A Hirzebruch proportionality principle in Arakelov geometry

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December 18, 2001

Abstract

We describe a tautological subring in the arithmetic Chow ring of bases of abelian schemes of relative dimension d. Among the results are an Arakelov version of the Hirzebruch proportionality principle and a purely analytical formula for the global height of complete bases of dimension $d(d-1)/2$.

2000 Mathematics Subject Classification: 14G40, 58J52, 20G05, 20G10, 14M17

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1 Introduction

The purpose of this note is to exploit some implications of a fixed point formula in Arakelov geometry when applied to the action of the (-1) involution on abelian schemes of relative dimension d. It is shown that the fixed point formula's statement in this case is equivalent to giving the values of arithmetic Pontrjagin classes of the Hodge bundle $\overline{E} := (R^1\pi_*\mathcal{O}, \|\cdot\|_{L^2})^*$, where these Pontrjagin classes are defined as polynomials in the arithmetic Chern classes defined by Gillet and Soulé. The resulting formula is (see Theorem 3.4)

$$\widehat{p}_k(\overline{E}) = (-1)^k \left(\frac{2\zeta'(1-2k)}{\zeta(1-2k)} + \sum_{j=1}^{2k-1} \frac{1}{j} - \frac{2\log 2}{1-4^{-k}} \right) (2k-1)!\, a(\mathrm{ch}(E)^{[2k-1]}) \ . \tag{1}$$

When combined with the statement of the Gillet-Soulé's non-equivariant arithmetic Grothendieck-Riemann-Roch formula ([GS8],[F]), one obtains a formula for the class $\hat{c}_1^{1+d(d-1)/2}$ of the Hodge bundle in terms of topological classes and a certain special differential form γ (Theorem 5.1). This can be regarded as a formula for the height of complete cycles of codimension d in the moduli space. Finally we derive an Arakelov version of the Hirzebruch proportionality principle, namely a ring homomorphism from the Arakelov Chow ring of Lagrangian Grassmannians to the arithmetic Chow ring of bases of abelian schemes (Theorem 5.4).

A fixed point formula for maps from arithmetic varieties to Spec D has been proven by Roessler and the author in [KR1], where D is a regular arithmetic ring. In [KR2, Appendix] we described a conjectural generalization to flat equivariantly projective maps between arithmetic varieties over D. The missing ingredient to the proof of this conjecture was the equivariant version of Bismut's formula for the behavior of analytic torsion forms under the composition of immersions and fibrations [B4], i.e. a merge of [B3] and [B4]. This formula has meanwhile been shown by Bismut and Ma [BM].

We work only with regular schemes as bases; extending these results to moduli stacks and their compactifications remains an open problem, as Arakelov geometry for such situations is not yet developed. In particular one could search an analogue of the full Hirzebruch-Mumford proportionality principle in Arakelov geometry. When this article was almost finished, we learned about recent related work by van der Geer concerning the classical Chow ring of the moduli stack of abelian varieties and its compactifications [G]. The approach there to determine the tautological subring uses the non-equivariant Grothendieck-Riemann-Roch Theorem applied to line bundles associated to theta divisors. Thus it might be possible to avoid the use of the fixed point formula in our situation by mimicking this method, possibly by extending the methods of Yoshikawa [Y]; but computing the occurring objects related to the theta divisor is presumably not easy.

According to a conjecture by Oort, there are no complete subvarieties of codimension d in the complex moduli space for $d \geq 3$. Thus a possible application of our formula for $\hat{c}_1^{1+d(d-1)/2}$ of the Hodge bundle could be a proof of this conjecture by showing that the height of potential subvarieties would be lower than the known lower bounds for heights. Van der Geer [G, Cor. 7.2] used the degree with respect to the Hodge bundle to show that complete subvarieties have codimension $\geq d$.

Alas with the methods used in this article we get the value of the height only up to rational multiples of $\log 2$, which of course makes estimates impossible. This is caused by the automorphism of order 2 we work with, which prevents us from considering flat fixed point subschemes with fibers over 2. We believe that these rational multiples can be shown to vanish.

Results extending some parts of this article (in particular not involving the $\log 2$ ambiguity) have been announced in [MR].

Acknowledgements: I thank A. Johan de Jong, Damian Roessler, Christophe Soulé, Harry Tamvakis, Emanuel Ullmo and Thorsten Wedhorn for helpful discussions. Also I thank the Deutsche Forschungsgemeinschaft which supported me with a Heisenberg fellowship during the preparation of this article.

2 Torsion forms

Let $\pi : E^{1,0} \to B$ denote a d-dimensional holomorphic vector bundle over a complex manifold. Let Λ be a lattice subbundle of the underlying real vector bundle $E^{1,0}_{\mathbf{R}}$ of rank $2d$. Thus the quotient bundle $M := E^{1,0}/\Lambda \to B$ is a holomorphic fibration by tori Z. Let

$$\Lambda^* := \{\mu \in (E^{1,0}_{\mathbf{R}})^* \,|\, \mu(\lambda) \in 2\pi\mathbf{Z}\,\forall\lambda \in \Lambda\}$$

denote the dual lattice bundle. Assume that $E^{1,0}$ is equipped with an Hermitian metric such that the volume of the fibers is constant. Any polarization induces such a metric.

Let N_V be the number operator acting on $\Gamma(Z, \Lambda^q T^{*0,1}Z)$ by multiplication with q. Let Tr_s denote the supertrace with respect to the $\mathbf{Z}/2\mathbf{Z}$-grading on $\Lambda T^*B \otimes \mathrm{End}(\Lambda T^{*0,1}Z)$. Let ϕ denote the map acting on $\Lambda^{2p}T^*B$ as multiplication by $(2\pi i)^{-p}$. We write $\widetilde{\mathfrak{A}}(B)$ for $\widetilde{\mathfrak{A}}(B) := \bigoplus_{p\geq 0}(\mathfrak{A}^{p,p}(B)/(\mathrm{Im}\,\partial + \mathrm{Im}\,\overline{\partial}))$, where $\mathfrak{A}^{p,p}(B)$ denotes the C^∞ differential forms of type (p,p) on B.

We shall denote a vector bundle F together with an Hermitian metric h by $\overline{F}$. Then $\mathrm{ch}_g(\overline{F})$ shall denote the Chern-Weil representative of the equivariant Chern character associated to the restriction of (F,h) to the fixed point subvariety. Recall also that $\mathrm{Td}_g(\overline{F})$ is the differential form

$$\frac{c_{\mathrm{top}}(\overline{F}^g)}{\sum_{i\geq 0}(-1)^k \mathrm{ch}_g(\Lambda^k \overline{F})}\ .$$

In [K, Section 3], a superconnection A_t acting on the infinite-dimensional vector bundle $\Gamma(Z, \Lambda T^{*0,1}Z)$ over B had been introduced, depending on $t \in \mathbf{R}^+$. For a fibrewise acting holomorphic isometry g the limit $\lim_{t\to\infty} \phi \mathrm{Tr}_s g^* N_H e^{-A_t^2} =: \omega_\infty$ exists and is given by the respective trace restricted to the cohomology of the fibers. The equivariant analytic torsion form $T_g(\pi, \mathcal{O}_M) \in \widetilde{\mathfrak{A}}(B)$ was defined there as the derivative at zero of the zeta function with values in differential forms on B given by

$$-\frac{1}{\Gamma(s)}\int_0^\infty (\phi \mathrm{Tr}_s g^* N_H e^{-A_t^2} - \omega_\infty) t^{s-1}\, dt$$

for Re $s > d$ (for more general fibrations π, one has to be more careful with the convergence of this integral). In the case of a Kähler fibration by tori (e.g. a fibration by polarized abelian varieties), $T_g(\pi, \mathcal{O}_M)$ coincides with the equivariant torsion forms investigated by Xiaonan Ma [Ma] for general Kähler fibrations. This extends the definition of torsion forms [BK] to an equivariant situation.

