THE STRUCTURE OF REGULAR GENUS-ONE CURVES OVER IMPERFECT FIELDS

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ABSTRACT. Working over imperfect fields, we give a comprehensive classification of genus-one curves that are regular but not geometrically regular, extending the known case of geometrically reduced curves. The description is given intrinsically, in terms of twisted forms of standard models with respect to infinitesimal group scheme actions, and not via extrinsic equations. The main new idea is to analyze and exploit moduli for fields of representatives in Cohen's Structure Theorem. The results serve for the understanding of genus-one fibrations over higher-dimensional bases.

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INTRODUCTION

Algebraic curves fall into three classes of rather unequal size, each determined by the ampleness properties of the dualizing sheaf: The projective line, elliptic curves, and curves of higher genus. This trichotomy is expected to hold true in dimensions $d \geq 2$ in the following sense: Up to passing to suitable birational models, a proper normal scheme Z is either covered by projective lines, or admits a fibration $f : Z \to B$ over a lower-dimensional base where the generic fiber satisfies $c_1 = 0$, or

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is of general type. This is indeed true for algebraic surfaces, in light of the Enriques classification.

To understand the case of a fibration $f : Z \to B$ whose generic fiber $Y = f^{-1}(\eta)$ satisfies the condition $c_1 = 0$, one inevitably is lead to study proper normal schemes Y of dimension $n \geq 1$ over function fields F such that the structure sheaf satisfies $h^0(\mathscr{O}_Y) = 1$ and the dualizing sheaf ω_Y is numerically trivial. The goal of this paper is to carry out an in-depth analysis for the simplest relevant case, when Y is a proper normal curve with numerical invariants $h^0(\mathscr{O}_Y) = h^1(\mathscr{O}_Y) = 1$ and dualizing sheaf $\omega_Y = \mathscr{O}_Y$. We also say that Y is a *genus-one curve*.

The situation is completely understood in characteristic zero, when such an Y is a principal homogeneous space over an elliptic curve. I like to say that Y is *para-elliptic*, compare [23] and [34]. One also understands the case when Y is geometrically singular and geometrically reduced: Then the curve is a twisted form of the rational cuspidal curve $\operatorname{Spec} k[u^2, u^3] \cup \operatorname{Spec} k[u^{-1}]$, and this happens precisely in characteristic is $p \leq 3$. The occurring equations where analyzed by Queen ([26], [27]), and one also says that Y is *quasi-elliptic*. Note that the occurrence of such curves is arguably the key phenomenon in the extension of the Enriques classification to positive characteristics by Bombieri and Mumford ([4], [5]). In arithmetic settings, the understanding of quasi-elliptic fibrations at the primes two or three is often a key step to unravel the situation over fields of characteristic zero (for example in [35]).

Throughout this paper, we investigate the general case, *when the field F is imperfect of characteristic $p > 0$, and Y is regular but not geometrically regular*. Note that this includes the geometrically non-reduced curves, and our results mainly pertain to them. I already touched upon the topic in [32], Section 6, but with very limited results.

Here we use a completely different and novel approach, which relies on a connection between commutative algebra and algebraic geometry that to my best knowledge was not explored so far: The *fields of representatives* occurring in Cohen's Structure Theorems for complete local noetherian rings are usually non-unique, and thus have *moduli in the sense of algebraic geometry*. The main idea of this paper is to study and exploit such moduli in a systematic fashion. I expect that this should be useful in many other contexts as well.

To state our main result, we have to introduce certain curves $C = C^{(i)}_{r,F,\Lambda}$ called *standard models*. They are birational to the projective line $\mathbb{P}^1_R$ over the local Artin ring $R = F[W_1, \ldots, W_r]/(W_1^p, \ldots, W_r^p)$, in which the field F and the integer $r \geq 0$ enter. The birational morphisms $\mathbb{P}^1_R \to C$ is specified in terms of certain F-subalgebras Λ lying inside the ring of dual numbers $R[\epsilon]$ or the product ring $R \times R$, subject to conditions that among other things ensure $h^0(\mathscr{O}_C) = h^1(\mathscr{O}_C) = 1$. The upper index pertains to this subring, and is a symbol that could take the values $0 \leq i \leq r$ or $i = (1,1)$. Note that one views the ring of dual numbers or the product ring as coordinate rings of some effective Cartier divisor on $\mathbb{P}^1_R$, and that the subring Λ has moduli. For details of the construction we refer to Section 2. We now can state our first main result:

Theorem. *(see Thm. 2.3) Let Y be a genus-one curve that is regular but not geometrically regular. Then the following holds:*

(i) *The field F has characteristic at most three, and p-degree at least $r + 1$.*

(ii) *In characteristic two the second Frobenius base-change $Y^{(p^2)}$ is a twisted form of some standard model $C_{r,F,\Lambda}^{(i)}$ with $0 \leq i \leq 2$ or $i = (1,1)$.*

(iii) *In characteristic three, the first Frobenius base-change $Y^{(p)}$ is isomorphic to some standard model $C_{r,F,\Lambda}^{(i)}$ with $0 \leq i \leq 1$.*

In particular, the base-change $Y \otimes F^{\mathrm{alg}}$ is one of the specified standard models over the algebraic closure F^{alg}. Concerning existence, we have the following:

Theorem. *(see Thm. 2.4) Let $r \geq 0$ be some integer. Suppose characteristic and p-degree of the ground field F, and the symbol i are as in one of the rows of the following table:*

$\mathrm{char}(F)$	$\mathrm{pdeg}(F)$	i
3	$\geq r + 1$	$0, 1$
2	$\geq r + 1$	$0, 1, 2$
2	$\geq r + 2$	$(1,1)$

Then for some Λ the standard model $C_{r,\Lambda,F}^{(i)}$ admits a twisted form Y that is regular.

I find these results surprising on at least two counts: First, due to moduli for the subalgebras Λ inside $R[\epsilon]$ or $R \times R$, the genus-one curves Y are *usually not twisted forms of some standard model without making a ground field extension.* This is in marked contrast two the case of para-elliptic curves, which are twisted forms of their Jacobian, and also for quasi-elliptic curves, all of which are twisted forms of the rational cuspidal curve. Second, the existence of genus-one curves that are regular but geometrically non-reduced remains *confined to characteristic two and three*: This is as with quasi-elliptic curves, a state of affairs that I did not expect.

Let me point out that our approach emphasizes intrinsic aspects of the curves, and also sheds further light on their geometry. I hope that future research will lead to equations as well. However, I expect that such equations to be rather complicated and of extrinsic nature, and thus perhaps of limited practical value.

The idea for the proof for the above theorems is as follows: Using a result from [32], the problem is reduced to the case that for some simple height-one extension $F' = F(\omega^{1/p})$, the base-change $Y' = Y \otimes_F F'$ becomes singular, yet stays integral. We then argue that the normalization X' must be a genus-zero curve, over some further height-one extension $F' \subset E'$, a phenomenon called *constant field extension* in classical parlance. Using the comparatively simple structure of regular genus-zero curves, combined with the arithmetic of the height-one extensions, we can unravel the geometry of the *conductor square* that accompanies the normalization map. It is precisely this interplay that yields the connection to the standard forms $C_{r,F,\Lambda}^{(i)}$. We also get an explicit description of the sheaf $\Omega^1_{Y'/F'}$, which has to be locally free modulo its torsion part. Using the techniques already developed in [31] and [9], this allows to discards all primes except $p \leq 3$.

Any classification of proper smooth schemes X with $c_1 = 0$ of dimension $n \geq 3$ over an algebraically closed ground field k of positive characteristics is likely to involve the above classification, in parallel to the Enriques classification by Bombieri and Mumford. More general, it should be important for the understanding fibrations $f : X \to B$, of relative dimension one or higher, whose generic fiber has $c_1 = 0$, as appearing in the Minimal Model Program and Mori Theory. One could also hope to generalize the Ogg–Shafarevich Theory or the description of multiple fibers (compare [11], [14], [38]).

The paper is organized as follows: In Section 1 we collect some foundational facts on denormalization and twisting that are used throughout. The main results of the paper are formulated in Section 2. Section 3 establishes some relevant facts on intersections of subrings in Artin rings. We then start to construct regular genus-one curves that are not geometrically regular: In Section 4 we use pairs of subfields, a particularly simple way with direct geometric appeal, which however works only in characteristic two. Section 5 contains our construction relying on non-standard field of representatives. This also involves the choice of certain additive vector fields. An analysis of the restricted Lie algebra of all global vector fields is given in Section 6. Section 7 contains the most challenging construction of genus-one curves, where nilpotent elements play a crucial role in the denormalization. This relies on some technical observations from commutative algebra verified in Section 8. After Section 9, where we collect some results on ribbons and their twisted forms in connection to genus-zero curves, we examine genus-one curves with singularities in Section 10. The paper culminates in Section 11, which comprises the proofs for our main results.

1. Generalities and recollections

Let F be a ground field, for the moment of arbitrary characteristic $p \geq 0$. Throughout, the term *curve* refers to a proper scheme X that is equi-dimensional, of dimension one. It comes with the numerical invariants $h^i(\mathscr{O}_X) = \dim_F H^i(X, \mathscr{O}_X)$, for $0 \leq i \leq 1$. We say that X is a *curve of genus g* if these invariants take the values $h^0(\mathscr{O}_X) = 1$ and $h^1(\mathscr{O}_X) = g$. Clearly, the condition is stable under ground field extensions. Moreover, such curves are geometrically connected and without embedded components, hence Cohen–Macaulay. Note that we do not impose any other conditions whatsoever on the singularities, and in particular allow nilpotent elements in the structure sheaf.

For $g = 0$ we say that X is a *genus-zero curve*. Examples are, of course, the projective line, or quadric curves in $\mathbb{P}^2$, or infinitesimal extensions $X = \mathbb{P}^1 \oplus \mathscr{I}$, where $\mathscr{I}$ is a locally free sheaf on $\mathbb{P}^1$ with splitting type $(-1, \ldots, -1)$. For $g = 1$ we say that X is a *genus-one curve*. Here examples are elliptic curves, or cubic curves in $\mathbb{P}^2$, or infinitesimal extensions $X = E \oplus \mathscr{I}$, where E is an elliptic curve and $\mathscr{I}$ is a sum of invertible sheaves that are non-trivial of degree zero, or $X = \mathbb{P}^1 \oplus \mathscr{J}$, where $\mathscr{J}$ has splitting type $(-2, -1, \ldots, -1)$.

Let X be an arbitrary curve, and assume for simplicity that X is irreducible and has no embedded components. Let $f : X \to Y$ be a finite birational morphism between such curves. The *branch scheme* $B \subset Y$ is the closed subscheme corresponding to the *sheaf of conductor ideals* $\mathscr{I} \subset \mathscr{O}_Y$, which is defined as the annihilator of $f_*(\mathscr{O}_X)/\mathscr{O}_Y$. The preimage $A = f^{-1}(B)$ is called the *ramification scheme*. It is the closed subscheme of Y corresponding to sheaf of ideals $\mathscr{I} \subset f_*(\mathscr{O}_X)$. Indeed, by the very definition of $\mathscr{I} \subset \mathscr{O}_Y$, this is also a sheaf of ideals inside $f_*(\mathscr{O}_X)$. Moreover, the resulting cartesian diagram

(1)
$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array}$$

is also cocartesian. Then $0 \to \mathscr{O}_Y \to f_*(\mathscr{O}_X) \oplus \mathscr{O}_B \to f_*(\mathscr{O}_A) \to 0$ is exact, where the first arrow is the diagonal map, and the second arrow is the difference map. In turn, we have a long exact sequence

(2) $0 \to \Gamma(\mathscr{O}_Y) \to \Gamma(\mathscr{O}_X) \oplus \Gamma(\mathscr{O}_B) \to \Gamma(\mathscr{O}_A) \to H^1(Y, \mathscr{O}_Y) \to H^1(X, \mathscr{O}_X) \to 0.$

Note that both maps $\mathscr{O}_Y \to f_*(\mathscr{O}_X)$ and $\mathscr{O}_B \to f_*(\mathscr{O}_A)$ are injective. Let us record the following fundamental facts:

Proposition 1.1. *In the above situation, the following holds:*

   (i) *The numerical invariants of the schemes are related by the formula*

   $$h^0(\mathscr{O}_Y) + h^0(\mathscr{O}_A) + h^1(\mathscr{O}_X) = h^0(\mathscr{O}_X) + h^0(\mathscr{O}_B) + h^1(\mathscr{O}_Y).$$

   (ii) *Suppose the restriction map $\Gamma(\mathscr{O}_X) \to \Gamma(\mathscr{O}_A)$ is injective. Then the ring $\Gamma(\mathscr{O}_Y)$ equals the intersection $\Gamma(\mathscr{O}_X) \cap \Gamma(\mathscr{O}_B)$ inside $\Gamma(\mathscr{O}_A)$.*
   (iii) *The finite morphism $f : X \to Y$ is a universal homeomorphism if and only if this holds for $f : A \to B$.*
   (iv) *If the one-dimensional scheme Y is Gorenstein then for each point $b \in B$ the formula $\sum_{a \in f^{-1}(b)} h^0(\mathscr{O}_{A,a}) = 2h^0(\mathscr{O}_{B,b})$ holds. The converse is true provided that X is Gorenstein and $A \subset X$ is Cartier.*

Proof. Statement (i) follows from the additivity of Euler characteristics and the exact sequence (2). The latter also gives (ii). Assertion (iii) is a consequence of [19], Corollary 18.12.11, and (iv) follows from [12], Proposition A.2 and A.3, compare also the references discussed there.   □

We will use the following consequence frequently, where here and throughout the symbol ϵ denotes an indeterminate subject to the condition $\epsilon^2 = 0$:

Corollary 1.2. *Suppose Y is Gorenstein and $f : X \to Y$ is a universal homeomorphism. Let $a \in A$ be a closed point, with image $b \in B$ and residue field E. Assume that $\mathscr{O}_{A,a} = E[\epsilon]$ is a ring of dual numbers. Then the subring $\mathscr{O}_{B,b}$ takes the form $\Lambda = L + H\epsilon$, where $L \subset E[\epsilon]$ is a subfield, and $H\epsilon \subset E\epsilon$ is an L-linear subspace of codimension one.*

Proof. Write $L = \kappa(b)$ for the residue field, such that we have extensions $F \subset L \subset E$. By Cohen's Structure Theorem ([8], Chapter IX, §4, No. 3, Theorem 1), the residue

class map $\mathscr{O}_{B,b} \to L$ admits a section, giving an inclusion of L into $\mathscr{O}_{B,b}$ and $\mathscr{O}_{A,a}$. Write $\mathfrak{m}_b$ and $\mathfrak{m}_a$ for the maximal ideals in these local Artin rings. The inclusion $\mathfrak{m}_b \subset \mathfrak{m}_a = E\epsilon$ shows that $\mathfrak{m}_b^2 = 0$, hence the $\mathscr{O}_{B,b}$-module structures comes from a structure of L-vector spaces. Now write $\mathfrak{m}_b = H\epsilon$. Clearly

$$h^0(\mathscr{O}_{A,a}) = 2[E:L] \cdot [L:F] \quad \text{and} \quad h^0(\mathscr{O}_{B,b}) = (1 + \dim_L(H\epsilon)) \cdot [L:F].$$

According to the proposition, we have $h^0(\mathscr{O}_{A,a}) = 2h^0(\mathscr{O}_{B,b})$. Combining with the above equations, we arrive at $[E:L] = 1 + \dim_L(H\epsilon)$, and thus $H\epsilon \subset E\epsilon$ must be an L-hyperplane. $\qquad\square$

Note that the sum $\Lambda = L + H\epsilon$ is direct because $L \cap H\epsilon = 0$. However, this sum is usually not compatible with the canonical decomposition $E[\epsilon] = E \oplus E\epsilon$. Also note that the local Artin ring $\mathscr{O}_{B,b}$ is a field if and only if $H\epsilon = 0$, which in turn means that the subfield $L \subset \mathscr{O}_{A,a}$ is a field of representatives.

Recall that to form diagram (1), we started with a finite birational morphism $f : X \to Y$, which then determines B and $A = f^{-1}(B)$. Conversely, if one begins with a curve X, for simplicity assumed to be irreducible and without embedded components, together with a finite closed subscheme $A \subset X$, a finite scheme B, and an inclusion $\Gamma(\mathscr{O}_B) \subset \Gamma(\mathscr{O}_A)$, one obtains a curve Y by forming the cocartesian diagram (1). The push-out Y indeed exists as an algebraic space ([1], Theorem 6.1), which here is actually a scheme ([13], Theorem 7.1). Such constructions are also called *pinchings*. It is not difficult to characterize those pinchings where A and B are actually defined by the sheaf of conductor ideals:

Lemma 1.3. *In the above situation, the diagram* (1) *is a conductor square if and only if the schematic support of* $f_*(\mathscr{O}_A)/\mathscr{O}_B$ *coincides with* B.

