 0$, thus the curves $R, \tilde{R} \subset Y$ must be disjoint.

Let $\psi : Y \to \mathbb{P}^1$ be the genus-one fibration with $\psi^{-1}(0) = 2\tilde{R}$. The fiber is indeed multiple, because $(\tilde{R} \cdot C_0) = 1$. The curves $C_1 + \ldots + C_7 + R$ support a curve of canonical type III^* disjoint from $\tilde{R}$, and we can assume that $\psi^{-1}(\infty)$ is the corresponding fiber. The latter is simple, because $(C_0 \cdot \tilde{R}) = 1$ and $(C_0 \cdot \psi^{-1}_{\mathrm{ind}}(\infty)) = (C_0 \cdot 2C_2) = 2$. The dual graph (13) gives $\psi^{-1}(\infty) \cdot \varphi^{-1}(\infty) = 4$. According to Proposition 11.4, the configuration of fibers for φ is either $\mathrm{I}_4^* + E_4 + E_2$ or $\mathrm{I}_4^* + \mathrm{II} + \mathrm{II}$. In both cases we find some half fiber $\tilde{F}$ for φ that is integral and whose regular locus contains exactly two rational points. These rational points must be the intersections with the two half-fibers $\psi^{-1}_{\mathrm{red}}(0)$ and $\psi^{-1}_{\mathrm{red}}(1)$. However, we have $2 = (\tilde{F} \cdot \psi^{-1}(\infty)) = (\tilde{F} \cdot 2R)$, thus $\tilde{F} \cap R$ must be a third rational point in the regular locus of $\tilde{F}$, contradiction. ∎

In a similar way, we establish:

Proposition 12.3. *There is no configuration $C_0 + \ldots + C_8 + R$ of ten (-2)-curves having simple normal crossings, with exception $(C_0 \cdot R) = 2$, and the following dual graph:*

(15)


Proof. Seeking a contradiction, we assume that such a configuration exists. The subconfigurations $C_0 + C_1 + \ldots + C_8$ and $R + C_1 + \ldots + C_8$ support curves of canonical type I_4^*. Let $\varphi : Y \to \mathbb{P}^1$ and $\varphi' : Y \to \mathbb{P}^1$ be the resulting genus-one fibrations. Without loss of generality we may assume

$$\varphi^{-1}(\infty) = C_0 + C_1 + 2(C_2 + \ldots + C_6) + C_7 + C_8,$$
$$\varphi'^{-1}(\infty) = R + C_1 + 2(C_2 + \ldots + C_6) + C_7 + C_8.$$

Indeed, these fibers must be simple by Proposition 11.4, and $\varphi^{-1}(\infty) \cdot \varphi'^{-1}(\infty) = 4$. Choose some half-fibers $F, F' \subset Y$ for the respective fibrations, with intersection number $(F \cdot F') = 1$. Let $S \subset \mathrm{Num}(Y)$ be the subgroup generated by $C_1, \ldots, C_8, F', F$. As in the preceding proof, we exploit properties of the resulting Gram matrix $N \in \mathrm{Mat}_{10}(\mathbb{Z})$. Note that this is the lattice $U \oplus D_8$, as one referee pointed out. Again $S \subset \mathrm{Num}(Y)$ has index two, and the discriminant group $A_S = S^*/S$ is elementary abelian of order four. Recall that the inverse

$$N^{-1} = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1/2 & -1/2 & 0 & 0 \\ -1 & -2 & -2 & -2 & -2 & -2 & -1 & -1 & 0 & 0 \\ -1 & -2 & -3 & -3 & -3 & -3 & -3/2 & -3/2 & 0 & 0 \\ -1 & -2 & -3 & -4 & -4 & -4 & -2 & -2 & 0 & 0 \\ -1 & -2 & -3 & -4 & -5 & -5 & -5/2 & -5/2 & 0 & 0 \\ -1 & -2 & -3 & -4 & -5 & -6 & -3 & -3 & 0 & 0 \\ -1/2 & -1 & -3/2 & -2 & -5/2 & -3 & -2 & -3/2 & 0 & 0 \\ -1/2 & -1 & -3/2 & -2 & -5/2 & -3 & -3/2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{16}$$