Theorem 2.1 *Let an isometry g act fibrewise with isolated fixed points on the fibration by tori $\pi : M \to B$. Then the equivariant torsion form $T_g(\pi, \mathcal{O}_M)$ vanishes.*

Proof: Let $f_\mu : M \to \mathbf{C}$ denote the function $e^{i\mu}$ for $\mu \in \Lambda^*$. As is shown in [K, §5] the operator A_t^2 acts diagonally with respect to the Hilbert space decomposition

$$\Gamma(Z, \Lambda T^{*0,1}Z) = \bigoplus_{\mu\in\Lambda^*} \Lambda E^{*0,1} \otimes \{f_\mu\}\ .$$

As in [KR4, Lemma 4.1] the induced action by g maps a function f_μ to a multiple of itself if only if $\mu = 0$ because g acts fixed point free on $E^{1,0}$ outside the zero section. In that case, f_μ represent an element in the cohomology. Thus the zeta function defining the torsion vanishes. **Q.E.D.**

Remark: As in [KR4, Lemma 4.1], the same proof shows the vanishing of the equivariant torsion form $T_g(\pi, \overline{\mathcal{L}})$ for coefficients in a g-equivariant line bundle $\overline{\mathcal{L}}$ with vanishing first Chern class.

We shall also need the following result of [K] for the non-equivariant torsion form $T(\pi, \mathcal{O}_M) := T_{\mathrm{id}}(\pi, \mathcal{O}_M)$: Assume for simplicity that π is Kähler. Consider for Re $s < 0$ the zeta function with values in $(d-1, d-1)$-forms on B

$$Z(s) := \frac{\Gamma(2d-s-1)\mathrm{vol}(M)}{\Gamma(s)(d-1)!} \sum_{\lambda \in \Lambda \setminus \{0\}} (\frac{\overline{\partial}\partial}{4\pi i} \|\lambda^{1,0}\|^2)^{\wedge(d-1)} (\|\lambda^{1,0}\|^2)^{s+1-2d}$$

where $\lambda^{1,0}$ denotes a lattice section in $E^{1,0}$ (in [K], the volume is equal to 1). Then the limit $\gamma := \lim_{s \to 0^-} Z'(0)$ exists and it transgresses the Chern-Weil form $c_d(\overline{E^{0,1}})$ representing the Euler class $c_d(E^{0,1})$

$$\frac{\overline{\partial}\partial}{2\pi i}\gamma = c_d(\overline{E^{0,1}}) \ .$$

In [K, Th. 4.1] the torsion form is shown to equal

$$T(\pi, \mathcal{O}_M) = \frac{\gamma}{\mathrm{Td}(\overline{E^{0,1}})}$$

in $\widetilde{\mathfrak{A}}(B)$. The differential form γ was intensively studied in [K].

3 Abelian schemes and the fixed point formula

We shall use the Arakelov geometric concepts and notation of [SABK] and [KR1]. In this article we shall only give a brief introduction to Arakelov geometry, and we refer to [SABK] for details. Let D be a regular arithmetic ring, i.e. a regular, excellent, Noetherian integral ring, together with a finite set $\mathcal{S}$ of ring monomorphism of $D \to \mathbf{C}$, invariant under complex conjugation. We shall denote by μ_n the diagonalizable group scheme over D associated to $\mathbf{Z}/n\mathbf{Z}$. We choose once and for all a primitive n-th root of unity $\zeta_n \in \mathbf{C}$. Let $f : Y \to \mathrm{Spec}\ D$ be an equivariant arithmetic variety, i.e. a regular integral scheme, endowed with a μ_n-projective action over Spec D. The groups of n-th roots of unity acts on $Y(\mathbf{C})$ by holomorphic automorphisms and we shall write g for the automorphism corresponding to ζ_n.

We write f^{μ_n} for the map $Y_{\mu_n} \to \mathrm{Spec}\ D$ induced by f on the fixed point subvariety. Complex conjugation induces an antiholomorphic automorphism of $Y(\mathbf{C})$ and $Y_{\mu_n,\mathbf{C}}$, both of which we denote by F_∞. The space $\widetilde{\mathfrak{A}}(Y)$ is the subspace of $\widetilde{\mathfrak{A}}(Y(\mathbf{C}))$ of classes of differential forms ω such that $F_\infty^* \omega = (-1)^p \omega$.

Let $\widehat{\mathrm{CH}}^*(Y)$ denote the Gillet-Soulé arithmetic Chow ring, consisting of arithmetic cycles and suitable Green currents on $Y(\mathbf{C})$. Let $\mathrm{CH}^*(Y)$ denote the classical Chow ring. Then there is an exact sequence in any degree p

$$\mathrm{CH}^{p,p-1}(Y) \xrightarrow{\rho} \widetilde{\mathfrak{A}}^{p-1,p-1}(Y) \xrightarrow{a} \widehat{\mathrm{CH}}^p(Y) \xrightarrow{\zeta} \mathrm{CH}^p(Y) \to 0 \tag{2}$$

with ρ being a Beilinson regulator map. For Hermitian vector bundles $\overline{E}$ on Y Gillet and Soulé defined arithmetic Chern classes $\widehat{c}_p(\overline{E}) \in \widehat{\mathrm{CH}}^*(Y)_{\mathbf{Q}}$.

By "product of Chern classes", we shall understand in this article any product of at least two equal or non-equal Chern classes of degree larger than 0 of a given vector bundle.

Lemma 3.1 *Let*

$$\widehat{\phi} = \sum_{j=0}^{\infty} a_j \widehat{c}_j + \text{products of Chern classes}$$

denote an arithmetic characteristic class with $a_j \in \mathbf{Q}$, $a_j \neq 0$ for $j > 0$. Assume that for a vector bundle $\overline{F}$ on an arithmetic variety Y, $\widehat{\phi}(\overline{F}) = m + a(\beta)$ where β is a differential form on $Y(\mathbf{C})$ with $\partial\overline{\partial}\beta = 0$ and $m \in \widehat{\mathrm{CH}}^0(Y)_{\mathbf{Q}}$. Then

$$\sum_{j=0}^{\infty} a_j \widehat{c}_j(\overline{F}) = m + a(\beta) \ .$$

Proof: By induction: for the term in $\widehat{\mathrm{CH}}^0(Y)_{\mathbf{Q}}$, the formula is clear. Assume now for $k \in \mathbf{N}_0$ that

$$\sum_{j=0}^{k} a_j \widehat{c}_j(\overline{F}) = m + \sum_{j=0}^{k} a(\beta)^{[j]} \ .$$

Then $\widehat{c}_j(\overline{F}) \in a(\ker \partial\overline{\partial})$ for $1 \leq j \leq k$, thus products of these $\widehat{c}_j$'s vanish by [SABK, Remark III.2.3.1]. Thus the term of degree $k+1$ of $\widehat{\phi}(\overline{F})$ equals $a_{k+1}\widehat{c}_{k+1}(\overline{F})$. **Q.E.D.**

We define **arithmetic Pontrjagin classes** $\widehat{p}_j \in \widehat{\mathrm{CH}}^{2j}$ of arithmetic vector bundles by the relation

$$\sum_{j=0}^{\infty} (-z^2)^j \widehat{p}_j := (\sum_{j=0}^{\infty} z^j \widehat{c}_j)(\sum_{j=0}^{\infty} (-z)^j \widehat{c}_j) \ .$$

Thus,

$$\widehat{p}_j(\overline{F}) = (-1)^j \widehat{c}_{2j}(\overline{F} \oplus \overline{F}^*) = \widehat{c}_j^2(\overline{F}) + 2\sum_{l=1}^{j} (-1)^l \widehat{c}_{j+l}(\overline{F})\widehat{c}_{j-l}(\overline{F})$$

for an arithmetic vector bundle $\overline{F}$ (compare [MiS, §15]). Similarly to the construction of Chern classes via the elementary symmetric polynomials, the Pontrjagin classes can be constructed using the elementary symmetric polynomials in the squares of the variables. Thus many formulae for Chern classes have an easily deduced analogue for Pontrjagin classes. In particular, Lemma 3.1 holds with Chern classes replaced by Pontrjagin classes.