Proof. Let $B' \subset B$ be the schematic support of the coherent sheaf $f_*(\mathscr{O}_A)/\mathscr{O}_B$. The sequence $0 \to \mathscr{O}_Y \to f_*(\mathscr{O}_X) \oplus \mathscr{O}_B \to f_*(\mathscr{O}_A) \to 0$ is exact, to the left because Y has no embedded components and f is birational, in the middle by the universal property of cocartesian squares, and to the right because $A \subset X$ is a closed embedding. Applying [20], Lemma 8.3.11 to the resulting cocartesian square of abelian sheaves on Y, we see that the canonical map $f_*(\mathscr{O}_X)/\mathscr{O}_Y \to f_*(\mathscr{O}_A)/\mathscr{O}_B$ is bijective. It immediately follows that the condition is necessary. For the converse, suppose that $B' = B$. It remains to check that the inclusion $A \subset f^{-1}(B)$ is an equality. Applying loc. cit. again, we see that the canonical map

$$\mathrm{Ker}(\mathscr{O}_Y \to \mathscr{O}_B) \longrightarrow \mathrm{Ker}(f_*(\mathscr{O}_X) \to f_*(\mathscr{O}_A))$$

is bijective, hence $A = f^{-1}(B)$. $\qquad\square$

The prime objective of the paper is to understand the structure of genus-one curves Y that are regular but not geometrically regular. The following well-known facts will be important:

Proposition 1.4. *Each genus-one curve Y that is integral and Gorenstein has dualizing sheaf $\omega_Y = \mathscr{O}_Y$. Each genus-zero curve X that is integral and Gorenstein is isomorphic to a quadric curve in $\mathbb{P}^2$. If it admits an invertible sheaf of degree one, we have $X \simeq \mathbb{P}^1$.*

Proof. The invertible sheaf $\mathscr{N} = \omega_Y$ has degree $d = -2\chi(\mathscr{O}_Y) = 0$. It admits a global section $s \neq 0$, because $h^0(\omega_Y) = h^1(\mathscr{O}_Y) = 1$. The map $s : \mathscr{O}_Y \to \mathscr{N}$ is injective because Y is integral, and thus must be bijective since $d = 0$.

The invertible sheaf $\mathscr{L} = \omega_X^{\otimes -1}$ has degree $d = 2\chi(\mathscr{O}_X) = 2$. The usual arguments with Serre Duality and Riemann–Roch show that $\mathscr{L}$ is globally generated with $h^0(\mathscr{L}) = 3$, and we obtain a morphism $f : X \to \mathbb{P}^2$ with $\mathscr{L} = f^*\mathscr{O}_{\mathbb{P}^2}(1)$. Let $C \subset \mathbb{P}^2$ be the image, which is an integral curve of degree at least two. The Degree Formula $\deg(\mathscr{L}) = \deg(f) \cdot \deg(C)$ reveals that f is birational and C is a quadric. Forming the conductor square (1) and applying Proposition 1.1 with $f : X \to C$, we see that $h^0(\mathscr{O}_A) = h^0(\mathscr{O}_B)$, thus f is an isomorphism.

If there is an invertible sheaf $\mathscr{N}$ of degree one, the usual arguments show that $\mathscr{L}$ is globally generated with $h^0(\mathscr{N}) = 2$, and that the resulting morphism $g : X \to \mathbb{P}^1$ with $\mathscr{N} = g^*(\mathscr{O}_{\mathbb{P}^1}(1))$ is an isomorphisms. $\qquad\square$

We close this section with some general observations on *twisting*. Let Y be a scheme over our ground field F. Another scheme $\tilde{Y}$ is called a *twisted form* if $\tilde{Y} \otimes E \simeq Y \otimes E$ for some field extension $F \subset E$. In characteristic $p > 0$, such twisted forms may arise as follows: Write $\Theta_{Y/F} = \underline{\mathrm{Hom}}(\Omega^1_{Y/F}, \mathscr{O}_Y)$ for the tangent sheaf. Let $\mathfrak{g} \subset H^0(X, \Theta_{X/F})$ be a finite-dimensional subspace that is stable with respect to Lie bracket $[D, D']$ and p-map $D^{[p]}$. Then $\mathfrak{g}$ is a finite-dimensional *restricted Lie algebra*, and thus corresponds to an infinitesimal group scheme G of height one with $\mathrm{Lie}(G) = \mathfrak{g}$. Moreover, the inclusion into $H^0(X, \Theta_{X/F})$ is nothing but a faithful G-action on Y, all this by the Demazure–Gabriel correspondence, see [10], Chapter II, §7, Theorem 3.5. We also refer to [29], Section 1 for more background. If $T = \mathrm{Spec}(E)$ is a G-torsor, one may form the twisted form

$$\tilde{Y} = T \wedge^G Y = G\backslash(T \times Y),$$

where the G-action on the product is given by $\sigma \cdot (t, y) = (\sigma t, \sigma y)$. Note that the G-action is free, so the quotient exists as an algebraic space (for example [23], Lemma 1.1), which here must be a scheme ([25], Theorem 6.2.2, compare also [36], Lemma 4.1). For examples of general twists that become non-schematic one may consult [33].

We now discuss a criterion that such twisted forms become regular. Given a quasicoherent sheaf of ideals $\mathscr{I}_0 \subset \mathscr{O}_Y$, there is a largest $\mathfrak{g}$-stable sheaf of ideals $\mathscr{I}$ contained in $\mathscr{I}_0$. It is actually quasicoherent, and the corresponding closed subscheme is the *orbit* $Z = G \cdot Z_0$ for the closed subscheme $Z_0 \subset Y$ defined by $\mathscr{I}_0$.

Lemma 1.5. *In the above situation, suppose that Y is noetherian. Then the noetherian scheme $\tilde{Y}$ is regular if the following three conditions hold:*

(i) *The twisted form $\tilde{Z} = T \wedge^G Z$ is regular.*
(ii) *The sheaf of ideals for $Z \subset Y$ is locally generated by a regular sequence.*
(iii) *For the open set $U = Y \smallsetminus Z$, the base-change $U \otimes E$ is regular.*

Proof. The projection $U \otimes E \to U$ is faithfully flat, so U is regular by [18], Corollary 6.5.2. Now fix some $b \in \tilde{Y}$ that belongs to the closed subscheme $\tilde{Z} = \tilde{Y} \smallsetminus \tilde{U}$. Then the local ring $\mathscr{O}_{\tilde{Y},b}$ is regular according to [17], Chapter 0, Corollary 17.1.9. $\qquad\square$

Note that the above principal already appeared in somewhat different form in in [30], Proposition 2.2. Also note that the situation becomes particularly simply for the restricted Lie algebra $\mathfrak{g} = k$, where both bracket and p-map are zero. The inclusion into $H^0(Y, \Theta_{Y/F})$ corresponds to a global vector field D that is non-zero and satisfies $D^{[p]} = 0$, and the group scheme $G = \alpha_p$ is the Frobenius kernel for the additive group $\mathbb{G}_a$. Moreover, the sheaf of ideals $\mathscr{I} \subset \mathscr{O}_Y$ defining the orbit $Z = G \cdot Z_0$ is the intersection for the kernels for the composite maps $D^i : \mathscr{I}_0 \to \mathscr{O}_Y/\mathscr{I}_0$, $1 \le i \le p-1$. Likewise is the case $\mathfrak{g} = \mathfrak{gl}_1(k)$, which stands for the vector space k endowed with the trivial bracket and p-map given by $\lambda^{[p]} = \lambda^p$. Now the inclusion into $H^0(Y, \Theta_{Y/F})$ is given by a non-zero global vector field D satisfying $D^{[p]} = D$.

2. FORMULATION OF THE MAIN RESULTS

In this section we state our main results on the structure of regular genus-one curves. For this we have to introduce certain highly singular genus-one curves, the so-called *standard models* $C = C_{r,F,\Lambda}^{(i)}$. Let F be a ground field of characteristic $p > 0$. Recall that a *p-basis* for a finite F-algebra R is a set of elements $\omega_1, \ldots, \omega_r \in R$ such that $\omega_i^p \in F$, and that the resulting map $F[T_1, \ldots, T_r]/(T_1^p - \omega_1^p, \ldots, T_r^p - \omega_r^p) \to R$ given by $T_i \mapsto \omega_i$ is bijective. In other words, each member from R can be written in a unique way as a polynomial $P(\omega_1, \ldots, \omega_r)$, with all exponents at most $p-1$ and all coefficients from F. In such a setting, we say that $P(T_1, \ldots, T_r)$ is a *p-truncated polynomial.* For more on these notions, see [21], Section 2, compare also [6], Chapter V, §13, No. 1.

Now fix some integer $r \ge 0$, and consider $R = F[W_1, \ldots, W_r]/(W_1^p, \ldots, W_r^p)$. Clearly, the classes $w_i \in R$ of the indeterminates W_i yield a p-basis. We now form the relative projective line

$$\mathbb{P}_R^1 = \operatorname{Proj} R[T_0, T_1] = (\operatorname{Spec} R[u]) \cup (\operatorname{Spec} R[u^{-1}]),$$

where $u = T_1/T_0$ is the affine coordinate function.

The standard models $C = C_{r,F,\Lambda}^{(i)}$ are defined for $0 \le i \le r$ in terms of the closed subscheme $A = V_+(T_1^2)$. Its coordinate ring $R[u]/(u^2) = R[\epsilon]$ can be seen as a *ring of dual numbers*. We now consider subrings $\Lambda \subset R[\epsilon]$ of the form $\Lambda = L + H\epsilon$, where $L \subset R[\epsilon]$ is a subring admitting a p-basis of length $r - i$, and $H\epsilon \subset R\epsilon$ is a an L-submodule such that $R\epsilon/H\epsilon$ free of rank one as L-module. Furthermore, we demand that $L \cap R\epsilon = 0$, and that the intersection $\Lambda \cap R \subset R[\epsilon]$ coincides with F. Under all these assumptions we form the cocartesian square

$$\begin{array}{ccc}
\operatorname{Spec}(R[\epsilon]) & \longrightarrow & \mathbb{P}_R^1 \\
\downarrow & & \downarrow \\
\operatorname{Spec}(\Lambda) & \longrightarrow & C,
\end{array}$$

which defines the *i-th standard model* $C = C_{r,F,\Lambda}^{(i)}$. The following is a first indication of their relevance for the classification of regular genus-one curves:

Proposition 2.1. *The standard models $C_{r,F,\Lambda}^{(i)}$ are geometrically irreducible genus-one curves whose local rings are Gorenstein and geometrically unibranch.*

Proof. It suffices to prove this when the ground field F is algebraically closed. Set $X = \mathbb{P}^1_R$ and $Y = C^{(i)}_{r,F,\Lambda}$. Clearly X_{red} is the projective line, and the composite map $\mathbb{P}^1 \to X \to Y$ is bijective, and the residue fields extensions are equalities. Thus Y is irreducible, and the local rings $\mathscr{O}_{Y,y}$ are geometrically unibranch.

Obviously $h^0(\mathscr{O}_X) = p^r$ and $h^1(\mathscr{O}_X) = 0$ and $h^1(\mathscr{O}_A) = 2p^r$. So Proposition 1.1 gives the formula $h^0(\mathscr{O}_Y) + p^r = h^0(\mathscr{O}_B) + h^1(\mathscr{O}_Y)$. By our assumptions on the subring $\Lambda = L + H\epsilon$, we have $[L : F] = p^{r-i}$ and $\dim_F(R\epsilon/H\epsilon) = [L : F] = p^{r-i}$. Using $\dim_F(R\epsilon) = [R : F] = p^r$ we get $h^0(\mathscr{O}_B) = p^{r-i} + (p^r - p^{r-i}) = p^r$, and conclude $h^0(\mathscr{O}_Y) = h^1(\mathscr{O}_Y)$. To see that Y is a genus-one curve it thus suffices to verify $h^0(\mathscr{O}_Y) = 1$. Clearly, the canonical map $\Gamma(\mathscr{O}_X) \to \Gamma(\mathscr{O}_A)$ is injective, so Proposition 1.1 gives $\Gamma(\mathscr{O}_Y) = \Gamma(\mathscr{O}_X) \cap \Gamma(\mathscr{O}_B)$, where the intersection takes place inside $\Gamma(\mathscr{O}_A)$. Again by our assumption on Λ, the intersection coincides with F, and we get $h^0(\mathscr{O}_Y) = 1$. This shows that Y is a genus-one curve.

One immediately sees that $h^0(\mathscr{O}_{A,a}) = 2h^0(\mathscr{O}_{B,b})$ for every $a \in A$, with $b = f(a)$. By Proposition 1.1, we see that Y is Gorenstein. $\square$

There is another relevant standard model $C = C^{(1,1)}_{r,F,\Lambda}$. It is defined in terms of the disconnected closed subscheme $A = V_+(T_0T_1)$, whose coordinate ring is $R \times R = R[u]/(u) \times R[u^{-1}]/(u^{-1})$. Consider subrings $\Lambda \subset R \times R$ of the form $\Lambda = \Lambda' \times \Lambda''$, where each factor admits a p-basis of length $r - 1$, and $\Lambda' \cap \Lambda'' \subset R$ coincides with F. The ensuing cocartesian diagram

$$\begin{array}{ccc}
\mathrm{Spec}(R \times R) & \longrightarrow & \mathbb{P}^1_R \\
\downarrow & & \downarrow f \\
\mathrm{Spec}(\Lambda' \times \Lambda'') & \longrightarrow & C,
\end{array}$$

defines the standard model $C = C^{(1,1)}_{r,F,\Lambda' \times \Lambda''}$. We also designate it by $C^{(i)}_{r,F,\Lambda}$, with upper index $i = (1,1)$ and $\Lambda = \Lambda' \times \Lambda''$. Note that this curve fails to be Gorenstein for $p \neq 2$, because $h^0(\mathscr{O}_{A,a}) = ph^0(\mathscr{O}_{B,b})$, with $a \in A$ and $b = f(a)$. However, as above one checks:

Proposition 2.2. *In characteristic two, the standard models $C^{(1,1)}_{r,F,\Lambda' \times \Lambda''}$ are geometrically irreducible genus-one curves whose local rings are Gorenstein and geometrically unibranch.*

For each F-scheme Y and $t \geq 0$, we write $Y^{(p^t)} = Y \otimes_F F$ for the base-change with respect to the t-fold Frobenius map $\lambda \mapsto \lambda^{p^t}$ on the ground field, and call them the *Frobenius base-changes*. We are now ready to formulate the main results of this paper.

Theorem 2.3. *Let Y be a genus-one curve that is regular but not geometrically regular, and $r = \mathrm{edim}(\mathscr{O}_{Y,\eta}/F)$ be its geometric generic embedding dimension. Then the following holds:*

(i) *The ground field F has characteristic $p \leq 3$, and p-degree at least $r + 1$.*

(ii) *In characteristic two, the second Frobenius base-change $Y^{(p^2)}$ is isomorphic to some standard model $C^{(i)}_{r,F,\Lambda}$ with $0 \leq i \leq 2$ or $i = (1,1)$.*

(iii) *In characteristic three, the first Frobenius base-change $Y^{(p)}$ is isomorphic to some standard model $C_{r,F,\Lambda}^{(i)}$ with $0 \le i \le 1$.*

Recall that the vector space dimension of $\Omega_{F/\mathbb{Z}}^1$ is called *p-degree* of the field F, and written as $n = \mathrm{pdeg}(F)$. This invariant is also determined by the formula $[F : F^p] = p^n$, and serves as a measure for imperfectness. Note that $\mathrm{pdeg}(F) = \dim(W) = \mathrm{trdeg}_k(F)$ provided that F is the function field of an integral scheme W of finite type over a perfect field k. The *geometric generic embedding dimension* $r = \mathrm{edim}(\mathscr{O}_{Y,\eta}/F)$ is the embedding dimension of the local Artin ring $\mathscr{O}_{Y,\eta} \otimes_F F^{\mathrm{perf}}$. This was introduced in [12], Section 1 as a measure of geometric non-reducedness. The statement $\mathrm{pdeg}(F) > \mathrm{edim}(\mathscr{O}_{Y,\eta}/F)$ in the above theorem is a general fact established in [32], Theorem 2.3, included here for the sake of clarity and completeness. As to existence, we have:

Theorem 2.4. *Let $r \ge 0$ be some integer. Suppose characteristic and p-degree of the ground field F, and the symbol i are as in one of the lines of the following table:*

$\mathrm{char}(F)$	$\mathrm{pdeg}(F)$	i
3	$\ge r+1$	$0, 1$
2	$\ge r+1$	$0, 1, 2$
2	$\ge r+2$	$(1,1)$

Then for some Λ the ensuing standard model $C_{r,\Lambda,F}^{(i)}$ admits a twisted form Y that is regular.

