Mathematisches Institut
Heinrich-Heine-Universität Düsseldorf
Prof. Dr. Stefan Schröer

WS 22/23

Algebraic Surfaces

Sheet 1

Exercise 1. Let X be a normal integral surface, and

$$f : X \longrightarrow C$$

be a morphism to a curve, with $\mathscr{O}_C = f_*(\mathscr{O}_X)$. Show that the map f is surjective, that the curve C is integral and regular, that every closed fiber $D = f^{-1}(c)$ is an effective Cartier divisor, and that the resulting invertible sheaf $\mathscr{L} = \mathscr{O}_X(D)$ is numerically trivial on D. Visualize the situation.

Exercise 2. Let (E, Φ) be a lattice, $E' \subset E$ a subgroup of finite index, and

$$\Phi' : E' \times E' \longrightarrow \mathbb{Z}, \quad (x, y) \longmapsto \Phi(x, y)$$

the induced symmetric bilinear form. Relate the rank, discriminant, signature and type of the two lattices (E, Φ) and (E', Φ'). Consider explicit examples.

Deadline: Monday, October 24.

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Algebraic Surfaces

Sheet 2

Exercise 1. Let X be a projective normal integral surface with Picard number $\rho \leq 1$. Show that there is no surjective morphism $f : X \to C$ to a curve C, by examining intersection numbers for ample sheaves and closed fibers.

Exercise 2. Let X be a surface, $D \subset X$ an effective Cartier divisor, and $\mathscr{L} = \mathscr{O}_X(D)$ the resulting invertible sheaf. Assume that $\mathscr{L}|D$ is globally generated, and that $H^1(X, \mathscr{O}_X) = 0$. Deduce that $\mathscr{L}$ is globally generated. Also give an example to show that the conlusion fails without the assumption on cohomology.

Deadline: Monday, November 14.

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Sheet 3

Exercise 1. Let $f : X \to \mathbb{P}^1$ be a Hirzebruch surfaces with invariant $e > 0$, and

$$\omega_{X/\mathbb{P}^1} = \omega_X \otimes f^*(\omega_{\mathbb{P}^1})^{\otimes -1}$$

be the *relative dualizing sheaf*. Show that the class $-K_{X/\mathbb{P}^1}$ in the numerical group $\mathrm{Num}(X)$ can be written as the sum of two disjoint sections $D, E \subset X$.

Exercise 2. Let C be a curve, and $\mathscr{E}$ be a locally free sheaf of rank $r = 2$, and

$$X = \mathbb{P}(\mathscr{E}) = \mathrm{Proj}(\mathrm{Sym}^\bullet \mathscr{E})$$

the resulting surface. Show that the structure morphism $f : X \to C$ admits two disjoint sections $D, D' \subset X$ if and only if there is a decomposition $\mathscr{E} \simeq \mathscr{L} \oplus \mathscr{L}'$.

Deadline: Monday, November 28.

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Algebraic Surfaces

Sheet 4

Exercise 1. Compute the blowing-up $X = \mathrm{Bl}_Z(\mathbb{A}^2)$ of the affine plane $\mathbb{A}^2 = \operatorname{Spec} k[x, y]$ with respect to the center Z defined via the ideal $\mathfrak{a} = (x, y^2)$, by looking at the x-chart and the y^2-chart. Show that the X contains a unique singularity $x \in X$, by considering the partial derivatives on the the charts. Also determine the exceptional divisor $E = f^{-1}(Z)$.

Exercise 2. Let $X = \mathbb{P}(\mathscr{E})$ be the Hirzebruch surface with invariant $e \geq 1$, and $E \subset X$ be the section with $E^2 = -e$. Show in detail that there is a contraction $f : X \to Y$ of the curve E, arguing as in the lecture course.

Deadline: Monday, December 12.

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Algebraic Surfaces

Sheet 5

Exercise 1. Let X be a scheme, and $Z \subset X$ be a closed subscheme such that the corresponding quasicoherent sheaf of ideals $\mathscr{I} \subset \mathscr{O}_X$ has the property $\mathscr{I}^2 = 0$. Verify that X and Z have the same underlying topological space, that the canonical map $\psi : \mathscr{O}_X^\times \to \mathscr{O}_Z^\times$ is surjective, that the map $\varphi : \mathscr{I} \to \mathscr{O}_X^\times$ given by $s \mapsto 1+s$ is a homomorphism of abelian sheaves, and that the resulting sequence

$$0 \longrightarrow \mathscr{I} \stackrel{\varphi}{\longrightarrow} \mathscr{O}_X^\times \stackrel{\psi}{\longrightarrow} \mathscr{O}_Z^\times \longrightarrow 1$$

is exact.

Exercise 2. Consider the ring $R = k[x,y]$ and the ideal $\mathfrak{a} = (x, y^n)$, for some $n \geq 2$. Compute the two homogeneous localizations

$$R[\mathfrak{a}T]_{(xT)} \quad \text{and} \quad R[\mathfrak{a}T]_{(y^nT)}$$

of the Rees ring $R[\mathfrak{a}T]$, and translate this into geometric statements on the blowing-up

$$f : X = \mathrm{Bl}_\mathfrak{a}(R) \longrightarrow \mathrm{Spec}(R) = Y$$

and the exceptional divisor $E \subset X$.

Deadline: Monday, January 9.

Marry Christmas and Happy New Year!

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Algebraic Surfaces

Sheet 6

Exercise 1. Let X be a regular surface whose dualizing sheaf ω_X is numerically trivial. Let $E, F \subset X$ be integral curves with $E^2 < 0$ and $F^2 = 0$. What are the possible values for $h^i(\mathscr{O}_E)$ and $h^i(\mathscr{O}_F)$?

Exercise 2. Let $X \subset \mathbb{P}^4$ be a regular surface that is the intersection of two hypersurfaces $H_i \subset \mathbb{P}^4$ of degree $d_i \geq 1$. For which values of $d_i = \deg(H_i)$ is X of general type? For which values is ω_X numerically trivial?

Deadline: Monday, January 23.

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Algebraic Surfaces

Sheet 7

Exercise 1. Suppose the ground field k is algebraically closed. Let X be a regular surface whose minimal model Y is a K3 surface or an abelian surface. Use the Canonical Bundle Formula for blowing-ups to show that $h^2(\mathscr{O}_X) = h^0(\omega_X) = 1$. What can be said about the zero locus of a global section $s \neq 0$ for the dualizing sheaf ω_X?

Exercise 2. Suppose the ground field k is algebraically closed. Let X be regular surface of general type that is minimal, $P = P(X, \omega_X)$ its canonical model, and $E \subset X$ be an integral curve contracted by the canonical morphism $f : X \to P$. Use the Adjunction Formula to show that E is a (-2)-*curve*, that is,

$$E = \mathbb{P}^1 \quad \text{and} \quad E^2 = -2.$$

Deadline: Monday, January 30.