Now let Y, B be μ_N-equivariant arithmetic varieties over some fixed arithmetic ring D and let $\pi : Y \to B$ be a map over D, which is flat, μ_N-projective and smooth over the complex numbers. Fix an $\mu_N(\mathbf{C})$-invariant Kähler metric on $Y(\mathbf{C})$. We recall [KR1, Definition 4.1] extending the definition of Gillet-Soulé's arithmetic K_0-theory to the equivariant setting: The arithmetic equivariant Grothendieck group $\widehat{K}^{\mu_n}(Y)$ of Y is the sum of the abelian group $\widetilde{\mathfrak{A}}(Y_{\mu_n})$ and the free abelian group generated by the equivariant isometry classes of Hermitian vector bundles, together with the following relations: For every short exact sequence $\overline{\mathcal{E}} : 0 \to E' \to E \to E'' \to 0$ and any equivariant metrics on E, E', E'', $\widetilde{\mathrm{ch}}_g(\overline{\mathcal{E}}) = \overline{E}' - \overline{E} + \overline{E}''$ in $\widehat{K}^{\mu_n}(Y)$. $\widehat{K}^{\mu_n}(Y)$ has a natural ring structure. We denote the canonical map $\widetilde{\mathfrak{A}}(Y_{\mu_n}) \to \widehat{K}^{\mu_n}(Y)$ by a; the canonical trivial Hermitian line bundle $\overline{\mathcal{O}}$ shall often be denoted by 1.

If $\overline{E}$ is a π-acyclic (meaning that $R^k\pi_*E = 0$ if $k > 0$) μ_N-equivariant Hermitian bundle on Y, let $\pi_*\overline{E}$ be the direct image sheaf (which is locally free), endowed with its natural equivariant structure and L_2-metric. Consider the rule which associates the element $\pi_*\overline{E} - T_g(f, \overline{E})$ of $\widehat{K}_0^{\mu_N}(B)$ to every f-acyclic equivariant Hermitian bundle $\overline{E}$ and the element $\int_{Y(\mathbf{C})_g/B(\mathbf{C})_g} \mathrm{Td}_g(\overline{Tf})\eta \in \widetilde{\mathfrak{A}}(B_{\mu_N})$ to every $\eta \in \widetilde{\mathfrak{A}}(Y_{\mu_N})$. This rule induces a group homomorphism $\pi_! : \widehat{K}_0^{\mu_N}(Y) \to \widehat{K}_0^{\mu_N}(B)$ ([KR2, Prop. 3.1]).

Let $\mathcal{R}$ be a ring as appearing in the statement of [KR1, Th. 4.4] (in the cases considered in this paper, we can choose $\mathcal{R} = D[1/2]$). The following result was stated as a conjecture in [KR2, Conj. 3.2].

Theorem 3.2 *Set*

$$\mathrm{td}(\pi) := \frac{\lambda_{-1}(\pi^*\overline{N}^*_{B/B_{\mu_n}})}{\lambda_{-1}(\overline{N}^*_{Y/Y_{\mu_n}})}(1 - a(R_g(N_{Y/Y_{\mu_n}})) + a(R_g(\pi^* N_{B/B_{\mu_N}}))).$$

Then the following diagram commutes

$$\begin{array}{ccc} \widehat{K}_0^{\mu_n}(Y) & \stackrel{\mathrm{td}(\pi)\rho}{\longrightarrow} & \widehat{K}_0^{\mu_n}(Y_{\mu_n}) \otimes_{R(\mu_n)} \mathcal{R} \\ \downarrow\ \pi_! & & \downarrow\ \pi_!^{\mu_n} \\ \widehat{K}_0^{\mu_n}(B) & \stackrel{\rho}{\longrightarrow} & \widehat{K}_0^{\mu_n}(B_{\mu_N}) \otimes_{R(\mu_n)} \mathcal{R} \end{array}$$

where ρ denotes the restriction to the fixed point subscheme.

Sketch of proof: As explained in [KR2, conjecture 3.2] the proof of the main statement of [KR1] was already written with this general result in mind and it holds without any major change for this situation, when using the generalization of Bismut's equivariant immersion formula for the holomorphic torsion ([KR1, Th. 3.11]) to torsion forms. The latter has now been established by Bismut and Ma [BM]. The proof in [KR1] holds when using [BM] instead of [KR1, Th. 3.11] and [KR2, Prop. 3.1] instead of [KR1, Prop. 4.3].

Also one has to replace in sections 5, 6.1 and 6.3 the integrals over Y_g, X_g etc. by integrals over Y_g/B_g, X_g/B_g, while replacing the maps occurring there by corresponding relative versions. As direct images can occur as coherent sheaves, one has to consider at some steps suitable resolutions of vector bundles such that the higher direct images of the vector bundles in this resolution are locally free as e.g. on [F, p. 74]. **Q.E.D.**

Let $f : B \to \mathrm{Spec}\, D$ denote a quasi-projective arithmetic variety and let $\pi : Y \to B$ denote a principally polarized abelian scheme of relative dimension d. For simplicity, we assume that the volume of the fibers over $\mathbf{C}$ is scaled to equal 1; it would be 2^d for the metric induced from the polarization. We shall explain the effect of rescaling the metric later (after Theorem 5.1). Set $\overline{E} := (R^1\pi_*\mathcal{O}, \|\cdot\|_{L^2})^*$. This bundle $E = \mathbf{Lie}(Y/B)^*$ is the Hodge bundle. Then by [BBM, Prop. 2.5.2], the full direct image of $\mathcal{O}$ under π is given by

$$\overline{R^\bullet\pi_*\mathcal{O}} = \Lambda^\bullet\overline{E}^* \tag{3}$$

and the relative tangent bundle is given by

$$\overline{T\pi} = \pi^*\overline{E}^* \ . \tag{4}$$

See also [FC, Th. VI.1.1], where these properties are extended to toroidal compactifications. Both isomorphisms are no longer isometries if the volume is not 1. For an action of $G = \mu_N$ on Y Theorem 3.2 combined with the arithmetic Grothendieck-Riemann-Roch Theorem in all degrees for π^G states (analogue to [KR1, section 7.4])

Theorem 3.3

$$\widehat{\mathrm{ch}}_G(\overline{R^\bullet\pi_*\mathcal{O}}) - a(T_g(\pi_{\mathbf{C}}, \overline{\mathcal{O}})) = \pi_*^G(\widehat{\mathrm{Td}}_G(\overline{T\pi})(1 - a(R_g(T\pi_{\mathbf{C}}))))$$

where R_g denotes Bismut's equivariant R-class. We shall mainly consider the case where π^G is actually be a smooth covering, Riemannian over $\mathbf{C}$; thus the statement of the arithmetic Grothendieck-Riemann-Roch is in fact very simple in this case. We obtain the equation

$$\widehat{\mathrm{ch}}_G(\Lambda^\bullet\overline{E}^*) - a(T_g(\pi_{\mathbf{C}}, \overline{\mathcal{O}})) = \pi_*^G(\widehat{\mathrm{Td}}_G(\pi^*\overline{E}^*)(1 - a(R_g(\pi^*E^*_{\mathbf{C}})))) \ . \tag{5}$$

Using the equation

$$\widehat{\mathrm{ch}}_G(\Lambda^\bullet \overline{E}^*) = \frac{\widehat{c}_{\mathrm{top}}(\overline{E}^G)}{\widehat{\mathrm{Td}}_G(\overline{E})}$$

(5) simplifies to

$$\frac{\widehat{c}_{\mathrm{top}}(\overline{E}^G)}{\widehat{\mathrm{Td}}_G(\overline{E})} - a(T_g(\pi_{\mathbf{C}}, \overline{\mathcal{O}})) = \widehat{\mathrm{Td}}_G(\overline{E}^*)(1 - a(R_g(E^*_{\mathbf{C}})))\pi_*^G \pi^* 1$$

or, using that $a(\ker \bar{\partial}\partial)$ is an ideal of square zero,

$$\widehat{c}_{\mathrm{top}}(\overline{E}^G)(1 + a(R_g(E^*_{\mathbf{C}}))) - a(T_g(\pi_{\mathbf{C}}, \overline{\mathcal{O}})\mathrm{Td}_g(\overline{E}_{\mathbf{C}})) = \widehat{\mathrm{Td}}_G(\overline{E})\widehat{\mathrm{Td}}_G(\overline{E}^*)\pi_*^G \pi^* 1 \ . \tag{6}$$