is the Gram matrix of the induced $\mathbb{Q}$-valued form on S^* with respect to the dual basis $C_1^*, \ldots C_8^*, F'^*, F^*$. The generator for the extension $S \subset \mathrm{Num}(Y)$ corresponds to a non-zero isotropic vector in A_S, with respect to the quadratic form $q : A_R \to \mathbb{Q}/2\mathbb{Z}$. Glancing at (16), we see that

$$C_1^* \equiv C_3^* \equiv C_5^* \quad \text{and} \quad C_7^* \quad \text{and} \quad C_8^*.$$

are non-zero modulo S, whereas the other dual basis vectors vanish modulo S, and that $(C_1^*)^2 = -1$ and $(C_7^*)^2 = (C_8^*)^2 = -2$. So the isotropic vectors in the discriminant group come from C_7^* and C_8^*. Note that $F^* = F'$ and $F'^* = F$ belong to the sublattice $S \subset S^*$.

By Proposition 11.4, the half fiber F' is integral. We have $F' \cdot \varphi^{-1}(\infty) = 2$, hence F' is a two-section for $\varphi : Y \to \mathbb{P}^1$. Let $Y_K = \varphi^{-1}(\eta)$ be the generic fiber. As in the preceding proof, the two-section F' gives a splitting $\mathrm{Pic}(Y_K) = \mathrm{Pic}^0(Y_K) \oplus 2\mathbb{Z}$, we get an identification $\mathrm{Pic}(Y_K) = \mathrm{Num}(Y)/S$, and there must be another integral curve $\tilde{R} \subset Y$ that is a two-section, with $\mathrm{Num}(Y) = S + \mathbb{Z}\tilde{R}$. Furthermore, we may may assume $\tilde{R}^2 = 0$ or $\tilde{R}^2 = -2$. We now examine the possible non-zero intersection numbers with the components of $\varphi^{-1}(\infty)$: Since the discriminant group is annihilated by two, the case $(\tilde{R} \cdot C_i) = 2$ with $i \in \{0, 1, 7, 8\}$ would imply that $\tilde{R} \in S^*$ belongs to S, contradiction. Also $(\tilde{R} \cdot C_i) = 1$ with $i \in \{2, \ldots, 6\}$ is impossible, because then $\tilde{R}$ becomes zero or non-isotropic in $A_S = S^*/S$. Likewise, the two cases

$$(\tilde{R} \cdot C_0) = (\tilde{R} \cdot C_1) = 1 \quad \text{and} \quad (\tilde{R} \cdot C_7) = (\tilde{R} \cdot C_8) = 1$$

are impossible, because $C_0^* + C_1^* \equiv F^* + C_1^*$ and $C_7^* + C_8^*$ are non-isotropic. Up to renumeration, the only possibilities left are $(\tilde{R} \cdot C_0) = (\tilde{R} \cdot C_8) = 1$ and $(\tilde{R} \cdot C_1) = (\tilde{R} \cdot C_8) = 1$. If $\tilde{R}^2 = -2$ then $\tilde{R} = \mathbb{P}^1$ and in both cases get a curve of canonical type I_8, in contradiction to Proposition 11.4. Thus $\tilde{R}^2 = 0$, and $\tilde{R} \subset Y$ is a genus-one curve. But then the two cases where already discarded in the proof for Proposition 12.1. Summing up, the configuration (15) also does not exist. □

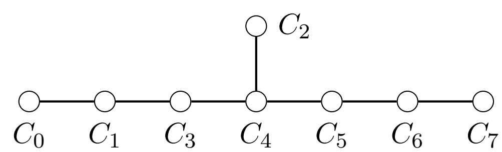
13. Elimination of III*-fibers

Let Y is a non-exceptional Enriques surface over the ground field $k = \mathbb{F}_2$, with constant Picard scheme. We now exclude the III^*-cases from the table in Proposition 11.4. We proceed in two steps, treating the case of multiple fibers first:

Proposition 13.1. *There is no genus-one fibration $\varphi : Y \to \mathbb{P}^1$ having a multiple fiber with Kodaira symbol* III^*.