Summing up, the above two results give a complete classification of the regular genus-one curves that are not geometrically regular. The proofs require extensive preparation, and will be given in Section 11.

3. Intersection algebras in Artin rings

In this section we establish some technical results on intersections of subalgebras, which already played a role in the definition of the standard models $C = C_{r,\Lambda,F}^{(i)}$.

Let F be a ground field of characteristic $p \ge 0$, and R be some finite F-algebra, endowed with ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_s$ together with i integers $n_1, \ldots, n_s \ge 0$. Given a ring extension $F \subset A$, we use index notation $R_A = R \otimes A$ for the base-change. We now consider the *Grassmann schemes* $\mathrm{Grass}_{R/F}^{n_i}$ whose A-valued points are the A-submodules $L_i \subset R_A$ whose quotients are locally free of rank $[R : F] - n_i$. Note that if $A = K$ is a field, this boils down to the vector subspaces $L_i \subset R_K$ of dimension $\dim_K(L_i) = n_i$. We refer to [15], Section 9.7 for a comprehensive treatment.

Proposition 3.1. *There is a locally closed set $Z \subset \prod_{i=1}^s \mathrm{Grass}_{R/F}^{n_i}$ such that for each field extension $F \subset K$, a K-valued point $(L_1, \ldots, L_s)$ belongs to Z if and only if the $L_i \subset R_K$ are subalgebras, the residue class maps $L_i \to (R/\mathfrak{a}_i)_K$ are injective, and $K = L_1 \cap \ldots \cap L_r$.*

Proof. First observe that a vector subspace $L \subset R$ is a subalgebra if and only if the map $L \otimes L \to R/L$ induced by multiplication is zero, and also the projection $F \to R/L$ is zero. These are closed conditions. Moreover, the residue class map

$L \to R/\mathfrak{a}_i$ is injective if an only $\Lambda^n(L) \to \Lambda^n(R/\mathfrak{a}_i)$ is non-zero, where $n = [L : F]$. This is an open condition. Given vector subspaces $L_1, \ldots, L_s \subset R$, the condition $F = L_1 \cap \ldots \cap L_s$ means that the diagonal map $\varphi : R \to R/L_1 \times \ldots \times R/L_r$ vanishes on F and has rank $d = [R : F] - 1$. This can be rephrased as $\varphi|F = 0$ and $\Lambda^d(\varphi) = 0$ and $\Lambda^{d-1}(\varphi) \neq 0$, a combination of closed and open conditions.

Using universal sheaves on $\prod_{i=1}^s \mathrm{Grass}^{n_i}(R)$, one easily sees that the combination of these open and closed conditions defines the desired locally closed set Z. $\qquad\square$

In positive characteristics, we have to deal with an additional condition:

Proposition 3.2. *Suppose $p > 0$. There is a locally closed set $Z' \subset Z$ such that for each field extension $F \subset K$, a K-valued point $(L_1, \ldots, L_s)$ belongs to Z' if and only if the K-algebras L_i admit p-bases.*

Proof. Let $A \subset B$ be a ring extensions that is finite and locally free. According to [21], Satz 6 the A-algebra B locally admits a p-basis if and only if $B^p \subset A$ and the B-module $\Omega^1_{B/A}$ is locally free. The former condition means that $g_1^p, \ldots, g_m^p \in A$, where $g_1, \ldots, g_m \in B$ are A-algebra generators. This in turn means that the A-linear map $B^{\oplus m} \to B/A$ given by $(\lambda_i) \mapsto [\sum \lambda_i g_i]$ is zero, which is a closed condition. Regarding the second condition, the subset inside $\mathrm{Spec}(B)$ where the sheaf attached to the module of finite presentation $\Omega^1_{B/A}$ is locally free is obviously open. Since $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a closed map, the subset inside $\mathrm{Spec}(A)$ over which $\Omega^1_{B/A}$ is locally free is likewise open. Our assertion follows by applying the preceding observations with the universal sheaves corresponding to L_i. $\qquad\square$

We remark in passing that a closer analysis reveals that our conditions put a canonical scheme structure on Z and Z', making them subscheme in the product of Grassmannians $\prod_{i=1}^s \mathrm{Grass}^{n_i}_{R/F}$. This, however, plays no role in later applications.

Obviously, the formation of the locally closed sets Z and Z' commutes with ground field extensions. By Hilbert's Nullstellensatz, they are uniquely determined by the conditions in the preceding propositions. Note, however, that they may well be empty. We now establish existence results in two particular cases that are relevant throughout.

Proposition 3.3. *Suppose R admits a p-basis of length $r \geq 2$. Then there are subalgebras $\Lambda_1, \Lambda_2 \subset R$ that admit p-bases of length $r - 1$, and satisfy $\Lambda_1 \cap \Lambda_2 = F$.*

Proof. Choose a p-basis $\omega_1, \ldots, \omega_r \in R$. Then $\omega_i' = \omega_i$ for $i \leq r - 1$ and $\omega_r' = \omega_r + 1$ constitute another p-basis, as one easily sees by taking differentials. It $1 + \omega_r$ is not a unit, it must belong to the maximal ideal of the local Artin ring, and thus $1 + \omega_r'$ is a unit. So without restriction of generality we may assume that $1 + \omega_r \in R^\times$. Now set $s = r - 1$ and consider

$$\Lambda_1 = F[\omega_1, \ldots, \omega_s] \quad \text{and} \quad \Lambda_2 = F[\omega_1 + \omega_1\omega_r, \ldots, \omega_s + \omega_s\omega_r].$$

Obviously the generators form a p-basis for the first subring. For the second, we observe $(\Lambda_2)^p \subset F$ and $d(\omega_i + \omega_i\omega_r) = d(\omega_i)(1 + \omega_r) + \omega_i d(\omega_r)$. These differentials are linearly independent in $\Omega^1_{R/F}$, because $1 + \omega_r \in R^\times$, and we conclude with [21], Satz 6 that the generators form a p-basis.

To see $\Lambda_1 \cap \Lambda_2 = F$, suppose we have $P, Q \in F[T_1, \ldots, T_s]$ with all exponents bounded by $p - 1$, such that $P(\omega_1, \ldots, \omega_s) = Q(\omega_1 + \omega_1\omega_r, \ldots, \omega_s + \omega_s\omega_r)$. Write

$$Q(T_1 + T_1 T_r, \ldots, T_s + T_s T_r) = \sum_{i=0}^{s} T_r^i Q_i(T_1, \ldots, T_s)$$

Then $Q_0 = Q$, and the Taylor expansion (compare [6], Chapter IV, §4, No. 5) gives $Q_1 = \sum_{i=1}^{s} T_i \frac{\partial Q}{\partial T_i}$. Combining these equations and comparing coefficients gives the equation $\sum_{i=1}^{s} \omega_i \frac{\partial Q}{\partial T_i}(\omega_1, \ldots, \omega_s) = 0$ in R. Since the ω_i are p-independent and the exponents in Q are bounded by $p-1$ we actually have $\sum_{i=1}^{s} T_i \frac{\partial Q}{\partial T_i} = 0$ in $F[T_1, \ldots, T_s]$. If Q is homogeneous of degree $0 \le d \le p - 1$, the polynomial $\sum_{i=1}^{s} T_i \frac{\partial Q}{\partial T_i}$ coincides with dQ, by Newton's Formula. For general Q, one may apply this fact to the homogeneous summands, infers that Q is constant, and thus $\Lambda_1 \cap \Lambda_2 = F$. $\square$

The preceding result is a bit paradoxical, because if R is a field, the subspaces $\Omega^1_{\Lambda_i/F} \otimes_{\Lambda_i} R$ inside the vector space $\Omega^1_{R/F}$ have an intersection of dimension at least $2(r-1) - r = r - 2$. In the above proof, we actually have not fully used that the last member ω_r is part of the p-basis. This leads to the following:

Proposition 3.4. *Suppose R admits a p-basis of length r, and let $0 \le s \le r$. Then inside the ring of dual numbers $R[\epsilon]$, there are subalgebras of the form $\Lambda = L + H\epsilon$ such that the following holds:*

(i) *The F-algebra L admits a p-basis of length s.*
(ii) *The residue class map $L \to R[\epsilon]/(\epsilon) = R$ is injective.*
(iii) *As L-module, the quotient $\epsilon R/\epsilon H$ is free of rank one.*
(iv) *The intersection $R \cap \Lambda$ inside $R[\epsilon]$ coincides with F.*

Proof. Choose a p-basis $\omega_1, \ldots, \omega_r \in E$ and set $L = F[\omega_1 + \omega_1\epsilon, \ldots, \omega_s + \omega_s\epsilon]$. Then $(\omega_i + \omega_i\epsilon)^p = 0$. As in the preceding proof, one verifies that the generators form a p-basis, and that the intersection $R \cap L$ inside $R[\epsilon]$ coincides with F. The former gives (i), and the latter ensures (ii).

To proceed, note that the F-module $R\epsilon$ is free, and the p^r monomials

$$(3) \qquad P_\nu = \prod_{i=1}^{r}(\omega_i + \omega_i\epsilon)^{\nu_i} \cdot \epsilon = \prod_{i=s+1}^{r} \omega_i^{\nu_i}\epsilon \quad (0 \le \nu_i \le p - 1)$$

form a basis. We see that $R\epsilon$ remains free as an L-module, with a basis formed by the monomials P_ν with $\nu_1 = \ldots = \nu_s = 0$. In particular, the submodule $L\epsilon \subset R\epsilon$ is generated by the monomial with $\nu_1 = \ldots = \nu_r = 0$. It admits a complement, for example the the submodule $H\epsilon \subset R\epsilon$ generated by the monomials where $\nu = (\nu_1, \ldots, \nu_r)$ is non-zero with $\nu_1 = \ldots = \nu_s = 0$. Clearly, $E\epsilon/H\epsilon = L\epsilon$ is free of rank one, hence (iii).

It remains to verify the statement (iv) about the resulting subring $\Lambda = L + H\epsilon$. For this we use the decomposition $R[\epsilon] = R \oplus R\epsilon$ of abelian groups and the corresponding projections. Note that these respect the module structures over R and in particular over F, but not over L. Clearly, $\mathrm{pr}_1 | L$ is injective. It follows that $L \cap H\epsilon = 0$, hence the sum $\Lambda = L + H\epsilon$ is direct, and $\dim_F(\Lambda) = p^s + p^{r-s}$. The intersection $R \cap \Lambda$ is

isomorphic to the kernel of $\mathrm{pr}_2\,|\Lambda$. To understand the latter, we examine the image $\mathrm{pr}_2(\Lambda) \subset R\epsilon$, which is generated by the monomials P_ν of the form

$$\nu = (\nu_1, \ldots, \nu_s, 0, \ldots, 0) \quad \text{or} \quad \nu = (0, \ldots, 0, \nu_{s+1}, \ldots, \nu_r).$$

These are $p^s + (p^{r-s} - 1)$ members of the F-basis (3), which gives the rank of $\mathrm{pr}_2\,|\Lambda$. It follows that $\dim_F(R \cap \Lambda) = \dim_F(\Lambda) - \mathrm{rank}(\mathrm{pr}_2\,|\Lambda) = 1$. In turn, the inclusion $F \subset R \cap \Lambda$ must be an equality. $\qquad\square$

4. CONSTRUCTIONS VIA PAIRS OF SUBFIELDS

In this section we describe our first construction that leads to the regular genus-one curves. In contrast to later constructions, it does not rely on more advanced techniques from commutative algebra, and has some immediate geometric appeal. The drawback is that it only works in characteristic two.

So let F be an imperfect ground field of characteristic $p = 2$, and $F \subset E$ be a height-one extension of degree $[E : F] = p^r$ for some $r \geq 1$. Let X be an regular curve with $H^0(X, \mathscr{O}_X) = E$. Note that we regard this as a scheme over F, such that $h^0(\mathscr{O}_X) = p^r$. In this section we assume that there are two E-valued point $a' \neq a''$ on X, and will describe certain denormalizations $f : X \to Y$ that are relevant in the construction of regular genus-one curves.

Let $L', L'' \subset E$ be subextensions of degrees p^{r-1}, with $L' \cap L'' = F$. Note that according to Proposition 3.3, such fields indeed exist. Set

$$A = \mathrm{Spec}(E \times E) = \{a', a''\} \quad \text{and} \quad B = \mathrm{Spec}(L' \times L'') = \{b', b''\},$$

and consider the resulting morphism $f : A \to B$ with $b' = f(a')$ and $b'' = f(a'')$. In turn, the cocartesian diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow f \\ B & \longrightarrow & Y \end{array}$$

defines a new integral curve Y with singular locus $\mathrm{Sing}(Y) = \{b', b''\}$. According to Proposition 1.1, we have $h^0(\mathscr{O}_Y) = 1$ and $h^1(\mathscr{O}_Y) = h^1(\mathscr{O}_X) + 1$. Moreover, the local rings of Y are geometrically unibranch and Gorenstein, the latter relying on our assumption $p = 2$.

The inclusion $L' \cup L'' \subset E$ is never an equality ([6], Chapter V, §7, No. 4, Lemma 1), so there is some $\omega \in E$ not belonging to either of the subfields. This $\omega \in E$ is a p-basis over both L' and L'', so the scalar $\lambda = \omega^2$ belongs to $L' \cap L'' = F$, and is a non-square in both intermediate fields L' and L''.

We next examine the complete local noetherian rings at the singularities. The situation is symmetric, so we fix some $b \in \{b', b''\}$, write $L \in \{L', L''\}$ for the residue field, and $a \in \{a', a''\}$ for the corresponding point on X. Fix a uniformizer $u \in \mathscr{O}_{X,a}^\wedge$, such that $\mathscr{O}_{X,a}^\wedge = E[[u]]$. Note that the subring $\mathscr{O}_{Y,b}^\wedge$ comprises the formal power series $\sum \gamma_i u^i$ whose constant term γ_0 belongs to the subfield $L \subset E$.

Proposition 4.1. *Disregarding the F-structure, the complete local noetherian ring $\mathscr{O}_{Y,b}^\wedge$ is isomorphic to $L[[x, y]]/(x^2 - \lambda y^2)$.*

Proof. Clearly the linear polynomials ωu and u belong to the subring $\mathscr{O}_{Y,b}^{\wedge}$, and satisfy the relation $(\omega u)^2 = \lambda \cdot (u)^2$. The inclusion $Lu^0 \subset \mathscr{O}_{Y,b}^{\wedge}$ is a field of representatives, depending on the choice of uniformizer. The assignment $x \mapsto \omega u$ and $y \mapsto u$ now defines a homomorphism $\varphi : L[[x,y]]/(x^2 - \lambda y^2) \to \mathscr{O}_{Y,b}^{\wedge}$. The ring $R = L[[x,y]]/(x^2 - \lambda y^2)$ is a one-dimensional integral domain, the latter because $\lambda y^2 \in L[[y]]$ is a non-square.

Suppose we have some $\gamma \in E$ and $i \geq 1$. Write $\gamma = \alpha + \beta\omega$ with $\alpha, \beta \in L$. Then $\gamma u^i = \alpha u^i + \beta u^{i-1} \cdot \omega u$ belongs to the image of φ, and one easily infers that the map is surjective. Since R is integral, of the same dimension as $\mathscr{O}_{Y,b}$, we can apply Krull's Principal Ideal Theorem and infer that φ is bijective. □

We now specialize to the case that

$$X = \mathbb{P}_E^1 = \operatorname{Proj} E[T_0, T_1] = \operatorname{Spec} E[u] \cup \operatorname{Spec} E[u^{-1}]$$

is the projective line over E, with inhomogeneous coordinate $u = T_1/T_0$. Without loss of generality $a' = (1:0)$ and $a'' = (0:1)$. The arguments in the preceding proof work with polynomials instead of power series. It follows that the affine open covering of X induces an affine open covering

$$Y = \operatorname{Spec} L'[u, \omega u] \cup \operatorname{Spec} L''[u^{-1}, \omega u^{-1}],$$

and the coordinate ring for the overlap $U = Y \smallsetminus \{b, b'\}$ is the Laurent polynomial ring $E[u^{\pm}]$. The differential $d\omega \in \Omega_{E/F}^1$ is non-zero, because it is non-zero in the quotients $\Omega_{E/L'}^1$. Thus we find a basis of the form $d\omega_1, \ldots, d\omega_r \in \Omega_{E/F}^1$, with $\omega_1 = \omega$ and some $\omega_i \in E$. Then $du, d\omega_1, \ldots, d\omega_r$ form a basis of $\Omega_{U/F}^1$, and we write $\partial/\partial u, \partial/\partial\omega_1, \ldots \partial/\partial\omega_r$ for the dual basis, which lives in the tangent sheaf $\Theta_{U/F}$. Consider the vector field

$$D = \partial/\partial u + \omega u^{-1}\partial/\partial\omega = u^{-2}\partial/\partial u^{-1} + \omega u^{-1}\partial/\partial\omega,$$

a priori defined outside $\operatorname{Sing}(Y) = \{b', b''\}$. One easily computes that the induced derivation on the function field has

$$(4) \qquad D(u^{-1}) = u^{-2}, \quad D(u) = 1, \quad D(\omega) = \omega u^{-1}, \quad D(\omega u^{-1}) = D(\omega u) = 0.$$

Consequently, $D : E(u) \to E(u)$ stabilizes the two local rings $\mathscr{O}_{Y,b'}$ and $\mathscr{O}_{Y,b''}$ inside $\Gamma(U, \mathscr{O}_Y)$, and $D^{[2]} = D \circ D$ is the zero map. It thus defines a non-zero additive p-closed global vector field on Y, which corresponds to an inclusion $\alpha_p \subset \operatorname{Aut}_{Y/F}$, in other words, a faithful α_p-action.