Remark: 1) If G acts fibrewise with isolated fixed points (over $\mathbf{C}$), by Theorem 2.1 the left hand side of equation (6) is an element of $\widehat{\mathrm{CH}}^0(B)_{\mathbf{Q}(\zeta_n)} + a(\ker \partial\overline{\partial})$. Set for an equivariant bundle F

$$\widehat{A}_g(F) := \mathrm{Td}_g(F) \exp(-\frac{c_1(F) + \mathrm{ch}_g(F)^{[0]}}{2}) \ ; \tag{7}$$

thus $\widehat{A}_g(F^*) = (-1)^{\mathrm{rk}(F/F^G)} \widehat{A}_g(F)$. For isolated fixed points, by comparing the components in degree 0 in equation (6) one obtains

$$\pi_*^G \pi^* 1 = (-1)^d (\widehat{A}_g(E)^{[0]})^{-2}$$

and thus by Theorem 2.1

$$1 + a(R_g(E^*_{\mathbf{C}})) = \left(\frac{\widehat{\widehat{A}_G}(\overline{E})}{\widehat{A}_g(E)^{[0]}} \right)^2 \ . \tag{8}$$

(compare [KR4, Prop. 5.1]). Both sides can be regarded as products over the occurring eigenvalues of g of characteristic classes of the corresponding bundles E_ζ. One can wonder whether the equality holds for the single factors, similar to [KR4]. Related work is announced by Maillot and Roessler in [MR].

2) If $G(\mathbf{C})$ does not act with isolated fixed points, then the right hand side vanishes, $c_{\mathrm{top}}(E^G)$ vanishes and we find

$$\widehat{c}_{\mathrm{top}}(\overline{E}^G) = a(T_g(\pi_{\mathbf{C}}, \overline{\mathcal{O}})\mathrm{Td}_g(\overline{E}_{\mathbf{C}})) \ . \tag{9}$$

As was mentioned in [K, eq. (7.8)], one finds in particular

$$\widehat{c}_d(\overline{E}) = a(\gamma) \ . \tag{10}$$

For this statement we need Gillet-Soulé's arithmetic Grothendieck-Riemann-Roch [GS8] in all degrees, which was mentioned to be proven in [S, section 4]; a proof of an analogue statement is given in [R, section 8]. Another proof was sketched in [F] using a possibly different direct image. In case the reader doubts the Theorem to hold, one can at least show the existence of some $(d-1, d-1)$ differential form γ' with $\widehat{c}_d(\overline{E}) = a(\gamma')$ the following way: The analogue proof of equation (10) in the classical algebraic Chow ring $\mathrm{CH}^*(B)$ using the classical Riemann-Roch-Grothendieck Theorem shows the vanishing of $c_d(E)$. Thus by the exact sequence

$$\widetilde{\mathfrak{A}}^{d-1,d-1}(B) \xrightarrow{a} \widehat{\mathrm{CH}}^d(B) \xrightarrow{\zeta} \mathrm{CH}^d(B) \to 0$$

we see that (10) holds with some form γ'. Thus until a full formal proof of the arithmetic Riemann-Roch in all degrees is available, one might read the rest of this paper with this γ' replacing the γ explicitly described in section 2.

Now we restrict ourself to the action of the automorphism (-1). We need to assume that this automorphism corresponds to a μ_2-action. This condition can always be satisfied by changing the base Spec D to Spec $D[\frac{1}{2}]$ ([KR1, Introduction] or [KR4, section 2]).

Theorem 3.4 *Let $\pi : Y \to B$ denote a principally polarized abelian scheme of relative dimension d over an arithmetic variety B. Set $\overline{E} := (R^1\pi_*\mathcal{O}, \|\cdot\|_{L^2})^*$. Then the Pontrjagin classes of $\overline{E}$ are given by*

$$\widehat{p}_k(\overline{E}) = (-1)^k \left(\frac{2\zeta'(1-2k)}{\zeta(1-2k)} + \sum_{j=1}^{2k-1} \frac{1}{j} - \frac{2\log 2}{1-4^{-k}} \right) (2k-1)!\, a(\mathrm{ch}(E)^{[2k-1]}) \ . \tag{11}$$

The log 2-term actually vanishes in the arithmetic Chow ring over Spec $D[1/2]$.
Remark: The occurrence of R-class-like terms in Theorem 3.4 makes it very unlikely that there is an easy proof of this result which does not use arithmetic Riemann-Roch-Theorems. This is in sharp contrast to the classical case over $\mathbf{C}$, where the analogues formulae are a trivial topological result: The underlying real vector bundle of $E_{\mathbf{C}}$ is flat, as the period lattice determines a flat structure. Thus the topological Pontrjagin classes $p_j(E_{\mathbf{C}})$ vanish.

Proof: Let $Q(z)$ denote the power series in z given by the Taylor expansion of

$$4(1+e^{-z})^{-1}(1+e^z)^{-1} = \frac{1}{\cosh^2 \frac{z}{2}}$$

at $z=0$. Let $\widehat{Q}$ denote the associated multiplicative arithmetic characteristic class. Thus by definition for $G=\mu_2$

$$4^d\widehat{\mathrm{Td}}_G(\overline{E})\widehat{\mathrm{Td}}_G(\overline{E}^*)=\widehat{Q}(\overline{E})$$

and $\widehat{Q}$ can be represented by Pontrjagin classes, as the power series Q is even. Now we can apply Lemma 3.1 for Pontrjagin classes to equation (6) of equation (8). By a formula by Cauchy [Hi3, §1, eq. (10)], the summand of $\widehat{Q}$ consisting only of single Pontrjagin classes is given by taking the Taylor series in z at $z=0$ of

$$Q(\sqrt{-z})\frac{d}{dz}\frac{z}{Q(\sqrt{-z})}=\frac{\frac{d}{dz}(z\cosh^2\frac{\sqrt{-z}}{2})}{\cosh^2\frac{\sqrt{-z}}{2}}=1+\frac{\sqrt{-z}}{2}\tanh\frac{\sqrt{-z}}{2} \tag{12}$$

and replacing every power z^j by $\widehat{p}_j$. The bundle $\overline{E}^G$ is trivial, hence $\widehat{c}_{\mathrm{top}}(\overline{E}^G)=1$. Thus we obtain by equation (6) with $\pi_*^G\pi^*1=4^d$

$$\sum_{k=1}^{\infty}\frac{(4^k-1)(-1)^{k+1}}{(2k-1)!}\zeta(1-2k)\widehat{p}_k(\overline{E})=-a(R_g(E_{\mathbf{C}}))\ .$$

Consider the zeta function $L(\alpha,s)=\sum_{k=1}^{\infty}k^{-s}\alpha^k$ for Re $s>1$, $|\alpha|=1$. It has a meromorphic continuation to $s\in\mathbf{C}$ which shall be denoted by L, too. Then $L(-1,s)=(2^{1-s}-1)\zeta(s)$ and the function

$$\widetilde{R}(\alpha,x):=\sum_{k=0}^{\infty}\left(\frac{\partial L}{\partial s}(\alpha,-k)+L(\alpha,-k)\sum_{j=1}^{k}\frac{1}{2j}\right)\frac{x^k}{k!}$$

by which the Bismut equivariant R-class is constructed in [KR1, Def. 3.6] verifies for $\alpha=-1$

$$\begin{aligned}\widetilde{R}(-1,x)-\widetilde{R}(-1,-x) &= \sum_{k=1}^{\infty}\Big[(4^k-1)(2\zeta'(1-2k)+\zeta(1-2k)\sum_{j=1}^{2k-1}\frac{1}{j})\\ &\qquad -2\log 2\cdot 4^k\zeta(1-2k)\Big]\frac{x^{2k-1}}{(2k-1)!}\ .\end{aligned} \tag{13}$$

Thus we finally obtain the desired result. **Q.E.D.**

The first Pontrjagin classes are given by

$$\widehat{p}_1=-2\widehat{c}_2+\widehat{c}_1^2,\qquad \widehat{p}_2=2\widehat{c}_4-2\widehat{c}_3\widehat{c}_1+\widehat{c}_2^2,\qquad \widehat{p}_3=-2\widehat{c}_6+2\widehat{c}_5\widehat{c}_1-2\widehat{c}_4\widehat{c}_2+\widehat{c}_3^2\ .$$

In general, $\widehat{p}_k=(-1)^k2\widehat{c}_{2k}$+products of Chern classes. Thus knowing the Pontrjagin classes allows us to express the Chern classes of even degree by the Chern classes of odd degree.