Proof. Seeking a contradiction, we assume that $\varphi^{-1}(\infty)$ is multiple with Kodaira symbol III^*. The dual graph is:

(17)



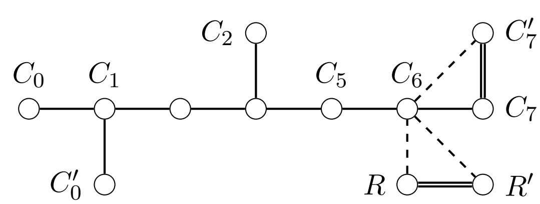
According to Proposition 11.4, there is a genus-one fibration $\psi : Y \to \mathbb{P}^1$ having $\psi^{-1}(\infty) \cdot \varphi^{-1}(\infty) = 4$. We may assume that $\psi^{-1}(\infty)$ has the largest number of irreducible components, among all fibers. Choose a multiple fiber $2F$ for ψ. After renumeration, we get $(F \cdot C_0) = 1$. The connected curve $C_1 + \ldots + C_7$ has dual graph $T_{2,3,4}$ and is thus contained in $\psi^{-1}(\infty)$. Its Kodaira symbol must be III^*, according to Proposition 11.4. Let $C_0' \subset \psi^{-1}(\infty)$ be the additional component, which has $(C_0' \cdot C_1) = 1$. From

$$2 \geq \varphi_{\mathrm{ind}}^{-1}(\infty) \cdot \psi_{\mathrm{ind}}(\infty) = (C_0 + 2C_1) \cdot C_0' = (C_0 \cdot C_0') + 2$$

we infer that C_0 and C_0' are disjoint. Thus $C_0' + C_0 + \ldots + C_5$ supports a curve of canonical type I_2^*. Let $\tau : Y \to \mathbb{P}^1$ be the resulting genus-one fibration that has this curve as reduced fiber over $t = \infty$. The fiber is multiple, because $\tau_{\mathrm{ind}}^{-1}(\infty) \cdot C_6 = 1$. According to Proposition 11.4, the fibers over $t = 0, 1$ have Kodaira symbol III. Clearly, the curve C_7 belongs to one such fiber. After applying an automorphism of $\mathbb{P}^1$ we obtain

$$\tau_{\mathrm{ind}}^{-1}(0) = C_7 + C_7' \quad \text{and} \quad \tau_{\mathrm{ind}}^{-1}(1) = R + R'.$$

The dual graph for the configuration $C_0' + C_0 + \ldots + C_7 + C_7' + R + R'$ takes the following shape:



The dashed edges indicate intersection numbers on which we can say the following: First note that the intersection number $(R \cdot C_6) = 1$ would lead to the configuration $C_0' + C_0 + \ldots + C_7 + R$, which contradicts Proposition 12.2. For the same reason $(R' \cdot C_6) = 1$ is impossible. On the other hand, we have

$$2 = C_6 \cdot \tau^{-1}(\infty) \geq C_6 \cdot \tau_{\text{ind}}^{-1}(1) = (C_6 \cdot R) + (C_6 \cdot R').$$

After interchanging R and R' if necessary, we may assume that $(C_6 \cdot R') = 2$ and $(C_6 \cdot R) = 0$. If follows that R belongs to the fibers of $\psi : Y \to \mathbb{P}^1$. According to Proposition 11.1, the fiber $\psi^{-1}(\infty)$ with Kodaira symbol III^* must be multiple. Hence

$$1 = \varphi_{\text{ind}}^{-1}(\infty) \cdot \psi_{\text{ind}}^{-1}(\infty) = (2C_1 \cdot C_0') = 2,$$

contradiction. □

Proposition 13.2. *There is no genus-one fibration $\varphi : Y \to \mathbb{P}^1$ having a simple fiber with Kodaira symbol* III^*.