Let $Z = \alpha_p \cdot Z_0$ be the orbit of the reduced closed subscheme $Z_0 = \{b', b''\}$. The geometry of our α_p-action has completely different features at the two singular points:

Proposition 4.2. *In the above setting, the α_p-scheme Z is equivariantly isomorphic to the disjoint union $(\alpha_p \times \operatorname{Spec} L') \cup (\operatorname{Spec} L'')$, where the action is described on the corresponding functors of k-algebras by the formula*

$$\sigma \cdot (\tau, z') = (\sigma\tau, z') \quad and \quad \sigma \cdot z'' = z''.$$

Moreover, the Weil divisor $Z \subset Y$ is Cartier at the point b', but not at b''.

Proof. Let $\mathscr{I}_0 \subset \mathscr{O}_Y$ be the sheaf of ideals for Z_0. Then the kernel $\mathscr{I}$ of the additive map $\mathscr{I}_0 \xrightarrow{D} \mathscr{O}_{Z_0}$ corresponds to the scheme of orbits $Z \subset Y$. From (4) we see that $\mathfrak{m}_{b''}$ is D-stable. It follows that the inclusion $Z_0 \subset Z$ is an equality at b''. Since this point is singular, the inclusion $Z \subset Y$ is not Cartier at b'. Furthermore, one sees that the induced derivation on the residue field $L'' = \kappa(b'')$ is trivial.

On the other hand, $\mathfrak{m}_{b'}$ is not D-stable, because $D(u) = 1$. Furthermore we have $\omega u \in \mathscr{I}_{b'}$, and see $\mathscr{O}_{Y,b'}/(\omega u) = L'[x,y]/(x^2 - \lambda y^2, x) = L'[y]/(y^2)$. Using that $\mathfrak{m}_{b'}/(\omega u)$ is a simple module we infer $\mathscr{I}_{b'} = (\omega u)$. This shows that $Z \subset Y$ is Cartier near b. Furthermore, (4) reveals that the α_p-action on Z at b is given by the derivation $\partial/\partial y$ on the coordinate ring $L'[y]/(y^2) = F[y]/(y^2) \otimes L'$, and the assertion follows. $\qquad\square$

By symmetry, the above analysis also applies to the vector field

$$\tilde{D} = \partial/\partial u^{-1} + \omega u \partial/\partial \omega = u^2 \partial/\partial u + \omega u \partial/\partial \omega,$$

with the roles of b', b'' interchanged. One easily checks that $[D, \tilde{D}] = 0$, and that the two derivations are F-linearly independent. In turn, the two global vector fields $D, \tilde{D}$ define a two-dimensional restricted Lie algebra $\mathfrak{g} = k^2$ inside $H^0(Y, \Theta_{Y/F})$, with trivial bracket and p-map. In other words, we have an action of $G = \alpha_p \times \alpha_p$ on Y.

Now F', F'' be two simple height-one extensions of F, and endow their spectra with the structure of an α_p-torsor. The affine scheme $T = \operatorname{Spec}(F' \otimes F'')$ becomes a torsor with respect to the group scheme $G = \alpha_p \times \alpha_p$. In turn, we obtain the twisted form

$$\tilde{Y} = T \wedge^G Y = G \backslash (T \times Y).$$

This defines another genus-one curve $\tilde{Y}$. Now recall that any purely inseparable field extensions of F uniquely embed into F^{alg}, so the concept of *linear disjointness* applies ([6], Chapter V, §2, No. 5), even without giving an ambient field.

Proposition 4.3. *Suppose the three height-one extensions $F \subset E, F', F''$ are linearly disjoint. Then the genus-one curve $\tilde{Y}$ is regular. Moreover, its Frobenius base-change $\tilde{Y}^{(p)}$ is is isomorphic to some standard model $C_{r,F,\Lambda}^{(1,1)}$.*

Proof. The orbit $Z = G \cdot Z_0$ of the reduced scheme $Z_0 = \{b', b''\}$ is G-stable. It is equivariantly isomorphic to the disjoint union $(\alpha_p \times \operatorname{Spec} L') \cup (\alpha_p \times \operatorname{Spec} L'')$, according to Proposition 4.2, and the G-action is given by

$$(\sigma_1, \sigma_2) \cdot (\tau, z') = (\sigma_1 \tau, z') \quad \text{and} \quad (\sigma_1, \sigma_2) \cdot (\tau, z'') = (\sigma_2 \tau, z'').$$

In turn, the twisted form $\tilde{Z} = Z \wedge^G T$ has coordinate ring $(F' \otimes L') \times (F'' \otimes L'')$. This Artin ring is regular, because by assumption the tensor products are fields. According to Proposition 4.2, the inclusion $Z \subset Y$ is an effective Cartier divisor, so the same holds for $\tilde{Z} \subset \tilde{Y}$. In turn, the scheme $\tilde{Y}$ is regular at the two points of the divisor.

Finally, consider the open set $U = \operatorname{Reg}(Y)$. The ensuing twisted form $\tilde{U}$ comes with a faithfully flat morphism from the spectrum of $E[u^{\pm 1}] \otimes F' \otimes F''$, which can be seen as the Laurent polynomial ring over $E \otimes F' \otimes F''$. Again, our assumption

ensures that this ring is regular, hence $\tilde{U}$ is regular. Thus Lemma 1.5 ensures that the genus-one curve $\tilde{Y}$ is regular.

The Frobenius pull-back $E \otimes_F F$ is isomorphic to $F[W_1, \ldots, W_r]/(W_1^p, \ldots, W_r^p)$, and likewise for the field extensions L', L'', F', F''. From this we see $Y^{(p)}$ is isomorphic to some standard model $C_{r,F,\Lambda}^{(1,1)}$. Moreover, $T^{(p)}$ becomes the trivial torsor with respect to $G = G^{(p)}$, thus Y and its twisted form $\tilde{Y}$ become isomorphic after Frobenius pull-back. Summing up, $\tilde{Y}^{(p)} \simeq C_{r,F,\Lambda}^{(1,1)}$. $\square$

5. Constructions with fields of representatives

In this section F is an imperfect ground field of arbitrary characteristic $p > 0$. Again X denotes a regular curve where $E = H^0(X, \mathscr{O}_X)$ is a height-one extension of degree $[E : F] = p^r$ for some $r \geq 1$. But now we fix only one E-valued point $a \in X$, and will describe another denormalization that is relevant in the construction of regular genus-one curves. Consider the local Artin ring $\mathscr{O}_{X,a}/\mathfrak{m}_a^2$. Throughout we will identify $E = \Gamma(\mathscr{O}_X)$ with its image in $\mathscr{O}_{X,a}/\mathfrak{m}_a^2$, which is the *standard field of representatives*.

We now fix *another field of representatives* $L \subset \mathscr{O}_{X,a}/\mathfrak{m}_a^2$, having the property $\Gamma(\mathscr{O}_X) \cap L = F$. According to Proposition 3.4 such fields indeed exist. Note that the projection to the residue field gives a canonical identification between $E = \Gamma(\mathscr{O}_X)$ and L, but it is crucial to view them as *different* fields inside $\mathscr{O}_{X,a}/\mathfrak{m}_a^2$. Set

$$A = \mathrm{Spec}(\mathscr{O}_{X,a}/\mathfrak{m}_a^2) = \{a\} \quad \text{and} \quad B = \mathrm{Spec}(L) = \{b\}.$$

The ensuing cocartesian diagram

(5)
$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y. \end{array}$$

defines a new integral curve Y, with $\mathrm{Sing}(Y) = \{b\}$. Arguing as in Proposition 2.1, one sees that $h^0(\mathscr{O}_Y) = 1$ and $h^1(\mathscr{O}_Y) = h^1(\mathscr{O}_X) + 1$. Furthermore, Y is locally unibranch and Gorenstein.

We now seek to understand the complete local ring $\mathscr{O}_{Y,b}^\wedge$ at the E-valued singularity $b \in Y$. To this end we fix a uniformizer $u \in \mathscr{O}_{X,a}^\wedge$, such that $\mathscr{O}_{X,a}^\wedge = E[[u]]$ as F-algebra. This turns $\mathscr{O}_{X,a}/\mathfrak{m}_a^2 = E[\epsilon]$ into a ring of dual numbers, where ϵ is the class of the uniformizer. Choose a p-basis $\omega_1, \ldots, \omega_r \in E$. The residue class bijection $L \to \kappa(b) = \kappa(a) = E$ takes the form

$$\omega_i + \alpha_i \epsilon \longmapsto \omega_i$$

for some uniquely determined $\alpha_i \in E$, and the elements $\omega_i + \alpha_i \epsilon \in E[\epsilon]$ form a p-basis for the subfield L. The cartesian square

(6)
$$\begin{array}{ccc} E[\epsilon] & \longleftarrow & E[u] \\ \uparrow & & \uparrow \\ F[\omega_1 + \alpha_1\epsilon, \ldots, \omega_r + \alpha_r\epsilon] & \longleftarrow & R \end{array}$$

defines a subring $R \subset E[u]$, which comprises the polynomials $\beta_0 + \beta_1 u + \ldots + \beta_n u^n$ whose truncation $\beta_0 + \beta_1 \epsilon$ is a polynomial in the $\omega_i + \alpha_i \epsilon$. In light of the Taylor expansion, this precisely means

$$(7) \qquad \beta_0 = P(\omega_1, \ldots, \omega_r) \quad \text{and} \quad \beta_1 = \sum_{i=1}^{r} \alpha_i \frac{\partial P}{\partial T_i}(\omega_1, \ldots, \omega_r)$$

for some $P \in F[T_1, \ldots, T_r]$.

Clearly the R-algebra $E[u]$ is a finite, hence $\mathfrak{m} = R \cap uE[u]$ is a maximal ideal, having residue field $R/\mathfrak{m} = L$. Moreover, R is a finitely generated F-algebra. It appears that the number of generators is excessive, so to improve the situation we have to pass to the formal completion $\widehat{R} = \varprojlim_n R/\mathfrak{m}^n$. The latter sits in a cartesian diagram analogous to (6), with $E[[u]] = \mathscr{O}^\wedge_{Y,a}$ in the top right corner, and is canonically identified with $\mathscr{O}^\wedge_{Y,b}$.

Proposition 5.1. *Disregarding the F-structure, the complete local ring $\widehat{R} = \mathscr{O}^\wedge_{Y,b}$ is isomorphic to $E[[x,y]]/(x^3 - y^2)$.*

Proof. Choose a section s for the residue class projection $\widehat{R} \to L$. It takes the form $s(\omega_i) = \omega_i + \alpha_i u + \ldots$. Set $\widehat{R}_0 = L[[x,y]]/(x^3 - y^2)$. The relation is irreducible because x^3 is not a square in $L[[x]]$. Hence $\widehat{R}_0$ is a one-dimensional integral domain. Clearly, $u^2, u^3 \in E[[u]]$ belong to $\mathfrak{m} = R \cap uE[u]$ and satisfy $(u^2)^3 = (u^3)^2$. Together with our field of representatives $s(L) \subset \widehat{R}$, the assignment $x \mapsto u^2$ and $y \mapsto u^3$ defines a map $\varphi : \widehat{R}_0 \to \widehat{R}$.

We claim that each formal power series $\sum_{i \geq 2} \lambda_i u^i$ with coefficients from E belongs to $\mathrm{Im}(\varphi)$. Write $\lambda_i = P_i(\omega_1, \ldots, \omega_r)$ as a p-truncated polynomial with coefficients from F. Then

$$\lambda_i u^i = P_i(\omega_1, \ldots, \omega_r) u^i = P_i(s(\omega_1), \ldots, s(\omega_r)) u^i + \ldots,$$

where the missing terms have order at least $i + 1$. The factor $P_i(s(\omega_1), \ldots, s(\omega_r))$ belongs to $\varphi(s(L))$, whereas $u^i = \varphi(y^{i/2})$ or $u^i = \varphi(xy^{(i-3)/2})$, depending on the parity of i. In any case, the exponent in the y-term grows linearly with $i \geq 2$, hence successive substitutions reveal that $\sum_{i \geq 2} \lambda_i u^i$ belongs to $\mathrm{Im}(\varphi)$.

In light of the cartesian diagram (6), the maximal ideal of $\widehat{R}$ consists of these $\sum_{i \geq 2} \lambda_i u^i$, and it follows that the map $\varphi : \widehat{R}_0 \to \widehat{R}$ is surjective. The rings have the same dimension and $\widehat{R}_0$ is integral. Using Krull's Principal Ideal Theorem, we infer that φ is bijective. $\qquad \square$

We next seek to understand $\widehat{R} = \mathscr{O}^\wedge_{Y,b}$ as F-algebra. Clearly, the $r+2$ polynomials

$$(8) \qquad u^3 \quad \text{and} \quad u^2 \quad \text{and} \quad \omega_i + \alpha_i u \quad (1 \leq i \leq r)$$

are contained in R. The p-powers $\lambda_i = \omega_i^p$ and $\mu_i = \alpha_i^p$ belong to the ground field F, and the above generators satisfy the $r+1$ *obvious relations*

$$(9) \qquad (u^3)^2 - (u^2)^3 = 0 \quad \text{and} \quad (\omega_i + \alpha_i u)^p - \lambda_i - \mu_i u^p = 0 \quad (1 \leq i \leq r).$$

Here one has to rewrite, for $p \geq 5$, the factor u^p as $(u^3) \cdot (u^2)^{(p-3)/2}$, to get an expression in terms of the generators. By abuse of notation, we also regard the

generators in (8) as indeterminates, and write $\tilde{R}$ for the resulting polynomial ring, formally completed with respect to the ideal (u^2, u^3). From universal properties we get a continuous homomorphism

$$\varphi : \tilde{R} = F[\omega_1 + \alpha_1 u, \ldots, \omega_r + \alpha_r u][[u^2, u^3]] \longrightarrow \widehat{R}.$$

The following gives the desired description as completed F-algebra:

Proposition 5.2. *The above map is surjective, the obvious relations in* (9) *form a regular sequence in* $\tilde{R}$, *and they generate the ideal* $\mathfrak{p} = \mathrm{Ker}(\varphi)$ *of all relations.*

Proof. Let $\mathfrak{a} \subset \tilde{R}$ be the ideal generated by the obvious relations in (9). The main task is to verify that the inclusion $\mathfrak{a} \subset \mathfrak{p}$ is an equality. One easily checks

$$\tilde{R}/(\mathfrak{a} + \tilde{R}u^2) = L[u^3]/(u^6) \quad \text{and} \quad (\tilde{R}/\mathfrak{a})_{u^2} = L[[u]]_{u^2} = L((u)).$$

Hence there are exactly two primes $\mathfrak{p}_1, \mathfrak{p}_2 \subset \tilde{R}$ containing $\mathfrak{a}$, one being the maximal ideal $\mathfrak{p}_1 = \mathfrak{a} + \tilde{R}u^2 + \tilde{R}u^3$, which has height $r+2$. By Krull's Principal Ideal Theorem, every minimal prime containing $\mathfrak{a}$ has height at most $r+1$. We infer $\dim(\tilde{R}/\mathfrak{a}) = 1$, which coincides with the difference $(r+2) - (r+1)$. It follows that the $r+1$ obvious relations from (9) in the $r+2$ generators in (8) form a regular sequence, and $\tilde{R}/\mathfrak{a}$ is Cohen–Macaulay ([37], Tag 02JN). In our situation, the latter simply means that the quotient has no embedded primes. Clearly $R_{u^2} = E[u^{\pm 1}]$, hence $\mathfrak{a} \subset \mathfrak{p}$ becomes an equality after localization of u^2. Thus $\mathfrak{p}/\mathfrak{a}$ is an $\tilde{R}$-module of finite length. It must vanish, because it coincides with the kernel of $\tilde{R}/\mathfrak{a} \to R$, and $\tilde{R}/\mathfrak{a}$ has no embedded primes.