Corollary 3.5 *The Chern-Weil forms representing the Pontrjagin classes vanish:*

$$c(\overline{E} \oplus \overline{E}^*) = 0 \qquad , \textit{ i.e.} \qquad \det(1 + (\Omega^E)^{\wedge 2}) = 0$$

for the curvature Ω^E of the Hodge bundle. The Pontrjagin classes in the algebraic Chow ring $\mathrm{CH}(B)$ *vanish:*

$$c(E \oplus E^*) = 0 \ .$$

Proof: These facts follow from applying the forget-functors $\omega : \widehat{\mathrm{CH}}(B) \to \mathfrak{A}(B(\mathbf{C}))$ and $\zeta : \widehat{\mathrm{CH}}(B) \to \mathrm{CH}(B)$. **Q.E.D.**

The first fact can also be deduced by "linear algebra", e.g. using the Mathai-Quillen calculus, but it is not that easy. The second statement was obtained in [G, Th. 2.5] using the non-equivariant Grothendieck-Riemann-Roch Theorem and the geometry of theta divisors.

4 A K-theoretical proof

The Pontrjagin classes form one set of generators of the algebra of even classes; another important set of generators is given by the Chern character in even degrees $2k$ times $(2k)!$. We give the value of these classes below. Let U denote the additive characteristic class associated to the power series

$$U(x) := \sum_{k=1}^{\infty} \left(\frac{\zeta'(1-2k)}{\zeta(1-2k)} + \sum_{j=1}^{2k-1} \frac{1}{2j} - \frac{\log 2}{1-4^{-k}} \right) \frac{x^{2k-1}}{(2k-1)!} \ .$$

Corollary 4.1 *The part of* $\widehat{\mathrm{ch}}(\overline{E})$ *in* $\widehat{\mathrm{CH}}^{\mathrm{even}}(B)_{\mathbf{Q}}$ *is given by the formula*

$$\widehat{\mathrm{ch}}(\overline{E})^{[\mathrm{even}]} = d - a(U(E)) \ .$$

Proof: The part of $\widehat{\mathrm{ch}}(\overline{E})$ of even degree equals

$$\widehat{\mathrm{ch}}(\overline{E})^{[\mathrm{even}]} = \frac{1}{2} \widehat{\mathrm{ch}}(\overline{E} \oplus \overline{E}^*) \ ,$$

thus it can be expressed by Pontrjagin classes. More precisely by Newton's formulae ([Hi3, §10.1]),

$$(2k)!\widehat{\mathrm{ch}}^{[2k]} - \widehat{p}_1 \cdot (2k-2)!\widehat{\mathrm{ch}}^{[2k-2]} + \cdots + (-1)^{k-1}\widehat{p}_{k-1} 2!\widehat{\mathrm{ch}}^{[2]} = (-1)^{k+1} k \widehat{p}_k$$

for $k \in \mathbf{N}$. As products of the arithmetic Pontrjagin classes vanish in $\widehat{\mathrm{CH}}(Y)_{\mathbf{Q}}$ by Lemma 3.4, we thus observe that the part of $\widehat{\mathrm{ch}}(\overline{E})$ in $\widehat{\mathrm{CH}}^{\mathrm{even}}(Y)_{\mathbf{Q}}$ is given by

$$\widehat{\mathrm{ch}}(\overline{E})^{[\mathrm{even}]} = d + \sum_{k>0} \frac{(-1)^{k+1}\widehat{p}_k(\overline{E})}{2(2k-1)!} \ .$$

Thus the result follows from Lemma 3.4. **Q.E.D.**

As Harry Tamvakis pointed out to the author, a similar argument is used in [T1, section 2] and its predecessors.

Now we show how to deduce Corollary 4.1 (and thus the equivalent Theorem 3.4) using only Theorem 3.2 without combining it with the arithmetic Grothendieck-Riemann-Roch Theorem as in Theorem 3.3. Of course the structure of the proof shall not be too different as the Grothendieck-Riemann-Roch Theorem was very simple in this case; but the following proof is quite instructive as it provides a different point of view on the resulting characteristic classes.

Theorem 3.2 applied to the abelian scheme $\pi : Y \to B$ provides the formula

$$\pi_! \overline{\mathcal{O}} = \pi_!^{\mu_2} \frac{1 - a(R_g(N_{Y/Y_{\mu_n}}))}{\lambda_{-1}(\overline{N}^*_{Y/Y_{\mu_n}})} \ .$$

In our situation, $\overline{N_{Y/Y_{\mu_n}}} = \overline{T\pi}$. Combining this with the fundamental equations (3), (4) and Theorem 2.1 yields

$$\lambda_{-1}\overline{E}^* = \pi_!^{\mu_2} \pi^* \frac{1 - a(R_g(E^*))}{\lambda_{-1}\overline{E}}$$

and using the projection formula we find

$$\lambda_{-1}\overline{E \oplus E^*} = 4^d(1 - a(R_g(E^*))) \ .$$

Let $\overline{E}'$ denote the vector bundle E equipped with the trivial μ_2-action. Now one can deduce from this that $\overline{E' \oplus E'^*}$ itself has the form $2d + a(\eta)$ with a $\bar{\partial}\partial$-closed form η: Apply the Chern character to both sides. Then use equation (12) and Lemma 3.1 to deduce by induction that all Chern classes of $\overline{E' \oplus E'^*}$ are in $a(\ker \bar{\partial}\partial)$. Thus using the fact that the arithmetic Chern character is an isomorphism up to torsion ([GS3, Th. 7.3.4]) $\overline{E' \oplus E'^*} = 2d + a(\eta)$ with $a(\eta)$ having even degrees, and $\overline{E \oplus E^*} = (2d + a(\eta)) \otimes (-1)$ in $\widehat{K}^{\mu_2}(B)_{\mathbf{Q}}$. One could use the γ-filtration instead to obtain this result; it would be interesting to find a proof which does not use any filtration.

For a $\beta \in \widetilde{\mathfrak{A}}^{p,p}(B)$, the action of the λ-operators can be determined as follows: The action of the k-th Adams operator is given by $\psi^k a(\beta) = k^{p+1} a(\beta)$ ([GS3, p. 235]). Then with $\psi_t := \sum_{k>0} t^k \psi^k$, $\lambda_t := \sum_{k\geq 0} t^k \lambda^k$ the Adams operators are related to the λ-operators via

$$\psi_t(x) = -t \frac{d}{dt} \log \lambda_{-t}(x)$$

for $x \in \widehat{K}^{\mu_N}(B)$. As $\psi_t(a(\beta)) = \mathrm{Li}_{-1-p}(t) a(\beta)$ with the polylogarithm Li, we find for $\beta \in \ker \bar{\partial}\partial$

$$\lambda_t(a(\beta)) = 1 - \mathrm{Li}_{-p}(-t) a(\beta)$$

or $\lambda^k a(\beta) = -(-1)^k k^p a(\beta)$ ($\mathrm{Li}_{-p}(\frac{t}{t-1})$ is actually a polynomial in t; in this context this can be regarded as a relation coming from the γ-filtration). In particular $\lambda_{-1} a(\beta) = 1 - \zeta(-p) a(\beta)$, and $\lambda_{-1}(a(\beta) \otimes (-1)) = \lambda_1 a(\beta) \otimes 1 = (1 + (1 - 2^{p+1})\zeta(-p) a(\beta)) \otimes 1$ in $\widehat{K}^{\mu_2} \otimes_{R_{\mu_2}} \mathbf{C}$.