Proof. Seeking a contradiction, we assume that $\varphi^{-1}(\infty)$ is simple with Kodaira symbol III^*, with dual graph as in (17). According to Proposition 11.4, there is another genus-one fibration $\psi : Y \to \mathbb{P}^1$ having $\psi^{-1}(\infty) \cdot \varphi^{-1}(\infty) = 4$. We may assume that $\psi^{-1}(\infty)$ has the largest number of irreducible components, among all fibers. Let $2F$ be one of the multiple fibers for ψ, such that $F \cdot \varphi^{-1}(\infty) = 2$. After renumeration, we have $(F \cdot C_2) = 1$ or $(F \cdot C_0) = (F \cdot C_7) = 1$ or $(F \cdot C_1) = 1$ or $(F \cdot C_0) = 2$, with otherwise zero intersection numbers. Our task is to rule out these four possibilities.

Suppose first that $(F \cdot C_2) = 1$. Then $C_0 + C_1 + C_3 + \ldots + C_7$ is a chain that is vertical with respect to ψ, hence belongs to $\psi^{-1}(\infty)$. Its Kodaira symbol must be III^*, by Propositions 11.4 and 12.1. We have $\psi_{\text{red}}^{-1}(\infty) = C_0 + C_1 + C_3 + \ldots + C_7 + R$, with $(R \cdot C_4) = 1$. This gives

$$4 \geq \varphi^{-1}(\infty) \cdot \psi_{\text{ind}}^{-1}(\infty) = (4C_4 + 2C_2) \cdot 2R = 8 + 4(C_2 \cdot R) \geq 8,$$

contradiction.

Next suppose that $(F \cdot C_0) = (F \cdot C_7) = 1$. Then $C_1 + \ldots + C_6$ has dual graph $T_{2,3,3}$ and is vertical with respect to ψ, whence is contained in $\psi^{-1}(\infty)$. As above, the Kodaira symbol is III^*, so $\psi_{\text{red}}^{-1}(\infty) = C_0' + C_1 + \ldots + C_6 + C_7'$, with $(C_0' \cdot C_1) = (C_7' \cdot C_6) = 1$. Then

$$4 \geq \varphi^{-1}(\infty) \cdot \psi_{\text{ind}}^{-1}(\infty) = (C_0 + 2C_1 + 2C_6 + C_7) \cdot (C_0' + C_7') \geq 4 + \sum (C_i \cdot C_j'),$$

where the sum runs over $i, j \in \{0, 7\}$. It follows that $C_0' \cup C_7'$ is disjoint from $C_0 \cup C_7$. Thus the curves $C_0' + C_0 + \ldots C_7 + C_7'$ forms a configuration as in Proposition 12.2, contradiction.

We come to the case $(F \cdot C_1) = 1$, which is the most challenging. The curve $C_2 + \ldots + C_7$ has dual graph $T_{2,2,4}$ and is ψ-vertical, thus contained in $\psi^{-1}(\infty)$. Now the possible Kodaira symbols are I_4^* or III^* or I_2^*. The former was already discarded in Proposition 12.1. Suppose the symbol is III^*. It is simple by Proposition 13.1, and with Proposition 11.1 we infer that also C_0 belongs to $\psi^{-1}(\infty)$. Consequently $\psi_{\text{red}}^{-1}(\infty) = C_0 + R + C_2 + \ldots + C_7$ for some additional component R with either $(R \cdot C_0) = (R \cdot C_3) = 1$ or $(R \cdot C_0) = (R \cdot C_2) = 1$. In the former case