It remains to check that the inclusion $\tilde{R}/\mathfrak{p} \subset R$ is an equality. One easily sees that it becomes an equality after localization of u^2, and also after formally completing with respect to (u^2), and the result follows. $\square$

Recall that the *Fitting ideals* $\mathrm{Fitt}_i(M)$ for an A-module M having a presentation $A^n \to A^m \to M \to 0$ are generated by the $(m-i)$-minors of the presentation matrix, compare the discussion in [9], Section 3. The above analysis enables us to understand Kähler differentials and their Fitting ideals, a method already used in loc. cit. in connection with automorphism group schemes.

Proposition 5.3. *The coherent sheaf* $\Omega^1_{Y/F}$ *has rank* $r+1$, *and the following are equivalent:*

    (i) *The torsion-free sheaf* $\mathrm{Fitt}_{r+1}(\Omega^1_{Y/F})$ *is locally free.*
    (ii) *The characteristic satisfies* $p \leq 3$.

If these equivalent conditions hold, $\Omega^1_{Y/F}$ *is locally free near* $b \in Y$.

Proof. First note that on the regular locus $Y \smallsetminus \{b\}$, any torsion-free sheaf is locally free. So we only have to understand the situation for the local ring $\mathscr{O}_{Y,b}$ or its formal completion $\mathscr{O}^\wedge_{Y,b}$. We get identifications

$$\Omega^1_{Y/F,b} \otimes_{\mathscr{O}_{Y,b}} \mathscr{O}^\wedge_{Y,b} = (\Omega^1_{Y/F,b})^\wedge_{\mathfrak{m}_b} = (\Omega^1_{R/F})^\wedge_{\mathfrak{m}} = \Omega^1_{R/F} \otimes_R \widehat{R},$$

the outer ones by [7], Chapter III, §3, No. 4, Theorem 3, the middle one by the compatible equalities $\mathscr{O}_{Y,b}/\mathfrak{m}^{n+1}_b = R/\mathfrak{m}^{n+1}$ stemming from Proposition 5.2. According to Lemma 5.5 below, the finitely presented module $(\Omega^1_{R/F})^\wedge_{\mathfrak{m}}$ has a presentation with

generators $d(u^2), d(u^3), d(\omega_i + \alpha_i u)$, and relations obtained by expressing the differentials of the relations (9) in terms of the generators. Hence $\Omega^1_{R/F} \otimes_R \widehat{R}$ has a presentation matrix in block form

$$P = \left(\begin{array}{c|ccc} 2u^3 & P_1 & \cdots & P_r \\ -3(u^2)^2 & Q_1 & \cdots & Q_r \\ \hline 0 & & 0 & \end{array} \right) \in \mathrm{Mat}_{(r+2)\times(r+1)}(\widehat{R}).$$

For $p \geq 5$, the entries in the upper right block are the coefficients from the differential of the relation $(\omega_i + \alpha_i u)^p - \lambda_i - \mu_i \cdot (u^3) \cdot (u^2)^{(p-3)/2}$, so

$$P_i = -\mu_i(u^2)^{(p-3)/2} \quad \text{and} \quad Q_i = -\frac{p-3}{2}\mu_i(u^3)(u^2)^{(p-5)/2}.$$

For the remaining characteristics we get

$$(P_i, Q_i) = \begin{cases} (0, -\mu_i) & \text{if } p = 2; \\ (-\mu_i, 0) & \text{if } p = 3. \end{cases}$$

One quickly computes that in all cases the 2-minors $P_iQ_j - Q_iP_j$ and $2u^3Q_i + 3(u^2)^2P_i$ vanish, and immediately sees that the presentation matrix has non-zero entries. For the Fitting ideals, this means $\mathrm{Fitt}_r(\Omega^1_{Y/F})_b = 0$ and $\mathrm{Fitt}_{r+1}(\Omega^1_{Y/F})_b \neq 0$, and it follows that the sheaf $\Omega^1_{Y/F}$ has rank r.

Suppose now $p = 3$. Then all but the first row of P vanish, and this row is $-(u^3, \mu_1, \ldots, \mu_r)$. Our standing assumption $\Gamma(\mathscr{O}_X) \cap L = F$ ensures that the $\mu_i = \alpha_i^p$ are non-zero, hence $\mathrm{Fitt}_{r+1}(\Omega^1_{Y/F})_b$ is the unit ideal, and in particular free as a module. Thus the numerical function $y \mapsto \dim_{\kappa(y)}(\Omega^1_{Y/F} \otimes \kappa(y))$ is constant on $\{b, \eta\}$, and it follows that the stalk $\Omega^1_{Y/F,b}$ is free. The argument for $p = 2$ is analogous. This establishes (ii)⇒(i).

It remains to prove (i)⇒(ii). Recall that $\mathfrak{b} = \mathrm{Fitt}_{r+1}(\Omega^1_{R/F} \otimes_R \widehat{R})$ is generated by the entries of the presentation matrix P, and write $\mathfrak{b}'$ for the induced ideal in the normalization $\widehat{R}' = E[[u]]$. Let us first consider the case $p \geq 7$, which ensures $2, 3 \in F$ are non-zero, and $(u^2)^2 \mid P_i$ and $u^3 \mid Q_i$. Then $\widehat{R}'/\mathfrak{b}' = E[u]/(u^3)$, and the relation (9) reveals $\widehat{R}/\mathfrak{b} = L[u^2]/(u^4)$. For $p = 5$ one similarly has $\mathfrak{b} = (u^3, u^2)$, giving $\widehat{R}'/\mathfrak{b}' = E[u]/(u^2)$ and $R/\mathfrak{b} = L$. In both cases

$$(10) \qquad\qquad \dim_F(\widehat{R}/\mathfrak{b}) < \dim_F(\widehat{R}'/\mathfrak{b}').$$

Seeking a contradiction, we now assume that that $p \geq 5$, and that $\mathfrak{b}$ is locally free. Being a non-zero ideal, it must be invertible, and thus defines an effective Cartier divisor $D \subset Y$ supported at the singularity $b \in Y$. By the Degree Formula, it has the same degree as its preimage $f^{-1}(D) \subset X$, in contradiction to (10). □

From now on, we assume $p \leq 3$, and specialize to the case that

$$X = \mathbb{P}^1_E = \mathrm{Proj}\, E[T_0, T_1] = \mathrm{Spec}\, E[u] \cup \mathrm{Spec}\, E[u^{-1}]$$

is the projective line over our height-one extension $F \subset E$, with inhomogeneous coordinate $u = T_1/T_0$, and that our chosen point is the origin $a = (0 : 1)$. This gives an affine open covering

$$Y = \mathrm{Spec}\, R \cup \mathrm{Spec}\, E[u^{-1}].$$

Suppose now that there is $D \in H^0(Y, \Theta_{Y/F})$ with either $D^{[p]} = 0$ or $D^{[p]} = D$, that the $\omega_k + \alpha_k u$ belong to $\mathrm{Ker}(D)$, and that D does not stabilized the maximal ideal $\mathfrak{m}_b \subset \mathscr{O}_{Y,b}$. According to Corollaries 6.3 and 6.5 below, such global vector fields indeed exists provided that the field of representatives L is chooses in a special way. Our $D \in H^0(Y, \Theta_{Y/F})$ corresponds to a faithful action of $G = \alpha_p$ or $G = \mu_p$. Let $F \subset F'$ be a simple height-one extension, endow its spectrum $T = \mathrm{Spec}(F')$ with the structure of an G-torsor, and consider the resulting twisted form $\tilde{Y} = T \wedge^G Y$.

Proposition 5.4. *Assumptions as above. If the height-one extensions E, F' are linearly disjoint over F, then the genus-one curve $\tilde{Y}$ is regular. Moreover, its Frobenius base-change $\tilde{Y}^{(p)}$ is isomorphic to some standard model $C_{r,F,\Lambda}^{(1)}$.*

Proof. Consider the singular locus $Z_0 = \{b\}$ as reduced closed subscheme, and write $Z = G \cdot Z_0$ for its orbit. We claim that Z is equivariantly isomorphic to $G \times \mathrm{Spec}(L)$, where the action is given by $\sigma \cdot (\tau, z) = (\sigma\tau, z)$, and that the inclusion $Z \subset Y$ is an effective Cartier divisor. To see this, write $\mathscr{I} \subset \mathscr{I}_0$ for the respective sheaf of ideals for the closed subschemes $Z \supset Z_0$. In order to give a uniform treatment use the Kronecker delta $\delta = \delta_{p,2}$. Obviously we have

$$u^p, \omega_1 + \alpha_1 u, \ldots, \omega_r + \alpha_r u \in \mathrm{Ker}(D) \quad \text{and} \quad D(u^{2+\delta}) = \pm 1.$$

It follows that $Z \subset Y$ is defined by $u^p \in \mathscr{O}_{Y,b}$, with resulting coordinate ring $L[u^{2+\delta}]/(u^{p(2+\delta)})$. The induced derivation on the coordinate ring vanishes on the $\omega_i + \alpha_i u$, and takes the unit value on the generator $u^{2+\delta}$. It follows that Z is equivariantly isomorphic to $G \times \mathrm{Spec}(L)$. Thus its twisted form $\tilde{Z}$ has coordinate ring $F' \otimes L$. We also see that the inclusion $Z \subset Y$ is an effective Cartier divisor.

Next, we consider the complementary open set $U = Y \smallsetminus \{b\}$. Its coordinate ring is the polynomial ring $E[u^{-1}]$, and its base change becomes $(F' \otimes E)[u^{-1}]$. Our assumptions ensure that the rings $F' \otimes E$ and hence also $F' \otimes L$ are fields. Thus Lemma 1.5 applies, and we conclude that the curve $\tilde{Y}$ is regular. One argues as in Proposition 4.3 to see that the Frobenius pull-back $Y^{(p)}$ is isomorphic to some standard model $C_{r,F,\Lambda}^{(1)}$. $\square$

In the proof for Proposition 5.3 we have used a useful general fact on Kähler differentials: Let R be a finitely generated F-algebra, $\mathfrak{m}$ be a maximal ideal, and $\widehat{R}$ be the resulting formal completion. Suppose we have elements $f_1, \ldots, f_m \in R$ and $g_1, \ldots, g_n \in \mathfrak{m}$ such that the former generate the the residue field $\kappa = R/\mathfrak{m}$ over F, and the latter generate the cotangent space $\mathfrak{m}/\mathfrak{m}^2$ over κ. We then have a homomorphism

$$(11) \qquad \varphi : F[x_1, \ldots, x_n, y_1, \ldots, y_n] \longrightarrow R$$

given by the assignments $x_i \mapsto f_i$ and $y_j \mapsto g_j$. Now suppose we have polynomials $h_1, \ldots, h_r$ in the indeterminates x_i and y_j that generate the kernel for the induced continuous map $\widehat{\varphi} : F[x_1, \ldots, x_m][[y_1, \ldots, y_n]] \to \widehat{R}$.

Lemma 5.5. *In the above situation, the $\widehat{R}$-module $(\Omega_{R/F}^1)_{\mathfrak{m}}^{\wedge}$ has a presentation where the generators are the differentials df_i, dg_j and the relations arise from writing the dh_k in terms of the generators.*

Proof. To better conform with the cited literature, we temporarily change notation and set

$$A = F \quad \text{and} \quad B = F[x_1, \ldots, x_m, y_1, \ldots, y_n] \quad \text{and} \quad C = R.$$

Consider the formal completions $\widehat{B} = \varprojlim_i B/\mathfrak{b}^i$ and $\widehat{C} = \varprojlim_i C/\mathfrak{m}^i$, where $\mathfrak{b} = (y_1, \ldots, y_n)$, and $\mathfrak{m} \subset C$ is the given maximal ideal. Our homomorphism (11) becomes $\varphi : B \to C$. The induced continuous map $\widehat{\varphi} : \widehat{B} \to \widehat{C}$ is surjective. To see this one argues as in the local case ([8], Chapter IX, §2, No. 5, Lemma 3), by observing that the induced map on associated graded rings is surjective, and applying [7], Chapter III, §2, No. 8, Corollary 2 for Theorem 1.

Sept $I = \text{Ker}(\widehat{\varphi})$. According to [17], Chapter 0, Theorem 20.5.12 we have an exact sequence

$$I/I^2 \longrightarrow \Omega^1_{\widehat{B}/A} \otimes_{\widehat{B}} \widehat{C} \longrightarrow \Omega^1_{\widehat{C}/A} \longrightarrow 0,$$

where the map on the left is given by $x \mapsto dx \otimes 1$. But note that the modules on the right are usually not finitely generated. To remedy this, we pass to $\widehat{\Omega}^1_{\widehat{B}/A} = \varprojlim \Omega^1_{B_i/A}$ where $B_i = B/\mathfrak{b}^i$. This module is separated and complete, the ensuing derivation $d : \widehat{B} \to \widehat{\Omega}^1_{\widehat{B}/A}$ is continuous, and actually universal for continuous derivation to separated and complete modules. Likewise, we form $\widehat{\Omega}^1_{\widehat{C}/A} = \varprojlim \Omega^1_{C_i/A}$ with $C_i = C/\mathfrak{m}^i$. As discussed in [17], 20.7.14 we still have a sequence

$$(12) \qquad I/I^2 \longrightarrow \widehat{\Omega}^1_{\widehat{B}/A} \otimes_{\widehat{B}} \widehat{C} \longrightarrow \widehat{\Omega}^1_{\widehat{C}/A} \longrightarrow 0.$$

Such sequences can be defined for every metrisable topological ring $\widehat{B}$ with a closed ideal I. As remarked in loc. cit. 20.7.17 and 20.7.20 the images are still dense in the kernels, but otherwise the above sequence may loose its exactness property. We now argue that this does not happen here, by using the work of Kunz:

According to [22], the ring $\widehat{B}$ admits a derivation $\widehat{B} \to \widetilde{\Omega}^1_{\widehat{B}/A}$ to a finitely generated module that is universal foe derivations to finitely generated modules, and the same holds for $\widehat{C}$. Moreover, in our situation we have canonical identifications

$$\widetilde{\Omega}^1_{\widetilde{B}/A} = \widehat{\Omega}^1_{\widehat{B}/A} \quad \text{and} \quad \widetilde{\Omega}^1_{\widetilde{C}/A} = \widehat{\Omega}^1_{\widehat{C}/A},$$

according to loc. cit. Corollary 12.5. Using loc. cit. Corollary 11.10 we see that (12) is indeed exact.

One directly checks that the $\widehat{B}$-modules $\widehat{\Omega}^1_{\widehat{B}/A}$ is freely generated by the dx_i and dy_j. By assumption we have $I = (h_1, \ldots, h_r)$. The only remaining task is to identify $\widehat{\Omega}^1_{\widehat{C}/A} = \widehat{\Omega}^1_{\widehat{R}/F}$ with $(\Omega^1_{R/F})^\wedge_\mathfrak{m}$. To see this set $R_i = R/\mathfrak{m}^i$, and consider the exact sequences $\mathfrak{m}^i/\mathfrak{m}^{2i} \to \Omega^1_{R/F}/\mathfrak{m}^i\Omega^1_{R/F} \to \Omega^1_{R_i/F} \to 0$. The image of the map on the left is denoted by M_i. Passing to inverse limits, we get an exact sequence

$$\varprojlim M_i \longrightarrow (\Omega^1_{R/F})^\wedge_\mathfrak{m} \longrightarrow \widehat{\Omega}^1_{\widehat{R}/F} \xrightarrow{\partial} R^1 \varprojlim M_i.$$

Clearly, the transition maps $M_{2i} \to M_i$ are zero, hence the term on the left vanishes. Since the M_i are modules of finite length, the inverse system $(M_i)_{i \geq 0}$ automatically satisfy the Mittag-Leffler Condition, so the connecting map ∂ is zero ([16], Proposition 13.2.2, confer also [28] and [24]). $\qquad\square$

6. Analysis of the Lie algebra

We keep the setting of the previous section, with the genus-one curve

$$Y = \operatorname{Spec} R \cup \operatorname{Spec} E[u^{-1}]$$

obtained as denormalizations of $X = \mathbb{P}^1_E$ by introducing a singularity $b \in Y$ on the first chart, for the moment with $p > 0$ arbitrary. We now seek to gather information on $\mathfrak{g} = H^0(Y, \Theta_{Y/F})$. This is a finite-dimensional restricted Lie algebra contained in $\operatorname{Der}_F(E[u^{-1}])$, via the restriction map to the regular locus $U = \operatorname{Spec} E[u^{-1}]$. Each element of $\Gamma(U, \Theta_{Y/F}) = \operatorname{Der}_F(E[u^{-1}])$ takes the form

$$D = -P\frac{\partial}{\partial u^{-1}} + \sum_{i=1}^r Q_i \frac{\partial}{\partial \omega_i} = u^2 P \frac{\partial}{\partial u} + \sum_{i=1}^r Q_i \frac{\partial}{\partial \omega_i},$$

with unique polynomials $P, Q_i \in E[u^{-1}]$. Every such D induces an F-derivation of the ring of formal Laurent series $E((u))$, and we have $D \in \mathfrak{g}$ if and only if this induced map stabilizes the subring $\widehat{R} = \mathscr{O}^{\wedge}_{Y,b}$. We now seek to characterize the condition $D \in \mathfrak{g}$ in terms of the coefficients of P and Q_i. Let us first corroborate that $\mathfrak{g}$ is finite-dimensional:

Proposition 6.1. *If $D \in \mathfrak{g}$ then $\deg(P) \leq 4$ and $\deg(Q_i) \leq p$. In odd characteristics we actually have $\deg(P) \leq 3$.*

Proof. One immediately sees

$$D(u^2) = 2u^3 P \quad \text{and} \quad D(u^3) = 3u^4 P \quad \text{and} \quad D(\omega_k u^p) = u^p Q_k \quad (1 \leq k \leq r).$$

These Laurent polynomials belong to the singular subring $\mathscr{O}^{\wedge}_{Y,b}$, because this holds for the arguments $u^2, u^3, \omega_k u^p$, and in particular lie in $E((u))$. As a consequence, they belong to $E[u] = E[u^{\pm 1}] \cap E((u))$, and the assertion follows. $\square$

Recall that $\omega_1, \ldots, \omega_r \in E$ form a p-basis, hence each $\varphi \in E$ can be uniquely written as p-truncated polynomial $\varphi = \varphi(\omega_1, \ldots, \omega_r)$ with coefficients from F. In turn, we may regard E as a D-*module*, that is, a module over the *Weyl algebra* $A_r(F)$ formed with the symbols x_i, ∂_i. Explicitly, the symbols act via $x_i \cdot \varphi = \omega_i \varphi$ and $\partial_i \cdot \varphi = \frac{\partial_i \varphi}{\partial x_i}(\omega_1, \ldots, \omega_r)$. The D-module is annihilated by the central elements $x_i^p - \omega_i^p$, so we may as well work with the residue class ring

$$(13) \qquad A_r(F)/(x_1^p - \omega_1^p, \ldots, x_r^p - \omega_r^p) = E[\partial_1, \ldots, \partial_r].$$

Note that this is the associative E-algebra given by the relations $\partial_i \omega_j - \omega_j \partial_i = \delta_{ij}$.