By comparing

$$\lambda_{-1}(a(\eta) \otimes (-1)) = a(\sum_{k>0} \zeta(1-2k)(1-4^k)\eta^{[2k-1]}) \otimes 1 = a(R_{-1}(E^*)) \otimes 1$$

we finally derive $a(\eta) = a(-2U(E))$ and thus

$$\overline{E' \oplus E'^*} = 2d - 2a(U(E)) \ .$$

From this Corollary 4.1 follows.

5 A Hirzebruch proportionality principle and other applications

The following formula can be used to express the height of complete subvarieties of codimension d of the moduli space of abelian varieties as an integral over differential forms.

Theorem 5.1 *There is a real number $r_d \in \mathbf{R}$ and a Chern-Weil form $\phi(\overline{E})$ on $B_{\mathbf{C}}$ of degree $(d-1)(d-2)/2$ such that*

$$\widehat{c}_1^{1+d(d-1)/2}(\overline{E}) = a(r_d \cdot c_1^{d(d-1)/2}(E) + \phi(\overline{E})\gamma) \ .$$

The form $\phi(\overline{E})$ is actually a polynomial with integral coefficients in the Chern forms of $\overline{E}$. See Corollary 5.5 for a formula for r_d.

Proof: Consider the graded ring R_d given by $\mathbf{Q}[u_1, \ldots, u_d]$ divided by the relations

$$(1 + \sum_{j=1}^{d-1} u_j)(1 + \sum_{j=1}^{d-1} (-1)^j u_j) = 1, \qquad u_d = 0 \tag{14}$$

where u_j shall have degree j ($1 \leq j \leq d$). This ring is finite dimensional as a vector space over $\mathbf{Q}$ with basis

$$u_{j_1} \cdots u_{j_m}, \qquad 1 \leq j_1 < \cdots < j_m < d \ , 1 \leq m < d \ .$$

In particular, any element of R_d has degree $\leq \frac{d(d-1)}{2}$. As the relation (14) is verified for $u_j = \widehat{c}_j(\overline{E})$ up to multiples of the Pontrjagin classes and $\widehat{c}_d(\overline{E})$, any polynomial in the $\widehat{c}_j(\overline{E})$'s can be expressed in terms of the $\widehat{p}_j(\overline{E})$'s and $\widehat{c}_d(\overline{E})$ if the corresponding polynomial in the u_j's vanishes in R_d.

Thus we can express $\widehat{c}_1^{1+d(d-1)/2}(\overline{E})$ as the image under a of a topological characteristic class of degree $\frac{d(d-1)}{2}$ plus γ times a Chern-Weil form of degree $\frac{(d-1)(d-2)}{2}$. As any element of degree $\frac{d(d-1)}{2}$ in R_d is proportional to $u_1^{d(d-1)/2}$, the Theorem follows. **Q.E.D.**

Any other arithmetic characteristic class of $\overline{E}$ vanishing in R_d can be expressed in a similar way.

Example: We shall compute $\widehat{c}_1^{1+d(d-1)/2}(\overline{E})$ explicitly for small d. Define topological cohomology classes r_j by $\widehat{p}_j(\overline{E}) = a(r_j)$ via Lemma 3.4. For $d = 1$, clearly

$$\widehat{c}_1(\overline{E}) = a(\gamma) \ .$$

In the case $d = 2$ we find by the formula for $\widehat{p}_1$

$$\widehat{c}_1^2(\overline{E}) = a(r_1 + 2\gamma) = a\Big[(-1 + \frac{8}{3}\log 2 + 24\zeta'(-1))c_1(E) + 2\gamma\Big] \ .$$

Combining the formulae for the first two Pontrjagin classes we get

$$\widehat{p}_2 = 2\widehat{c}_4 - 2\widehat{c}_3\widehat{c}_1 + \frac{1}{4}\widehat{c}_1^4 - \frac{1}{2}\widehat{c}_1^2\widehat{p}_1 + \frac{1}{4}\widehat{p}_1^2.$$

Thus for $d = 3$ we find, using $c_3(E) = 0$ and $c_1^2(E) = 2c_2(E)$,

$$\begin{aligned}\widehat{c}_1^4(\overline{E}) &= a(2c_1^2(E)r_1 + 4r_2 + 8c_1(E)\gamma) \\ &= a\Big[(-\frac{17}{3} + \frac{48}{5}\log 2 + 48\zeta'(-1) - 480\zeta'(-3))c_1^3(E) + 8c_1(\overline{E})\gamma\Big] \ .\end{aligned}$$

For $d = 4$ one obtains

$$\begin{aligned}\widehat{c}_1^7(\overline{E}) &= a\Big[64c_2(E)c_3(E)r_1 - (8c_1(E)c_2(E) + 32c_3(E))r_2 + 64c_1(E)r_3 \\ &\quad +16(7c_1(\overline{E})c_2(\overline{E}) - 4c_3(\overline{E}))\gamma\Big] \ .\end{aligned}$$

As in this case $\mathrm{ch}(E)^{[1]} = c_1(E)$, $3!\mathrm{ch}(E)^{[3]} = -c_1^3(E)/2 + 3c_3(E)$ and $5!\mathrm{ch}(E)^{[5]} = c_1^5(E)/16$, we find

$$\begin{aligned}\widehat{c}_1^7(\overline{E}) &= a\Big[(-\frac{1063}{60} + \frac{1520}{63}\log 2 + 96\zeta'(-1) - 600\zeta'(-3) + 2016\zeta'(-5))c_1^6(E) \\ &\quad +16(7c_1(\overline{E})c_2(\overline{E}) - 4c_3(\overline{E}))\gamma\Big] \ .\end{aligned}$$

For $g = 5$ on gets

$$\begin{aligned}\widehat{c}_1^{11}(\overline{E}) &= a\Big[2816c_2(3c_1c_3 - 8c_4)(\overline{E})\gamma + (-\frac{104611}{2520} + \frac{113632}{2295}\log(2) \\ &\quad +176\zeta'(-1) - 760\zeta'(-3) + 2352\zeta'(-5) - 3280\zeta'(-7))c_1^{10}(E)\Big]\end{aligned}$$

and for $g=6$

$$\begin{aligned}\widehat{c}_1^{16}(\overline{E}) &= a\Big[425984(11c_1c_2c_3c_4-91c_2c_3c_5+40c_1c_4c_5)(\overline{E})\gamma \\ &+(-\frac{3684242}{45045}+\frac{3321026752}{37303695}\log(2)+\frac{3264\,\zeta'(-1)}{11}-\frac{136320\,\zeta'(-3)}{143} \\ &+\frac{395136\,\zeta'(-5)}{143}-\frac{526080\,\zeta'(-7)}{143}+\frac{36096\,\zeta'(-9)}{13})c_1^{15}(E)\Big]\ .\end{aligned}$$

Remark: We shall shortly describe the effect of rescaling the metric for the characteristic classes described above. By the multiplicativity of the Chern character and using $\widehat{\mathrm{ch}}(\mathcal{O},\alpha|\cdot|^2)=1-a(\log\alpha)$, $\widehat{\mathrm{ch}}(\overline{E})$ changes by

$$\log\alpha\cdot a(\mathrm{ch}(E))$$

when multiplying the metric on E^* by a constant $\alpha\in\mathbf{R}^+$ (or with a function $\alpha\in C^\infty(B(\mathbf{C}),\mathbf{R}^+)$). Thus, we observe that in our case $\widehat{\mathrm{ch}}(\overline{E})^{[\mathrm{odd}]}$ is invariant under rescaling on E^* and we get an additional term

$$\log\alpha\cdot a(\mathrm{ch}(E)^{[\mathrm{odd}]})$$

on the right hand side in Corollary 4.1, when the volume of the fibers equals α^d instead of 1. Thus the right hand side of Theorem 3.4 gets an additional term

$$\frac{(-1)^{k+1}\log\alpha}{2(2k-1)!}a(\mathrm{ch}(E)^{[2k-1]})\ .$$

Similarly,

$$\hat{c}_d(\overline{E})=a(\gamma)+\log\alpha\cdot a(c_{d-1}(E))$$

for the rescaled metric. In Theorem 5.1, we obtain an additional

$$\log\alpha\cdot a(\frac{d(d-1)+2}{2}c_1^{d(d-1)/2}(E))$$

on the right hand side and this shows

$$\phi(E)c_{d-1}(E)=\frac{d(d-1)+2}{2}c_1^{d(d-1)/2}(E)\ . \tag{15}$$

Alternatively, one can show the same formulae by investigating directly the Bott-Chern secondary class of $R\pi_*\mathcal{O}$ for the metric change.