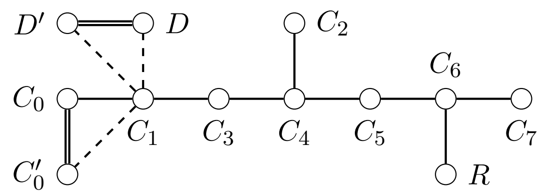
$$4 \geq \varphi^{-1}(\infty) \cdot \psi_{\text{ind}}^{-1}(\infty) = (C_0 + 2C_1 + 3C_3) \cdot 2R = 2 + 4(C_1 \cdot R) + 6 \geq 8,$$

contradiction, and the latter case is treated analogously. So our fiber has Kodaira symbol I_2^*, and $\psi_{\mathrm{red}}^{-1}(\infty) = C_2 + \ldots + C_7 + R$ for some additional component R with $(R \cdot C_6) = 1$. Now

$$4 \geq \varphi^{-1}(\infty) \cdot \psi_{\mathrm{ind}}^{-1}(\infty) = (2C_1 + 2C_6) \cdot R = 2(C_1 \cdot R) + 2.$$

This gives $(C_1 \cdot R) \leq 1$. But the case $(C_1 \cdot R) = 1$ yields a curve $R + C_1 + C_3 + \ldots + C_6$ of canonical type I_6, contradiction. Thus $C_1 \cap R = \varnothing$.

According to Proposition 11.4, ψ has $\mathrm{I}_2^* + \mathrm{III} + \mathrm{III}$ as configuration of fibers over rational points. The curve C_0 is ψ-vertical, and we can assume $\psi_{\mathrm{ind}}^{-1}(0) = C_0 + C_0'$. Write $\psi_{\mathrm{ind}}^{-1}(1) = D + D'$. The dual graph for all our curves takes the following form:

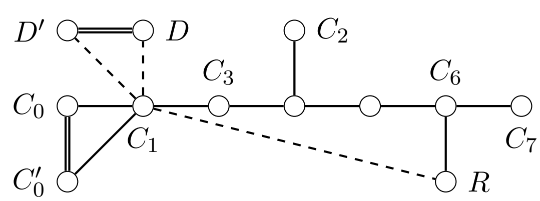


According to Proposition 11.1, the simple fiber $\varphi^{-1}(\infty)$ with Kodaira symbol III^* must intersect each (-2)-curve on Y. It follows that C_1 intersects D and D'. From $4 \geq \varphi^{-1}(\infty) \cdot \psi_{\mathrm{ind}}^{-1}(1) = 2C_1 \cdot (D + D')$ we infer $(C_1 \cdot D) = (C_1 \cdot D') = 1$. Thus $D + C_0 + \ldots + C_7 + R$ forms a configuration as in Proposition 12.2, again a contradiction.

It remains to deal with the case $(F \cdot C_0) = 2$. Then $C_1 + \ldots + C_7$ has dual graph $T_{2,3,4}$ and is vertical with respect to ψ, whence contained in $\psi^{-1}(\infty)$. Its Kodaira symbol must be III^*, by Proposition 11.4. Write $\psi_{\mathrm{red}}^{-1}(\infty) = C_0' + C_1 + \ldots + C_7$ for some additional component C_0' with $(C_0' \cdot C_1) = 1$. The fiber is simple, by Proposition 13.1, so

$$4 = \varphi^{-1}(\infty) \cdot \psi^{-1}(\infty) = \varphi^{-1}(\infty) \cdot C_0' = (C_0 + 2C_1) \cdot C_0' = (C_0 \cdot C_0') + 2.$$

Whence $(C_0 \cdot C_0') = 2$, and $C_0 + C_0'$ is a curve of canonical type, with Kodaira symbol $\mathrm{I}_2, \tilde{\mathrm{I}}_2$ or III. Let $\eta : Y \to \mathbb{P}^1$ be the ensuing genus-one fibration. By Proposition 11.4, we may assume that $\eta^{-1}(\infty)$ has symbol III^* or I_2^*. In the former case Proposition 13.1 ensures that it is simple so by Proposition 11.1 the configuration must be $\mathrm{III}^* + E_4 + E_4$, contradiction. In light of Proposition 11.1, the only remaining possible configuration is $\mathrm{I}_2^* + \mathrm{III} + \mathrm{III}$. Without restriction $\eta_{\mathrm{ind}}^{-1}(0) = C_0 + C_0'$ and $\eta_{\mathrm{ind}}^{-1}(1) = D + D'$. The curve $C_2 + \ldots + C_7$ has dual graph $T_{2,2,4}$ and thus belongs to $\eta^{-1}(\infty)$. Let R be the additional component. The dual graph of our curves takes the following form:



We have $\eta_{\mathrm{ind}}^{-1}(0) \cdot \varphi^{-1}(\infty) = (C_0 + C_0') \cdot 2C_1 = 2 + 2 = 4$. If $\eta^{-1}(0)$ is simple, each multiple fiber $2F'$ for $\eta : Y \to \mathbb{P}^1$ has $(F' \cdot C_1) = 1$, in contradiction to the previous paragraph. Thus $\eta^{-1}(0)$ is multiple, such that $\eta^{-1}(\infty)\cdot\varphi^{-1}(\infty)=8$. If the fiber $\eta^{-1}(\infty)$ is multiple as well, we get

$$4=\eta^{-1}_{\text{ind}}(\infty)\cdot\varphi^{-1}(\infty)=R\cdot(2C_1+2C_6)=2(R\cdot C_1)+2,$$

hence $C_3+\ldots+C_6+R+C_1$ is a curve of canonical type I_6, contradiction. Thus $\eta^{-1}(\infty)$ is simple, whereas $\eta^{-1}(1)$ must be the second multiple fiber. The latter gives

$$4=\eta^{-1}_{\text{ind}}(1)\cdot\varphi^{-1}(\infty)=(D+D')\cdot 2C_1=2(D\cdot C_1)+2(D'\cdot C_1).$$

According to Proposition 11.1, the fiber $\varphi^{-1}(\infty)$ intersects both D and D', so we have $(D\cdot C_1)=(D'\cdot C_1)=1$. Thus $D+C_1+\ldots+C_7$ gives a curve of canonical type III^*. This gives a further fibration on Y, in which the fiber of type III^* must be simple, according to Proposition 13.1. This gives

$$4\leq(D+2C_1+\ldots+2C_6+C_7)\cdot\varphi^{-1}(\infty)=D\cdot 2C_1=2,$$

contradiction. □

14. Surfaces with restricted configurations

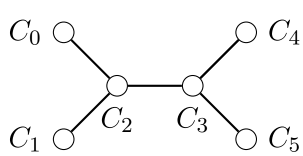
Let k be an algebraically closed ground field k of arbitrary characteristic $p\geq 0$, and Y be a classical Enriques surface. The goal of this section is to show that certain configurations of (-2)-curves cannot exist, once one imposes rather severe restrictions on the configuration of reducible fibers in genus-one fibrations.

Proposition 14.1. *There is no classical Enriques surface Y such that every genus-one fibration $\varphi:Y\to\mathbb{P}^1$ satisfies one of the following properties:*

(i) *The configuration of reducible fibers is $\mathrm{I}_1^*+\mathrm{I}_4$, where the I_1^*-fiber is multiple.*
(ii) *The configuration is $\mathrm{I}_2^*+\mathrm{III}+\mathrm{III}$, where two of these fibers are multiple.*

Proof. Seeking a contradiction, we assume that such a surface exists. Choose a genus-one fibration $\varphi:Y\to\mathbb{P}^1$. By [17], Theorem A the surface Y is non-exceptional. According to Proposition 4.4 there is another genus-one fibration $\psi:Y\to\mathbb{P}^1$, with $\varphi^{-1}(\infty)\cdot\psi^{-1}(\infty)=4$. Choose a multiple fiber $2F\subset Y$ for ψ. Without restriction, we may assume that the fibers of φ and ψ with Kodaira symbol I_n^* occur over $t=\infty$.