Suppose now $p = 2$, and consider derivations D whose coefficients take the form

$$P(u^{-1}) = \lambda_0 u^{-4} + \lambda_1 u^{-3} + \ldots + \lambda_4 \quad \text{and} \quad Q_i(u^{-1}) = \mu_0^{(i)} u^{-2} + \mu_1^{(i)} u^{-1} + \mu_2^{(i)}.$$

Consider, for $1 \leq k \leq r$, the following elements from the field E:

$$\Phi_k = \alpha_k \lambda_2 + \mu_2^{(k)} + \sum_{i=1}^r \mu_1^{(i)} \partial_i \alpha_k \quad \text{and} \quad \Psi_k = \alpha_k \lambda_3 + \sum_{i=1}^r \mu_2^{(i)} \partial_i \alpha_k.$$

Recall the denormalization $f : \mathbb{P}^1_E \to Y$ is defined by some non-standard field of representatives $L = F(\omega_1 + \alpha_1 \epsilon, \ldots, \omega_r + \alpha_r \epsilon)$, which in turn specifies the scalars $\alpha_i \in E$.

Proposition 6.2. *In the above setting, the derivation $D \in \mathrm{Der}_F(E[u^{-1}])$ belongs to $\mathfrak{g}$ if and only if the partial differential equations*

$$\lambda_1 = \Delta\lambda_0, \quad \mu_1^{(k)} = \Delta\mu_0^{(k)}, \quad \mu_0^{(k)} = \alpha_k\lambda_0, \qquad \Psi_k = \Delta\Phi_k$$

hold, with $\Delta = \sum_{i=1}^r \alpha_i\partial_i$ from residue class ring (13) *of the Weyl algebra.*

Proof. The first two equations express the respective conditions that

$$D(u^3) = \lambda_0 + \lambda_1 u + \ldots + \lambda_4 u^4 \quad \text{and} \quad D(\omega_k u^2) = \mu_0^{(k)} + \mu_1^{(k)}u + \mu_2^{(k)}u^2,$$

taken modulo u^2, belongs to the field of representatives $L \subset E[u]/(u^2)$. The third conditions means that in the Laurent polynomial

$$D(\omega_k + \alpha_k u) = \alpha_k u^2 P(u^{-1}) + Q_k(u^{-1}) + u\sum_{i=1}^r Q_i(u^{-1})\frac{\partial\alpha_k}{\partial\omega_i},$$

the term degree $d = -2$ vanishes. This, together with the first two equations, then implies that the above has no terms of degree $d < 0$, and is thus a polynomial in u. The final equation means that this polynomial, taken modulo u^2, belongs to $L \subset E[u]/(u^2)$. $\square$

The first three equations ensures that $\lambda_0 \in E$ already determines $\lambda_1, \mu_0^{(k)}, \mu_1^{(k)}$. The meaning of $\Delta\Phi_k = \Psi_k$ is less clear to me. However, the condition simplifies dramatically if we choose the field of representatives $L = F(\omega_1 + \alpha_1\epsilon, \ldots, \omega_r + \alpha_r\epsilon)$ inside the local Artin ring $\mathscr{O}_{Y,b}/\mathfrak{m}_b^2 = E[\epsilon]$ in a very special way:

Corollary 6.3. *Suppose we have $\alpha_k = \omega_k$ for $1 \leq k \leq r$. Then the derivation*

$$D = (u^{-2} + 1)\frac{\partial}{\partial u} + \sum_{i=1}^r \omega_i(u^{-2} + u^{-1})\frac{\partial}{\partial\omega_i}$$

inside $\mathrm{Der}_F(E[u^{-1}])$ belongs to $\mathfrak{g} = H^0(Y, \Theta_{Y/F})$. Furthermore, it has $D^{[2]} = 0$, satisfies $D(u^3) = 1 + u^2$, and vanishes on the $\omega_k + \alpha_k u$.

Proof. For the derivation at hand we get $\lambda_0 = \lambda_2 = 1$ and $\lambda_1 = \lambda_3 = \lambda_4 = 0$ and $\mu_0^{(k)} = \mu_1^{(k)} = \omega_k$ and $\Phi_k = \Psi_k = 0$. In turn, the partial differential equations in the proposition are obviously fulfilled. A direct computation reveals the stated conditions on D. $\square$

Suppose now $p = 3$. This is handled in an analogous way, we only state the result. Consider derivations D with coefficients $P(u^{-1}) = \lambda_0 u^{-3} + \lambda_1 u^{-2} + \ldots + \lambda_3$ and $Q_i(u^{-1}) = \mu_0^{(i)}u^{-3} + \mu_1^{(i)}u^{-2} + \ldots + \mu_3^{(i)}$, and set

$$\Phi_k = \alpha_k\lambda_0 + \mu_3^{(k)} + \sum_{i=1}^r \mu_2^{(i)}\partial_i\alpha_k \quad \text{and} \quad \Psi_k = \alpha_k\lambda_2 + \sum_{i=1}^r \mu_3^{(i)}\partial_i\alpha_k.$$

Proposition 6.4. *In the above setting, the derivation $D \in \mathrm{Der}_F(E[u^{-1}])$ belongs to $\mathfrak{g}$ if and only if the partial differential equations*

$$\lambda_1 = \Delta\lambda_0, \quad \mu_0^{(k)} = \mu_1^{(k)} = 0, \quad \mu_2^{(k)} = -\alpha_k\lambda_0, \quad \Psi_k = \Delta\Phi_k$$

hold, with $\Delta = \sum_{i=1}^r \alpha_i\partial_i$ from residue class ring (13) *of the Weyl algebra.*

Specializing the coefficients that determine $L = F(\omega_1 + \alpha_1\epsilon, \ldots, \omega_r + \alpha_r\epsilon)$ we get:

Corollary 6.5. *Suppose we have $\alpha_k = \omega_k$ for $1 \le k \le r$. Then the derivation*

$$D = (u - u^{-1})\frac{\partial}{\partial u} + \sum_{i=1}^{r} \omega_i(u^{-1} - 1)\frac{\partial}{\partial \omega_i}$$

inside $\mathrm{Der}_F(E[u^{-1}])$ belongs to $\mathfrak{g} = H^0(Y, \Theta_{Y/F})$. Furthermore, it has $D^{[3]} = D$, satisfies $D(u^2) = 1 - u^2$, and vanishes on the $\omega_k + \alpha_k u$.

7. Constructions involving nilpotents

As in the previous section, F denotes an imperfect ground field of characteristic $p > 0$, and X is a regular curve where $E = H^0(X, \mathscr{O}_X)$ is a height-one extension of degree $[E : F] = p^r$ for some $r \ge 1$. Again we fix an E-rational point $a \in X$, together with a uniformizer $u \in \mathscr{O}_{X,a}^{\wedge}$, and consider the local Artin ring $\mathscr{O}_{X,a}/\mathfrak{m}_a^2 = E[\epsilon]$.

We now choose a subfield $L \subset E[\epsilon]$ of degree $[L : F] = p^s$ for some $0 \le s < r$, and some L-vector subspace $H\epsilon \subset E\epsilon$ of codimension one. We also assume that $\Gamma(\mathscr{O}_X) \cap L = F$, and now form the subring $\Lambda = L + H\epsilon$. Note that the sum is direct, but the first summand is not aligned to the decomposition $E[\epsilon] = E \oplus E\epsilon$. Set

$$A = \mathrm{Spec}(E[\epsilon]) = \{a\} \quad \text{and} \quad B = \mathrm{Spec}(L + H\epsilon) = \{b\},$$

and consider the resulting cocartesian square

(14)
$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y. \end{array}$$

This defines a new integral curve Y with singular locus $\mathrm{Sing}(Y) = \{b\}$. Arguing as for Proposition 2.1, one sees $h^0(\mathscr{O}_Y) = 1$ and $h^1(\mathscr{O}_Y) = h^1(\mathscr{O}_X) + 1$. Furthermore, the local rings $\mathscr{O}_{Y,y}$ are unibranch and Gorenstein.

We now seek to understand the complete local ring $\mathscr{O}_{Y,b}^{\wedge}$ at the L-valued singularity $b \in Y$. Recall that $[E : F] = p^r$ and $[L : F] = p^s$. Thus $[\kappa(a) : L] = p^{r-s}$, and $\dim_L(H)$ coincides with the integer $n = p^{r-s} - 1$. We now choose a p-basis and an L-basis

$$\omega_1 + \alpha_1\epsilon, \ldots, \omega_s + \alpha_s\epsilon \in L \quad \text{and} \quad \beta_1\epsilon, \ldots, \beta_n\epsilon \in H,$$

all belonging to the over-ring $E[\epsilon]$. This defines scalars $\alpha_i, \beta_j \in E$ and p-independent $\omega_i \in E$. The ensuing cartesian square

$$\begin{array}{ccc} E[\epsilon] & \longleftarrow & E[u] \\ \uparrow & & \uparrow \\ F[\omega_i + \alpha_i\epsilon, \beta_j\epsilon] & \longleftarrow & R \end{array}$$

defines a ring R that is finitely generated over F. Note that the indices in the lower left corner are meant to run over $1 \le i \le s$ and $1 \le j \le n$, a convention that will apply throughout.

As in Section 5, we consider the maximal ideal $\mathfrak{m} = R \cap uE[u]$. The ensuing formal completion $\widehat{R}$ sits an an analogous cartesian square, with $E[[u]]$ instead of $E[u]$ in the top right corner, and comes with an identification $\mathscr{O}_{Y,b}^{\wedge} = \widehat{R}$. The following will be a key observation:

Proposition 7.1. *If $p^{r-s} \geq 5$ then the ring R is not locally of complete intersection.*

Proof. Clearly, the monomials $\beta_j u$ belong to $\mathfrak{m}$, and are linearly independent modulo $\mathfrak{m}^2$, because this holds for their images under the canonical map $R/\mathfrak{m}^2 \to L + H\epsilon$. We claim that these elements generate the cotangent space $\mathfrak{m}/\mathfrak{m}^2 = \widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2$, such that $\mathrm{edim}(R_\mathfrak{m}) = n$. To see this, consider a formal power series $\sum_{k \geq 1} \lambda_k u^k$ belonging to $\widehat{\mathfrak{m}}$. Subtracting an L-linear combination of the $\beta_j u$ we achieve $\lambda_1 = 0$. By Lemma 8.2 below, the canonical maps $\mathrm{Sym}_L^k(H\epsilon) \to \mathrm{Sym}_E^k(E\epsilon)$ are surjective for all $k \geq 2$. In particular we can write

$$\lambda_2 \epsilon^2 = \sum P_m(\omega_1 + \alpha_1 \epsilon, \dots, \omega_s + \alpha_s \epsilon)\beta_{m_1}\epsilon \cdot \beta_{m_2}\epsilon = \sum P_m(\omega_1, \dots, \omega_s)\beta_{m_1}\beta_{m_2}\epsilon^2,$$

with multi-indices $m = (m_1, m_2)$ and $P_m \in F[T_1, \dots, T_s]$. Note that in the outer terms, the symbol ϵ^2 signifies a monomial in the second symmetric power rather than a ring element. Comparing coefficients gives $\lambda_2 = \sum P_m(\omega_1, \dots, \omega_s)\beta_{m_1}\beta_{m_2}$, an equation in the field E. This in turn yields

$$\lambda_2 u^2 = \sum P_m(\omega_1 + \alpha_1 u, \dots, \omega_r + \alpha_s u) \cdot (\beta_{m_1} u)(\beta_{m_2} u) + \dots,$$

where the omitted terms have order ≥ 3. This reduces our task to $\lambda_2 = 0$, and likewise we achieve $\lambda_3 = 0$. Now both factors in $\sum_{k \geq 4} \lambda_k u^k = u^2 \cdot \sum_{k \geq 4} \lambda_k u^{k-2}$ belong to $\widehat{\mathfrak{m}}$, so the product lies in $\widehat{\mathfrak{m}}^2$. Summing up, the $\beta_j u$ form a basis form the cotangent space $\widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2$.

We now choose a section for the residue class map $R \to R/\mathfrak{m} = L$. The resulting continuous homomorphism

$$\varphi : L[[x_1, \dots, x_n]] \longrightarrow \widehat{R}, \quad x_j \longmapsto \beta_j u$$

is surjective, by the previous paragraph. Seeking a contradiction, we assume that R is locally of complete intersection. Then the ideal of relations $\mathfrak{a} = \mathrm{Ker}(\varphi)$ is generated by $n - 1$ elements. Write $\mathfrak{p} = (x_1, \dots, x_n)$ for the maximal ideal. Then $\mathfrak{a} \subset \mathfrak{p}^2$, so according to Lemma 7.1 below, we have

$$(15) \qquad \dim_L(\mathfrak{a} + \mathfrak{p}^3)/\mathfrak{p}^3 \leq \mathrm{edim}(\widehat{R}) - \dim(\widehat{R}) = n - 1.$$

Recall that the canonical map $\mathrm{Sym}_L^2(H\epsilon) \to \mathrm{Sym}_E^2(E\epsilon) \simeq E$ is surjective. Its kernel U is thus an L-vector space of dimension

$$(16) \qquad \dim_L(U) = \binom{n-1+2}{2} - (n+1) = \frac{n^2 - n - 2}{2}.$$

As explained above, each vector $\sum \lambda_m \beta_{m_1}\epsilon \cdot \beta_{m_2}\epsilon$ from U gives an equation of in the ring $\widehat{R} \subset E[[u]]$ of the form

$$0 = \sum P_m(\omega_1 + \alpha_1 u, \dots, \omega_s + \alpha_s u) \cdot (\beta_{m_1} u)(\beta_{m_2} u) + \dots,$$

where the missing terms have order ≥ 3. Thus $\dim_L(\mathfrak{a} + \mathfrak{p}^3)/\mathfrak{p}^3 \geq \dim_L(U)$. Substituting (15) and (16) we arrive at the inequality $n - 1 \geq (n^2 - n - 2)/2$, and thus $n \leq 3$. But $n = p^{r-s} - 1 \geq 4$ by assumption, giving the desired contradiction. $\qquad \square$

We are thus only interested in the situation $p^{r-s} \leq 4$. This gives three cases, and in particular we have $p \leq 3$ and $r - s \leq 2$.

Proposition 7.2. *Suppose $p^{r-s} \leq 4$. Then R is locally of complete intersection. Disregarding the F-structure, the formal completion $\widehat{R}$ is given by the following table, for certain scalars $\lambda, \mu, \gamma, \delta \in F$:*

p	$\widehat{R}$	s
3	$L[[x,y]]/(x^3 - \lambda y^3)$	$r-1$
2	$L[[x,y]]/(x^4 - \mu y^2)$	$r-1$
2	$L[[x,y,z]]/(x^2 - \gamma z^2, y^2 - \delta z^2)$	$r-2$

Proof. First note that $p^{r-s} \leq 4$ allows exactly for the three combinations of p and s occurring in the table. Now choose a section for the residue class projection $\widehat{R} \to L$.