Assume that the base space Spec D equals Spec $\mathcal{O}_K[\frac{1}{2}]$ for a number field K. We consider the push forward map

$$\widehat{\deg}:\widehat{\mathrm{CH}}(B)\to\widehat{\mathrm{CH}}(\mathrm{Spec}(\mathcal{O}_K[\frac{1}{2}]))\to\widehat{\mathrm{CH}}(\mathrm{Spec}(\mathbf{Z}[\frac{1}{2}]))\cong\mathbf{R}/(\mathbf{Q}\log 2)$$

where the last identification contains the traditional factor $\frac{1}{2}$. Using the definition $h(B) := \frac{1}{[K:\mathbf{Q}]}\widehat{\deg}\,\widehat{c}_1^{1+\dim B_{\mathbf{C}}}(\overline{E})$ of the **global height** (thus defined modulo rational multiples of $\log 2$ in this case) of a projective arithmetic variety we find

Corollary 5.2 *If* $\dim B_{\mathbf{C}} = \frac{d(d-1)}{2}$ *and* B *is projective, then the (global) height of* B *with respect to* $\det \overline{E}$ *is given by*

$$h(B) = \frac{r_d}{2} \cdot \deg B + \frac{1}{2}\int_{B_{\mathbf{C}}} \phi(\overline{E})\gamma \ .$$

with deg *denoting the algebraic degree.*

Corollary 5.3 *For* B *as in Corollary 5.2 set* $h'(B) := \frac{h(B)}{(\dim B_{\mathbf{C}}+1)\deg B}$. *The height of* B *changes under the action of the Hecke operator* $T(p)$ *by*

$$h'(T(p)B) = h'(B) + \frac{p^d-1}{p^d+1} \cdot \frac{\log p}{2} \ .$$

Proof: For this proof we need that γ is indeed the form determined by the arithmetic Riemann-Roch Theorem in all degrees (compare equation (10)). The action of Hecke operators on γ was investigated in [K, Section 7]. In particular it was shown that

$$T(p)\gamma = \prod_{j=1}^{d}(p^j+1)\left(\gamma + \frac{p^d-1}{p^d+1}\log p \cdot c_{d-1}(\overline{E})\right) \ .$$

The action of Hecke operators commutes with multiplication by a characteristic class, as the latter are independent of the period lattice in E. Thus by Theorem 5.2 the height of $T(p)B$ is given by

$$\begin{aligned} h(T(p)B) \;&=\; \prod_{j=1}^{d}(p^j+1)\Bigg(\frac{r_d}{2}\cdot \deg B_{\mathbf{C}} + \frac{1}{2}\int_{B_{\mathbf{C}}}\phi(\overline{E})\gamma \\ &\quad + \frac{p^d-1}{p^d+1}\frac{\log p}{2}\int_{B_{\mathbf{C}}}\phi(\overline{E})c_{d-1}(E)\Bigg) \ . \end{aligned}$$

Combining this with equation (15) gives the result. **Q.E.D.**

Similarly one obtains a formula for the action of any other Hecke operator using the explicit description of its action on γ in [K, equation (7.4)].

Now we are going to formulate an Arakelov version of Hirzebruch's proportionality principle. In [Hi2, p. 773] it is stated as follows: Let G/K be a non-compact irreducible symmetric space with compact dual G'/K and let $\Gamma \subset G$ be a cocompact subgroup such that $\Gamma\backslash G/K$ is a smooth manifold. Then there is an ring monomorphism

$$h : H^*(G'/K, \mathbf{Q}) \to H^*(\Gamma\backslash G/K, \mathbf{Q})$$

such that $h(c(TG'/K)) = c(TG/K)$ (and similar for other bundles F', F corresponding to K-representation V', V dual to each other). This implies in particular that Chern numbers on G'/K and $\Gamma\backslash G/K$ are proportional[Hi1, p. 345]. Now in our case think for the moment about B as the moduli space of principally polarized abelian varieties of dimension d. Its projective dual is the Lagrangian Grassmannian L_d over Spec $\mathbf{Z}$ parametrizing isotropic subspaces in symplectic vector spaces of dimension $2d$ over any field, $L_d(\mathbf{C}) = \mathbf{Sp}(d)/\mathbf{U}(d)$. But as the moduli space is a non-compact quotient, the proportionality principle must be altered slightly by considering Chow rings modulo certain ideals corresponding to boundary components in a suitable compactification. For that reason we consider the Arakelov Chow group $\mathrm{CH}^*(\overline{L}_{d-1})$, which is the quotient of $\mathrm{CH}^*(\overline{L}_d)$ modulo the ideal $(\widehat{c}_d(\overline{S}), a(c_d(\overline{S})))$ with $\overline{S}$ being the tautological bundle on L_d, and we map it to $\widehat{\mathrm{CH}}^*(Y)/(a(\gamma))$. Here L_{d-1} shall be equipped with the canonical symmetric metric. For the Hermitian symmetric space L_{d-1}, the Arakelov Chow ring is a subring of the arithmetic Chow ring $\widehat{\mathrm{CH}}(L_{d-1})$ ([GS2, 5.1.5]) such that the quotient abelian group depends only on $L_{d-1}(\mathbf{C})$. Instead of dealing with the moduli space, we continue to work with a general regular base B.

Theorem 5.4 *There is a ring homomorphism*

$$h : \mathrm{CH}^*(\overline{L}_{d-1})_{\mathbf{Q}} \to \widehat{\mathrm{CH}}^*(B)_{\mathbf{Q}}/(a(\gamma))$$

with

$$h(\widehat{c}(\overline{S})) = \widehat{c}(\overline{E}) \left(1 + a\left(\sum_{k=1}^{d-1}\left(\frac{\zeta'(1-2k)}{\zeta(1-2k)} - \frac{\log 2}{1-4^{-k}}\right)(2k-1)!\mathrm{ch}^{[2k-1]}(E)\right)\right)$$

and

$$h(a(c(\overline{S}))) = a(c(\overline{E})) \ .$$

Note that S^* and E are ample. One could as well map $a(c(\overline{S}^*))$ to $a(c(\overline{E}))$, but the correction factor for the arithmetic characteristic classes would have additional harmonic number terms.

Remark: For $d \leq 6$ one can in fact construct such a ring homomorphism which preserves degrees. Still this seems to be a very unnatural thing to do.

This is thus in remarkable contrast to the classical Hirzebruch proportionality principle.