Suppose first that $\varphi^{-1}(\infty)$ has Kodaira symbol I_1^*, and hence must be multiple. The dual graph is:



After renumeration, we may assume $(F\cdot C_0)=1$. The curve $C_1+\ldots+C_5$ has dual graph $T_{2,2,3}$ and is ψ-vertical, hence must belong to $\psi^{-1}(\infty)$. Suppose the latter has Kodaira symbol I_1^*. Then $\psi^{-1}_{\text{red}}(\infty)=R+C_1+\ldots+C_5$ for some additional component R with $(R\cdot C_2)=1$, and the fiber is multiple. This gives the contradiction

$$1=\varphi^{-1}_{\text{ind}}(\infty)\cdot\psi^{-1}_{\text{ind}}(\infty)=(C_0+2C_2)\cdot R=(C_0\cdot R)+2\geq 2.$$

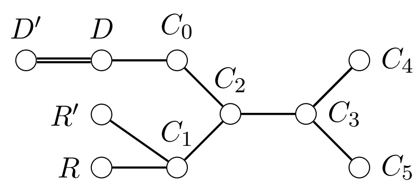
Thus $\psi^{-1}(\infty)$ has Kodaira symbol I_2^*, and we write $\psi_{\mathrm{ind}}^{-1}(\infty) = R + R' + C_1 + \ldots + C_5$ with additional components R and R', which have $(R \cdot C_1) = (R' \cdot C_1) = 1$. Then

$$2 \geq \varphi_{\mathrm{ind}}^{-1}(\infty) \cdot \psi_{\mathrm{ind}}^{-1}(\infty) = (C_0 + C_1) \cdot (R + R') = C_0 \cdot (R + R') + 1 + 1.$$

Hence $R \cup R'$ is disjoint from C_0, and $\psi^{-1}(\infty)$ must be simple. Without restriction, we may assume that $\psi^{-1}(0)$ is multiple with Kodaira symbol III. Write $\psi_{\mathrm{ind}}^{-1}(0) = D + D'$. Then

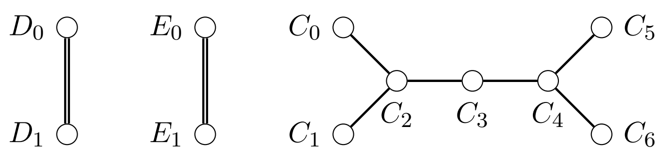
$$2(D + D') \cdot C_0 = \psi^{-1}(\infty) \cdot C_0 = 2C_2 \cdot C_0 = 2.$$

Without restriction, we may assume $(C_0 \cdot D) = 1$ and $(C_0 \cdot D') = 0$. Then the dual graph for our curves becomes:



Thus $R + D + C_0 + \ldots + C_4$ is a curve of canonical type IV^*, contradiction.

Summing up, every genus-one fibration on Y *has* $\mathrm{I}_2^* + \mathrm{III} + \mathrm{III}$ *as configuration of degenerate fibers, two of which are multiple.* We tacitly assume that the I_2^*-fiber lies over $t = \infty$, whereas the III-fibers appear over $t = 0, 1$. In particular, the dual graph for the reducible fibers of our $\varphi : Y \to \mathbb{P}^1$ is:



We first verify that $\varphi^{-1}(\infty)$ is simple. Seeking a contradiction, we assume that it is multiple, such that $F \cdot \varphi_{\mathrm{ind}}^{-1}(\infty) = 1$. After renumeration we may assume $(F \cdot C_0) = 1$. The connected curve $C_1 + \ldots + C_6$ has dual graph $T_{2,2,4}$, hence is contained in $\psi^{-1}(\infty)$. Let $R \subset \psi^{-1}(\infty)$ be the additional component, which has $(R \cdot C_2) = 1$. From

$$2 \geq \varphi_{\mathrm{ind}}^{-1}(\infty) \cdot \psi_{\mathrm{ind}}^{-1}(\infty) = (C_0 + 2C_2) \cdot R = (C_0 \cdot R) + 2$$

we infer $R \cap C_0 = \varnothing$. Thus $C_0 + \ldots + C_3 + R$ supports a curve of canonical type I_0^*, contradiction.

Summing up, in all genus-one fibrations the I_2^**-fiber is simple and the* III*-fibers are multiple.* Recall that $2F$ is a multiple fiber for our fibration $\psi : Y \to \mathbb{P}^1$ with $\varphi^{-1}(\infty) \cdot \psi^{-1}(\infty) = 4$. After renumeration, we may assume that $(F \cdot D_0) = (F \cdot E_0) = 1$ and $(F \cdot D_1) = (F \cdot E_1) = 0$. In particular, the curves D_0, E_0 are ψ-horizontal, whereas D_1, E_1 are ψ-vertical.