Suppose first that $p = 2$ and $s = r - 1$. Then $H\epsilon \subset E\epsilon$ is generated by a single element $\beta_1 \epsilon$ over L. One easily checks that the assignment $x \mapsto \beta_1 u$ and $y \mapsto u^2$ gives the desired presentation of $\widehat{R}$, with $\lambda = \beta_1^2$. For $p = 2$ and $s = r - 2$ the L-vector space $H\epsilon$ is three-dimensional, and one checks that $x \mapsto \beta_1 u$ and $y \mapsto \beta_2 u$ and $z \mapsto \beta_3 u$ leads to the desired description, with $\gamma = (\beta_3/\beta_1)^2$ and $\delta = (\beta_2/\beta_1)^2$. Finally, for $p = 3$ and $s = r-1$ we take $x \mapsto \beta_1 u$ and $y \mapsto \beta_2 u$, with $\lambda = (\beta_2/\beta_1)^3$. □

From now on, we specialize to the case that

$$X = \mathbb{P}_E^1 = \operatorname{Proj} E[T_0, T_1] = \operatorname{Spec} E[u] \cup \operatorname{Spec} E[u^{-1}]$$

is the projective line over E, with $u = T_1/T_0$ and $a = (0 : 1)$. Then

$$Y = \operatorname{Spec} R \cup \operatorname{Spec} E[u^{-1}].$$

By construction the $\omega_1, \ldots, \omega_s \in E$ are p-independent. So we can extend them to a p-basis $\omega_1, \ldots, \omega_r \in E$. The differentials $du, d\omega_i$ form a basis for the Kähler differentials on the regular locus $U = Y \smallsetminus \{b\}$, and we write $\partial/\partial u, \partial/\partial \omega_i$ for the dual basis in the tangent sheaf $\Theta_{U/F}$. It is now possible to calculate the Lie algebra $\mathfrak{g} = H^0(Y, \Theta_{Y/F})$, with the methods of Section 6. For the sake of brevity, we state the relevant findings:

Proposition 7.3. *Suppose $p^{r-s} \leq 4$ and $\operatorname{pdeg}(F) \geq 2r - s$. Then for suitable choices of $L + H\epsilon \subset E[\epsilon]$, the resulting genus-one curve Y admits a twisted form $\tilde{Y}$ that is regular, and whose Frobenius base-change $\tilde{Y}^{(p)}$ is isomorphic to some standard model $C = C_{r,\Lambda,F}^{(i)}$.*

Proof. We have to consider three cases, and start with $p = 3$ and $r - s = 1$. Then $n = 2$ and $\operatorname{pdeg}(F) \geq r + 1$. So there is a simple height-one extension $F \subset F'$ such that $E \otimes F'$ remains a field, and endow $T = \operatorname{Spec}(F')$ with the structure of an α_p-torsor. We choose a particular $L + H\epsilon \subset E[\epsilon]$ by setting $\alpha_i = \omega_i$ for $1 \leq i \leq r-1$ and $\beta_j = \omega_r^j$ for $1 \leq j \leq 2$. Now consider the derivation

$$D = -\omega_r^{-1}(1 + u)\frac{\partial}{\partial u} + (1 - u^{-1})\frac{\partial}{\partial \omega_r} + \omega_r^{-1} \sum_{i=1}^{r-1} \omega_i \frac{\partial}{\partial \omega_i}.$$

It belongs to $\operatorname{Der}_F(E[u^{-1}])$, and one directly computes

$$D(\omega_r u) = 1 \quad \text{and} \quad D(\omega_r^2 u) = \omega_r u \quad \text{and} \quad D(\omega_k + \omega_k u) = 0 \quad (1 \leq k \leq r - 1).$$

This ensures that D extends to a global vector field on Y. Furthermore, one computes $D^{[p]} = 0$. Consider the action of the group scheme $G = \alpha_p$ on Y corresponding to $D \in H^0(Y, \Theta_{Y/F})$. One sees that the orbit of the singular locus is Cartier, and isomorphic to the trivial G-torsor. The regular twisted form $\tilde{Y}$ is constructed as in Proposition 4.3.

Next suppose $p = 2$ and $r - s = 1$, such that $n = 1$. Again $\mathrm{pdeg}(F) \geq r + 1$, and we choose $T = \mathrm{Spec}(F')$ as above. The particular $L + H\epsilon \subset E[\epsilon]$ is obtained by setting $\alpha_1 = \omega_1, \ldots, \alpha_{r-1} = \omega_{r-1}$ and $\beta_1 = \omega_r$, and consider the derivation

$$D = \omega_r^{-1}(u^{-1} + u)\frac{\partial}{\partial u} + (u^{-2} + 1)\frac{\partial}{\partial \omega_r} + \omega_r^{-1}\sum_{i=1}^{r-1}\omega_i(u^{-1} + 1)\frac{\partial}{\partial \omega_i}.$$

This belongs to $\mathrm{Der}_F(E[u^{-1}])$, and one directly computes

$$D(\omega_r u) = 0 \quad \text{and} \quad D(\omega_r u^2) = 1 + u^2 \quad \text{and} \quad D(\omega_k + \omega_k u) = 0$$

for all $1 \leq k \leq r - 1$, and furthermore $D^{[p]} = 0$. The argument proceeds as in the previous paragraph.

Finally, suppose $p = 2$ and $r - s = 2$, which is the most challenging case. Now $n = 3$ and $\mathrm{pdeg}(F) \geq r + 2$. We choose $L + H\epsilon \subset E[\epsilon]$ by setting

$$\alpha_1 = \omega_1, \ldots, \alpha_{r-2} = \omega_{r-2} \quad \text{and} \quad \beta_1 = \omega_{r-1}, \quad \beta_2 = \omega_r, \quad \beta_3 = \omega_{r-1}\omega_r.$$

Consider the derivation

$$D = \omega_{r-1}^{-1}(1 + u)\frac{\partial}{\partial u} + \frac{\partial}{\partial \omega_{r-1}} + \omega_r\omega_{r-1}^{-1}(u^{-1} + 1)\frac{\partial}{\partial \omega_r} + \omega_{r-1}^{-1}\sum_{i=1}^{r-2}\omega_i\frac{\partial}{\partial \omega_i}.$$

It belongs to $\mathrm{Der}_F(E[u^{-1}])$, and one immediately checks

$$D(\omega_{r-1}u) = 1 \quad \text{and} \quad D(\omega_r u) = 0 \quad \text{and} \quad D(\omega_{r-1}\omega_r u) = \omega_r u,$$

and also $D(\omega_k + \omega_k u) = 0$ for all $1 \leq k \leq r - 2$. This ensures $D \in \Gamma(Y, \Theta_{Y/F})$. Furthermore, one computes $D^{[2]} = 0$. By symmetry, the same formula but with ω_{r-1} and ω_r interchanged defines another such $D' \in \Gamma(Y, \Theta_{Y/F})$. One computes $[D, D'] = 0$, and thus gets an inclusion of the restricted Lie algebra $k^2 \subset \Gamma(Y, \Theta_{Y/F})$, where on the left both bracket and p-map are trivial. This corresponds to an action of the group scheme $G = \alpha_p^{\oplus 2}$ on Y. The orbit of the singular point $b \in Y$ is the effective Cartier divisor defined by the element $\omega_{r-1}\omega_r u \in \mathscr{O}_{Y,b}$. The desired twisted form $\tilde{Y}$ that is regular arises as follows: Now choose two simple height-one extensions F' and F'' so that $E \otimes F' \otimes F''$ remains a field, endow their spectra with the structure of an α_p-torsor, consider the resulting diagonal G-torsor $T = \mathrm{Spec}(F' \otimes F'')$, and set $\tilde{Y} = T \wedge^G Y$. $\qquad\square$

I would like to point out that fixing a particular $L + H\epsilon \subset E[\epsilon]$ and finding suitable derivations $D \in \mathrm{Der}_F(E[u^{-1}])$ was a long and sometimes painful process of matrix computations and guesswork, not at all reflected in the above comparatively short arguments.

8. Verification of some technical facts

In this section we established the facts from commutative algebra and field theory used in the previous section. We start with the former. Recall that a noetherian ring R is called *locally of complete intersection* if for every prime $\mathfrak{p}$, the corresponding complete local ring $R_\mathfrak{p}^\wedge$ is isomorphic to $A/(f_1, \ldots, f_r)$, where A is some local noetherian ring that is complete and regular, and $f_1, \ldots, f_r \in \mathfrak{m}_A$ is a regular sequence. Note that if R itself is local, it suffices to verify this condition with the maximal ideal $\mathfrak{p} = \mathfrak{m}_R$, by [2], Corollary 1.

Suppose that R is a complete local noetherian ring, and write it as $R = A/\mathfrak{a}$ for some local noetherian ring A that is complete and regular, with $\mathrm{edim}(R) = \mathrm{edim}(A)$. Fix some $n \geq 0$ with $\mathfrak{a} \subset \mathfrak{m}_A^n$ and form $M = (\mathfrak{a} + \mathfrak{m}_A^{n+1})/\mathfrak{m}_A^{n+1}$. The latter is annihilated by $\mathfrak{m}_A$, and thus becomes a vector space over the residue field $k = A/\mathfrak{m}_A = R/\mathfrak{m}_R$. Intuitively speaking, all relations have order at least n, and the vector space M measures how many of them involve terms of order n.

Lemma 8.1. *If* $\dim_k(M) > \mathrm{edim}(R) - \dim(R)$*, then the ring R is not locally of complete intersection.*

Proof. Seeking a contradiction, we suppose that R is locally of complete intersection. By [19], Proposition 19.3.2 the ideal $\mathfrak{a}$ is generated by a regular sequence $f_1, \ldots, f_r$. Krull's Principal Ideal Theorem gives $r = \mathrm{edim}(R) - \dim(R)$, and the Nakayama Lemma ensures $\dim_k(\mathfrak{a} \otimes k) \leq r$. The Isomorphism Theorem gives an identification $M = \mathfrak{a}/(\mathfrak{a} \cap \mathfrak{m}_A^{n+1})$, and hence $\dim_k(M) \leq \dim_k(\mathfrak{a} \otimes_R k)$. Combing these inequalities we get the desired contradiction. $\square$

We also used a purely field-theoretic fact: Let $L \subset E$ be a finite field extension, $E\epsilon$ be a one-dimensional E-vector space, where ϵ denotes a basis vector, and $H\epsilon$ be some L-linear subspace of codimension one.

Lemma 8.2. *In the above setting, the canonical maps* $\mathrm{Sym}_L^k(H\epsilon) \to \mathrm{Sym}_E^k(E\epsilon)$*, $k \geq 2$ are surjective provided that $[E : L] \geq 3$.*

Proof. Without restriction the one-dimensional vector space is E itself, and we write $H \subset E$ for the L-linear subspace of codimension one. The canonical map between symmetric powers becomes the multiplication map $\mathrm{Sym}_L^k(H) \to E$. It suffices to treat the case $k = 2$. Write $H \cdot H \subset E$ for the image of the multiplication map. This has codimension at most one, because for each non-zero $a \in H$ the multiplication map $x \mapsto ax$ is injective. Seeking a contradiction, we assume that $H \cdot H \subset E$ is a hyperplane. It contains, for each non-zero $a \in H$, the subvector space aH. The latter is also a hyperplane, and thus $aH = H \cdot H$.

We claim that $H \subset H \cdot H$. If not, $H \cap (H \cdot H) \subset E$ has codimension two. Since $[E : L] \geq 3$, the intersection contains some $a \neq 0$. We just saw that we can write it in the form $a = ab$ for some $b \in H$. Then $b = 1$, and we infer that $H = 1H$ is contained in $H \cdot H$, contradiction. This actually establishes $H = H \cdot H$.

We then even have $1 \in H$: Since $\dim_L(H) = [E : L] - 1 \geq 2$, we find some non-zero $a \in H$, can write it as $a = ab$ for some $b \in H$, and infer $b = 1$. It follows that the subfield $L \subset E$ is contained in the hyperplane $H \subset E$, and thus $H \subset E$ is an L-subalgebra. It must be a field, because our field extensions are finite. Set

$d = [E : F]$. The Degree Formula $[E : L] = [E : H] \cdot [H : L]$ gives $[E : H] = d/(d-1)$. Rewriting the fraction as $1+1/(d-1)$ we conclude $d = 2$ or $d = 0$, contradiction. □

9. Twisted ribbons and genus-zero curves

Our analysis of genus-one curves depends on a thorough understanding of regular genus-zero curves and their behaviour under base-changes, and we collect the relevant observations in this section.

Let S be a base scheme, Z be a scheme, and $\mathscr{L}$ be an invertible sheaf. As introduced by Bayer and Eisenbud ([3], Section 1), a *ribbon for Z with respect to $\mathscr{L}$* is a triple (X, i, φ), where X is a scheme, $i : Z \to X$ is a closed embedding corresponding to a sheaf of ideals $\mathscr{I} \subset \mathscr{O}_X$ with $\mathscr{I}^2 = 0$, and $\varphi : \mathscr{L} \to \mathscr{I}$ is an isomorphism respecting the $\mathscr{O}_Z$-module structure. A ribbon is called *split* if the inclusion $i : Z \to X$ admits a retraction $r : X \to Z$. Each ribbon gives a short exact sequence

$$0 \longrightarrow \mathscr{L} \longrightarrow \Omega^1_{X/S}|Z \longrightarrow \Omega^1_{Z/S} \longrightarrow 0,$$

which in turn defines an extension class $\alpha \in \mathrm{Ext}^1(\Omega^1_{Z/S}, \mathscr{L})$. The ribbons from a category in the obvious way, and $(X, i, \varphi) \mapsto \alpha$ is functorial. As explained in loc. cit. Theorem 1.2 this functor induces a bijection between the set of isomorphism class of ribbons and the group $\mathrm{Ext}^1(\Omega^1_{Z/S}, \mathscr{L})$.

Now suppose that (X, i, φ) be a ribbon on Z with respect to $\mathscr{L}$, with class $\alpha \in \mathrm{Ext}^1(\Omega^1_{Z/S}, \mathscr{L})$. Let $D \subset Z$ be any effective Cartier divisor. Set $\mathscr{L}' = \mathscr{L}(D)$, and let $\alpha' \in \mathrm{Ext}^1(\Omega^1_{Z/S}, \mathscr{L}')$ be the image of α under the canonical map $\mathscr{L} \to \mathscr{L}'$. By the universal property, the blowing-up morphism $\mathrm{Bl}_D(Z) \to Z$ is an isomorphism. Now regard D as a closed subscheme on X. The blowing-up X' comes with a canonical morphisms $i' : Z = \mathrm{Bl}_D(Z) \to \mathrm{Bl}_D(X) = X'$. The following is a very useful observation, the proof being as for [3], Theorem 1.9:

Lemma 9.1. *In the above setting, the blowing-up $\mathrm{Bl}_D(X)$ is a ribbon on Z with respect to the invertible sheaf $\mathscr{L}'$, with class α'.*

Now let F be a ground field, for the moment of arbitrary characteristic $p \geq 0$. Clearly, a double line $2L \subset \mathbb{P}^2$ is a ribbon on the line $L = \mathbb{P}^1$ with respect to $\mathscr{L} = \mathscr{O}_{\mathbb{P}^1}(-1)$. From $\mathrm{Ext}^1(\Omega_L, \mathscr{O}_L(-1)) = H^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(1)) = 0$ we see that each such ribbon is split. We write it as $X_0 = \mathbb{P}^1 \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$. Clearly, this is a genus-zero curve, and the interpretation as double line shows that it is locally of complete intersection. Since $\mathscr{O}_{2L}(-1)$ coincides with the dualizing sheaf ω_{X_0}, we see that every twisted form X is also a quadric curve in $\mathbb{P}^2$. We need the following structural result:

Theorem 9.2. *Let X be a twisted form of $X_0 = \mathbb{P}^1 \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$ that is regular, and $a \in X$ a closed point whose residue field $E = \kappa(a)$ has degree two. Then the following holds:*

    (i) *The field extension $F \subset E$ is simple of height one, and in particular $p = 2$.*
    (ii) *The only singularity on $X \otimes E$ is the rational point corresponding to $a \in X$.*
    (iii) *The normalization of $X \otimes E$ is isomorphic to $\mathbb{P}^1_{E'}$, where $F \subset E'$ has height one and degree $[E' : F] = p^2$.*

Conversely, for every simple height-one extension $F \subset F'$ the base-change $X \otimes F'$ is singular if and only if $F' \simeq \kappa(a)$ for some point $a \in X$ as above.

Proof. (i) Since the ribbon X_0 is everywhere singular, the local ring $\mathscr{O}_{X,a}$ must be geometrically singular, hence $F \subset E$ is not separable, according to [12], Corollary 2.6. From $[E : F] = 2$ we deduce $p = 2$, and that the extension has height one.