By symmetry, we now write $\psi_{\mathrm{ind}}^{-1}(0) = R_0 + R_1$ such that R_0 is φ-horizontal and R_1 is φ-vertical. We claim that R_1 is not contained in $\varphi^{-1}(0)$. For $R_1 = D_1$ we get the contradiction

$$1 = (D_0 + D_1) \cdot (R_0 + R_1) = D_0 \cdot (R_0 + R_1) \geq (D_0 \cdot R_1) = (D_0 \cdot D_1) = 2,$$

whereas $R_1 = D_0$ leads to

$$1 = (D_0 + D_1) \cdot (R_0 + R_1) = D_1 \cdot (R_0 + R_1) = 0,$$

again a contradiction. Thus R_1 is disjoint from $\varphi^{-1}(0)$, which gives

$$(D_0 \cdot R_0) = D_0 \cdot (R_0 + R_1) = (D_0 + D_1) \cdot (R_0 + R_1) = 1.$$

By symmetry, we also get $(E_0 \cdot R_0) = 1$. Likewise, we write $\psi_{\text{ind}}^{-1}(1) = S_0 + S_1$ such that S_0 is φ-horizontal and S_1 is φ-vertical, and see $(D_0 \cdot S_0) = (E_0 \cdot S_0) = 1$. Thus $D_0 + R_0 + E_0 + S_0$ is a curve of canonical type I_4, contradiction. □

15. Proof of the main result

We are finally ready to prove Theorem 5.1, the main result of this paper. The task is to show that the fiber category $\mathscr{M}_{\text{Enr}}(\mathbb{Z})$ for the stack of Enriques surfaces is empty. Seeking a contradiction, we assume that that there is a family of Enriques surfaces $f : \mathfrak{Y} \to \text{Spec}(\mathbb{Z})$. According to Proposition 5.5, the Picard scheme $\text{Pic}_{\mathfrak{Y}/\mathbb{Z}}$ is constant. So the closed fiber $Y = \mathfrak{Y} \otimes \mathbb{F}_2$ also has constant Picard scheme. It is non-exceptional, by [57], Theorem 7.2, because this Enriques surface comes with a deformation over $W_2(\mathbb{F}_2) = \mathbb{Z}/4\mathbb{Z}$. Therefore it suffices to establish:

Theorem 15.1. *There is no Enriques surface Y over the prime field $k = \mathbb{F}_2$ that is non-exceptional and whose Picard scheme is constant.*

Proof. Suppose such a Y exists. By Theorem 5.6 there is a genus-one fibration $\varphi : Y \to \mathbb{P}^1$. For each geometric point $\bar{a} : \text{Spec}(\Omega) \to \mathbb{P}^1$ lying over a non-rational closed point $a \in \mathbb{P}^1$, the geometric fiber $Y_{\bar{a}}$ is irreducible, according to Proposition 3.1, thus the Kodaira symbol is $\mathrm{I}_0, \mathrm{I}_1$ or II. Moreover, it must be simple by Theorem 5.6. The possible configurations of fibers over the rational points $t = 0, 1, \infty$ are tabulated in Proposition 11.4. In Sections 12 and 13 we saw that the Kodaira symbols I_4^* and III^* actually do not occur. For the classical Enriques surface $\bar{Y} = Y \otimes_k k^{\text{alg}}$ this means that for every genus-one fibration, the configuration of degenerate fibers is either $\mathrm{I}_1^* + \mathrm{I}_4$ or $\mathrm{I}_2^* + \mathrm{III} + \mathrm{III}$. Furthermore, in the former case the I_1^*-fiber is multiple, whereas in the latter case two of the three fibers are multiple. We showed in Proposition 14.1 that such an Enriques surface does not exist. This gives the desired contradiction. □

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