(iii) By construction, the point $a' \in X \otimes E$ corresponding to $a \in X$ is rational, so its local ring is singular. According to [32], Lemma 1.3 the curve $X \otimes E$ is integral. Let $X' = \mathrm{Bl}_{a'}(X \otimes E)$ be the blowing-up with reduced center $\{a'\}$. Using that a corresponding blowing-up of the split ribbon $X_0 = \mathbb{P}^1 \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$ is the split ribbon $X'_0 = \mathbb{P}^1 \oplus \mathscr{O}_{\mathbb{P}^1}$ by Lemma 9.1, we see that X' is a twisted form of X'_0, and in particular $h^0(\mathscr{O}_{X'}) = 2$ and $h^1(\mathscr{O}_{X'}) = 0$. In light of [32], Proposition 1.4 the field $E' = H^0(X', \mathscr{O}_{X'})$ is a height-one extension of F. By construction, it contains E, and we infer $[E' : F] = [E' : E][E : F] = p^2$. Let $f : X' \to X \otimes E$ be the canonical morphism. Then $D = f^{-1}(a')$ is an effective Cartier divisor such that the canonical map $\Gamma(\mathscr{O}_{X'}) \to \Gamma(\mathscr{O}_D)$ is bijective, as one checks for the corresponding blowing-up $X'_0 \to X_0$. It follows that $X' \simeq \mathbb{P}^1_{E'}$.

(ii) Let A and B be the ramification and branch locus for $f : \mathbb{P}^1_{E'} \to X \otimes E$. Using Proposition 1.1 we get $h^0(\mathscr{O}_A) - h^0(\mathscr{O}_B) = 1$. Thus $f_*(\mathscr{O}_X)/\mathscr{O}_Y = f_*(\mathscr{O}_A)/\mathscr{O}_B$ is supported by a single point, which must be a'.

Finally, suppose that F' is a simple height-one extension so that $X \otimes F'$ becomes singular. The latter is integral by [32], Lemma 1.3, and we write X' for its normalization. Forming the conductor square for the normalization map $X' \to X \otimes F'$, we get an exact sequence

$$0 \longrightarrow H^0(\mathscr{O}_{X \otimes F'}) \longrightarrow H^0(\mathscr{O}_{X'}) \oplus H^0(\mathscr{O}_A) \longrightarrow H^0(\mathscr{O}_B) \longrightarrow 0.$$

The field $E' = H^0(\mathscr{O}_{X'})$ has height one, and we write $h^0(\mathscr{O}_{X'}) = 2^r$ and $h^0(\mathscr{O}_A) = d2^r$ for some $r \geq 0$ and $d \geq 1$. Then $h^0(\mathscr{O}_B) = d2^{r-1}$, according to Proposition 1.1. From the above exact sequence we get $1 - (2^r + d2^{r-1}) + d2^r = 0$, or in other words $(2 - d)2^{r-1} = 1$. The only solution is $d = 1$ and $r = 1$. Thus $B \subset X \otimes F'$ is the inclusion of an F'-valued point b'. The image $b \in X$ is not a rational point, hence the inclusion $\kappa(b) \subset \kappa(b') = F'$ must be an equality. $\qquad\square$

10. Genus-one curves with singularities

Fix a ground field F of characteristic $p > 0$, and let Y be a genus-one curve that is not geometrically regular. The ultimate goal is to understand the situation when Y is regular. As a preparation, we assume here that Y is integral, geometrically unibranch, locally of complete intersection, and that some local rings $\mathscr{O}_{Y,y}$ are singular. We furthermore assume that the Fitting ideals for $\Omega^1_{Y/F}$ are locally free, and that the purely inseparable field extension $E = H^0(X, \mathscr{O}_X)$ resulting from the normalization map $f : X \to Y$ has height-one. These conditions perhaps appears artificial, but is precisely the situation that happens on certain base-changes from the regular case.

Write $[E : F] = p^r$ and form the conductor square

$$
\begin{CD}
A @>>> X \\
@VVV @VV{f}V \\
B @>>> Y.
\end{CD}
$$

Proposition 10.1. *In the above situation, the following holds:*

(i) *The scheme X is a regular genus-zero curve over the field E.*
(ii) *The normalization map $f : X \to Y$ is a universal homeomorphism.*
(iii) *The finite E-scheme A has degree two.*
(iv) *For each $a \in A$, we have $h^0(\mathscr{O}_{A,a}) = 2h^0(\mathscr{O}_{B,f(a)})$.*
(v) *The field $F = \Gamma(\mathscr{O}_Y)$ is the intersection $\Gamma(\mathscr{O}_X) \cap \Gamma(\mathscr{O}_B)$ inside $\Gamma(\mathscr{O}_A)$.*

Proof. We prove (i) by contradiction: Suppose $H^1(X, \mathscr{O}_X) \neq 0$. By the exact sequence (2), this E-vector space is one-dimensional as F-vector space. It follows that $E = F$, and thus $h^i(\mathscr{O}_Y) = h^i(\mathscr{O}_X)$ for all $i \geq 0$. Now Proposition 1.1 gives $h^0(\mathscr{O}_B) = h^0(\mathscr{O}_A)$, hence the inclusion $\Gamma(\mathscr{O}_B) \subset \Gamma(\mathscr{O}_A)$ is an equality. In turn, $f : X \to Y$ is an isomorphism, so Y is regular, contradiction. This shows that X is a genus-zero curve over E. Condition (ii) holds because the local rings $\mathscr{O}_{Y,y}$ are unibranch by assumption.

The remaining statements also follow from Proposition 1.1: (iv) and (v) are immediate consequences. To see (iii) we use $h^0(\mathscr{O}_A) = 2h^0(\mathscr{O}_B)$ to obtain $h^0(\mathscr{O}_X) + h^0(\mathscr{O}_B) = h^0(\mathscr{O}_A)$. Combining these equation we see $h^0(\mathscr{O}_A) = 2h^0(\mathscr{O}_X)$, we conclude that the E-algebra $\Gamma(\mathscr{O}_A)$ has degree two. $\square$

From $[\Gamma(\mathscr{O}_A) : E] = 2$ and $\Gamma(\mathscr{O}_B) \subset \Gamma(\mathscr{O}_A)$ we see that there are only the four possibilities given by the columns of the following table:

A	non-reduced	non-reduced	disconnected	integral
B	non-reduced	reduced	disconnected	integral

We shall see that the first three possibilities lead to $X = \mathbb{P}^1_E$, and also the last case can be reduced to this by some further base-change.

Proposition 10.2. *Suppose B and A are non-reduced. Then*

$$p \leq 3 \quad and \quad X = \mathbb{P}^1_E \quad and \quad \Gamma(\mathscr{O}_A) = E[\epsilon] \quad and \quad \Gamma(\mathscr{O}_B) = L + H\epsilon,$$

for some subfield $L \subset \Gamma(\mathscr{O}_E)$ and some L-hyperplane $H\epsilon \subset E\epsilon$. Moreover, we have $p^{r-s} \leq 4$, where $[L : F] = p^s$.

Proof. Choose a non-zero nilpotent element $\epsilon \in \Gamma(\mathscr{O}_B)$. This gives a non-zero map $E[\epsilon] \to \Gamma(\mathscr{O}_A)$. Since the ring of dual numbers is a principal ideal ring, with only three ideals (ϵ^i), $0 \leq i \leq 2$, we infer that the non-zero map is injective. It actually is bijective, because both sides are vector spaces over E of the same dimension. It follows that both $A = \{a\}$ and $B = \{b\}$ are singletons. The statements on $\Gamma(\mathscr{O}_B)$ follow from Corollary 1.2. The genus-zero curve X contains an E-valued point, so by Proposition 1.4 it is isomorphic to $\mathbb{P}^1_E$. Finally, the statement $p^{r-s} \leq 4$ follows from Proposition 7.1, and in particular $p \leq 3$. $\square$

Proposition 10.3. *Suppose B is reduced and A is non-reduced. Then*

$$p \leq 3 \quad and \quad X = \mathbb{P}^1_E \quad and \quad \Gamma(\mathscr{O}_A) = E[\epsilon].$$

Moreover, the subrings $\Gamma(\mathscr{O}_B)$ and $\Gamma(\mathscr{O}_X)$ are fields of representatives inside the local Artin ring $\Gamma(\mathscr{O}_A)$.

Proof. Choose a non-zero nilpotent element $\epsilon \in \Gamma(\mathscr{O}_A)$. As in the preceding proof we infer that $\Gamma(\mathscr{O}_A) = E[\epsilon]$, that $X = \mathbb{P}^1_E$, and that both $A = \{a\}$ and $B = \{b\}$ are singletons. Now $L = \Gamma(\mathscr{O}_B)$ must be a field. From $2 = \mathrm{length}_{\mathscr{O}_{B,b}}(\mathscr{O}_{A,a}) = \dim_L E[\epsilon] = 2[E : L]$ we conclude that $L \subset E[\epsilon]$ is a field of representatives. Obviously, the same holds for the subfield $E = \Gamma(X, \mathscr{O}_X)$. $\qquad\square$

Proposition 10.4. *Suppose A is disconnected. Then*

$$p = 2 \quad and \quad X = \mathbb{P}^1_E \quad and \quad \Gamma(\mathscr{O}_A) = E \times E \quad and \quad \Gamma(\mathscr{O}_B) = L' \times L'',$$

where $L', L'' \subset E$ are subfields with $[E : L'] = [E : L''] = 2$.

Proof. Since the E-algebra $\Gamma(\mathscr{O}_A)$ has degree two, it must be isomorphic to $E \times E$. The genus-zero curve X over E contains an E-valued point, and is thus isomorphic to $\mathbb{P}^1_E$. Obviously $\Gamma(\mathscr{O}_B) = L' \times L''$. The inclusion $L' \subset E$ is strict. Using that Y is Gorenstein we infer $p^{r-s} = [E : L'] = 2$. $\qquad\square$

It remains the case that both B and A are integral. We shall reduce this to the preceding three cases. The field $\Gamma(\mathscr{O}_A) = \kappa(a)$ contains both $\Gamma(X, \mathscr{O}_X) = E$ and $\Gamma(\mathscr{O}_B) = \kappa(b)$. More precisely, E has degree two over both subfields $\kappa(a)$ and $\kappa(b)$, is purely inseparable over the latter, and $\Gamma(\mathscr{O}_Y) = F$ is the intersection of the two.

Proposition 10.5. *In the above situation, we have $p = 2$, and there is a quadratic extension $F \subset F'$ with the following properties:*

(i) *The base-changes $X \otimes F'$ and $Y \otimes F'$ remain integral.*
(ii) *The field of global sections $E' = \Gamma(\mathscr{O}_{X'})$ for the normalization X' of $X \otimes F'$ has height-one over F'.*
(iii) *The ramification scheme A' for the normalization map $X' \to Y \otimes F'$ is non-integral.*

Proof. The characteristic must be two, because $\kappa(b) \subset \kappa(a)$ has degree two and is purely inseparable. The inclusion of sets $E \cup \kappa(b) \subset \kappa(a)$ must be strict ([6], Chapter V, §7, No. 4, Lemma 1), and we choose some $\zeta \in \kappa(a)$ from the complement. This is a generator, over both $\kappa(a)$ and $\kappa(b)$, and we have $\zeta^2 \in \kappa(b)$. If $E \subset \kappa(a)$ is purely inseparable, ζ^2 is contained in E, and thus in $F = E \cap \kappa(b)$, and we conclude that $\xi = \zeta$ has degree two over F. If $E \subset \kappa(a)$ is separable, we have $\zeta^2 \notin E$, and there is an equation $\zeta^2 + \zeta + \lambda = 0$ for some $\lambda \in E$. Squaring the equation gives $\zeta^4 + \zeta^2 + \mu = 0$, where $\mu = \lambda^2$ belongs to F. Now $\xi = \zeta^2$ has degree two over F. In both cases we have a quadratic extension $F' = F(\xi)$ such that the minimal polynomial of ξ does not split over E, and thus $E(\xi) = F' \otimes E$ stays a field.

The base-change $X \otimes_F F' = X \otimes_E E(\xi)$ remains integral, according to [32], Lemma 1.3, and the same then follows for the birational curve $Y \otimes_F F'$, which establishes (i). Assertion (ii) is a consequence of loc. cit., Proposition 1.4. It remains to verify (iii). By construction, the generator ξ belongs to $\kappa(a) = \Gamma(\mathscr{O}_A)$, so the base-change

$A \otimes F'$ ceases to be integral. We are done if $X \otimes F'$ remains normal, so let us assume this is not the case. Then $F \subset F'$ is not separable, hence a height-one extension, and X is a twisted form of the double line. According to Theorem 9.2, the base-change $X \otimes F' = X \otimes_E E(\xi)$ acquires a singularity, and its normalization X' is a projective line over some E' containing ξ and being contained in $E^{1/p}$. The ramification scheme for $X' \to X \otimes F'$ is the schematic fiber over the singular point in $X \otimes F'$, and is non-reduced. It follows that the ramification locus for $X' \to Y \otimes F'$ is non-reduced as well.   □

11. Proof of the main results

This section contains the proofs for our main results, which where already stated in Section 2. Let F be a ground field of characteristic $p \geq 0$.

Proof of Theorem 2.3. Suppose that that Y is a genus-one curve that is regular but not geometric regular. Write $r = \operatorname{edim}(\mathscr{O}_{Y,\eta}/F)$ for its geometric generic embedding dimension. The task is to establish assertions (i)–(iii) from the enunciation of Theorem 2.3. First note that the ground field F must be imperfect ([17], Chapter 0, Theorem 22.5.8), and therefore $p > 0$. According to [32], Proposition 1.5 there is a subextension $F' \subset F^{1/p}$ such that the base-change $Y \otimes F'$ remains integral, and that its normalization X' has $\Gamma(X', \mathscr{O}_{X'}) = F^{1/p}$. The scheme X' is a regular genus-zero curve over $F^{1/p}$, according to Proposition 10.1. Write $B' \subset Y \otimes F'$ and $A' \subset X'$ for the branch scheme and the ramification scheme for the normalization map $X' \to Y \otimes F'$. the latter is an effective Cartier divisor, and its coordinate ring has degree two over the field $F^{1/p}$, again by Proposition 10.1. The height-one extension $F' \subset F^{1/p}$ is finite, and we set $[F^{1/p} : F'] = p^r$.

Suppose for the moment that the ramification scheme A' is non-integral. Then $X' = \mathbb{P}^1_{F^{1/p}}$ in light of the results from Section 10. More precisely, we have three possible cases, and the ensuing situations are as follows: If both A' and B' are non-reduced, then $p \leq 3$, according to Proposition 10.2. Moreover, in the description of the coordinate ring $\Gamma(\mathscr{O}_{B'}) = L' + H'\epsilon$ as a subring of $\Gamma(\mathscr{O}_{A'}) = F^{1/p}[\epsilon]$ we have $p^{r-s} \leq 4$, where $[L' : F'] = p^s$. It follows that $Y \otimes F'$ is a twisted form of the standard model $C^{(i)}_{r,\Lambda,F'}$ with $0 \leq i \leq 2$ in characteristic two, and $0 \leq i \leq 1$ in characteristic three. If A' is non-reduced and B' is reduced, then again $p \leq 3$, and $Y \otimes F'$ is a twisted form of $C^{(i)}_{r,\Lambda,F'}$ with $i = 0$, now by Proposition 10.3. If A' is disconnected, then Proposition 10.4 tells us that $p = 2$, and that $Y \otimes F'$ is a twisted form of some $C^{(1,1)}_{r,F',\Lambda}$.

Now suppose that A' and B' are integral. Proposition 10.5 then tells us that $p = 2$, and that we find some subextension $F'' \subset F^{1/p^2}$ containing F' such that $Y \otimes F''$ remains integral, and has a normalization whose ramification scheme becomes non-integral. In turn, the preceding paragraph applies with F'' instead of F'.   □

Proof of Theorem 2.4. Let $r \geq 0$ be an integer, and suppose that the characteristic and p-degree of the ground field F, together with thy symbol i are as in the table of the enunciation of the theorem. The task is to find a twisted form of $C = C^{(i)}_{r,\Lambda,F}$ that is regular, for some suitable Λ.

If $p = 2$ and $\text{pdeg}(F) \geq r + 2$ and $i = (1,1)$ then the desired twisted form exists by Proposition 4.3. Suppose now $p = 2$ and $\text{pdeg}(F) \geq r + 1$ and $0 \leq i \leq 2$. We then apply Proposition 7.3. Note that the case $i = 0$ is already clear by the theory of quasi-elliptic curves, and $i = 1$ is also a consequence of Proposition 5.4. For $p = 3$ and $\text{pdeg}(F) \geq r + 1$ and $0 \leq i \leq 1$ we again apply Proposition 7.3. $\qquad\square$

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