Proof: The Arakelov Chow ring $\mathrm{CH}^*(\overline{L}_{d-1})$ has been investigated by Tamvakis in [T1]. Consider the graded commutative ring

$$\mathbf{Z}[\widehat{u}_1, \dots, \widehat{u}_{d-1}] \oplus \mathbf{R}[u_1, \dots, u_{d-1}]$$

where the ring structure is such that $\mathbf{R}[u_1, \dots, u_{d-1}]$ is an ideal of square zero. Let $\widehat{R}_d$ denote the quotient of this ring by the relations

$$(1 + \sum_{j=1}^{d-1} u_j)(1 + \sum_{j=1}^{d-1} (-1)^j u_j) = 1$$

and

$$(1+\sum_{k=1}^{d-1} \widehat{u}_k)(1+\sum_{k=1}^{d-1} (-1)^k \widehat{u}_k) = 1-\sum_{k=1}^{d-1} \left(\sum_{j=1}^{2k-1} \frac{1}{j} \right) (2k-1)!\mathrm{ch}^{[2k-1]}(u_1, \dots, u_{d-1}) \tag{16}$$

where $\mathrm{ch}(u_1, \dots, u_{d-1})$ denotes the Chern character polynomial in the Chern classes, taken of $u_1, \dots, u_{d-1}$. Then by [T1, Th. 1], there is a ring isomorphism $\Phi : \widehat{R}_d \to \mathrm{CH}^*(\overline{L}_{d-1})$ with $\Phi(\widehat{u}_k) = \widehat{c}_k(\overline{S}^*)$, $\Phi(u_k) = a(c_k(\overline{S}^*))$. The Chern character term in (16), which could be written more carefully as $(0, \mathrm{ch}^{[2k-1]}(u_1, \dots, u_{d-1}))$, is thus mapped to $a(\mathrm{ch}^{[2k-1]}(c_1(\overline{S}^*), \dots, c_{d-1}(\overline{S}^*)))$. When writing the relation (16) as

$$\widehat{c}(\overline{S})\widehat{c}(\overline{S}^*) = 1 + a(\epsilon_1)$$

and the relation in Theorem 3.4 as

$$\widehat{c}(\overline{E})\widehat{c}(\overline{E}^*) = 1 + a(\epsilon_2)$$

we see that a ring homomorphism h is given by

$$h(\widehat{c}_k(\overline{S})) = \sqrt{\frac{1 + h(a(\epsilon_1))}{1 + a(\epsilon_2)}} \widehat{c}_k(\overline{E}) = (1 + \frac{1}{2} h(a(\epsilon_1)) - \frac{1}{2} a(\epsilon_2)) \widehat{c}_k(\overline{E})$$

(where h on im a is defined as in the Theorem). Here the factor $1+\frac{1}{2}h(a(\epsilon_1))-\frac{1}{2}a(\epsilon_2)$ has even degree, and thus

$$h(\widehat{c}_k(\overline{S}^*)) = \sqrt{\frac{1 + h(a(\epsilon_1))}{1 + a(\epsilon_2)}} \widehat{c}_k(\overline{E}^*)$$

which provides the compatibility with the cited relations. **Q.E.D.**

Remarks: 1) Note that this proof does not make any use of the remarkable fact that $h(a(\epsilon_1^{[k]}))$ and $a(\epsilon_2^{[k]})$ are proportional forms for any degree k.

2) It would be favorable to have a more direct proof of Theorem 5.4, which does not use the description of the tautological subring. The R-class-like terms suggest that one has to use an arithmetic Riemann-Roch Theorem somewhere in the proof; one could wonder whether one could obtain the description of $\mathrm{CH}^*(\overline{L}_{d-1})$ by a method similar to section 3. Also, one might wonder whether the statement holds for other symmetric spaces. Our construction relies on the existence of a universal proper bundle with a fibrewise acting non-trivial automorphism; thus it shall not extend easily to other cases.

In particular Tamvakis' height formula [T1, Th. 3] provides a combinatorial formula for the real number r_d occurring in Theorem 5.1. Replace each term $\mathcal{H}_{2k-1}$ occurring in [T1, Th. 3] by

$$-\frac{2\zeta'(1-2k)}{\zeta(1-2k)} - \sum_{j=1}^{2k-1}\frac{1}{j} + \frac{2\log 2}{1-4^{-k}}$$

and divide the resulting value by half of the degree of L_{d-1}. Using Hirzebruch's formula

$$\deg L_{d-1} = \frac{(d(d-1)/2)!}{\prod_{k=1}^{d-1}(2k-1)!!}$$

for the degree of L_{d-1} (see [Hi1, p. 364]) and the $\mathbf{Z}_+$-valued function $g^{[a,b]_{d-1}}$ from [T1] counting involved combinatorial diagrams, we obtain

Corollary 5.5 *The real number r_d occurring in Theorem 5.1 is given by*

$$\begin{aligned} r_d \;&=\; \frac{2^{1+(d-1)(d-2)/2}\prod_{k=1}^{d-1}(2k-1)!!}{(d(d-1)/2)!} \\ &\cdot \sum_{k=0}^{d-2}\left(-\frac{2\zeta'(-2k-1)}{\zeta(-2k-1)} - \sum_{j=1}^{2k+1}\frac{1}{j} + \frac{2\log 2}{1-4^{-k-1}}\right) \\ &\cdot \sum_{b=0}^{\min\{k,d-2-k\}} (-1)^b 2^{-\delta_{b,k}} g^{[k-b,b]_{d-1}} \end{aligned}$$

where $\delta_{b,k}$ is Kronecker's δ.

One might wonder whether there is a "topological" formula for the height of locally symmetric spaces similar to [KK, Theorem 8.1]. Comparing the fixed point version [KK, Lemma 8.3] of the topological height formula with the Schubert calculus expression [T1, Th. 3] for the height of Lagrangian Grassmannians, one finds the following relation for the numbers $g^{[k-b,b]_{d-1}}$:

$$\sum_{\epsilon_1,\dots,\epsilon_{d-1}\in\{\pm 1\}} \frac{1}{\prod_{i\le j}(\epsilon_i i+\epsilon_j j)} \sum_{\ell=1}^{\frac{d(d-1)}{2}} \sum_{i\le j} \frac{(\sum \epsilon_\nu \nu)^{\frac{d(d-1)}{2}} - (\sum \epsilon_\nu \nu)^{\frac{d(d-1)}{2}-\ell+1} (\sum \epsilon_\nu \nu - (2-\delta_{ij}(\epsilon_i i+\epsilon_j j))^\ell}{2\ell(\epsilon_i i+\epsilon_j j)}$$

$$= \sum_{k=0}^{d-2} \left(\sum_{j=1}^{2k-1} \frac{1}{j} \right) \sum_{b=0}^{\min\{k,d-2-k\}} (-1)^b 2^{-\delta_{b,k}} g^{[k-b,b]_{d-1}} .$$

Similarly one can compare with the classical Schubert calculus version of this height formula derived in [T2].

In [G, Th. 2.5] van der Geer shows that R_d embeds into the (classical) Chow ring $\mathrm{CH}^*(\mathcal{M}_d)_{\mathbf{Q}}$ of the moduli stack $\mathcal{M}_d$ of principally polarized abelian varieties. Using this result one finds

Lemma 5.6 *Let B be a regular finite covering of the moduli space $\mathcal{M}_d$ of principally polarized abelian varieties of dimension d. Then for any non-vanishing polynomial expression $p(u_1,\dots,u_{d-1})$ in R_d,*

$$h(p(\widehat{c}_1(\overline{S}),\dots,\widehat{c}_{d-1}(\overline{S}))) \notin \operatorname{im} a .$$

In particular, h is non-trivial in all degrees. Furthermore, h is injective iff $a(c_1(E)^{d(d-1)/2}) \neq 0$ in $\widehat{\mathrm{CH}}^{d(d-1)/2+1}(B)_{\mathbf{Q}}/(a(\gamma))$.

Proof: Consider the canonical map $\zeta : \widehat{\mathrm{CH}}^*(B)_{\mathbf{Q}}/(a(\gamma)) \to \mathrm{CH}^*(B)_{\mathbf{Q}}$. Then

$$\zeta(h(p(\widehat{c}_1(\overline{S}),\dots,\widehat{c}_{d-1}(\overline{S})))) = p(c_1(E),\dots,c_{d-1}(E)) ,$$

and the latter is non-vanishing according to [G, Th. 1.5]. This shows the first part.

If $a(c_1(E)^{d(d-1)/2}) \neq 0$ in $\widehat{\mathrm{CH}}^{d(d-1)/2+1}(B)_{\mathbf{Q}}/(a(\gamma))$, then by the same induction argument as in the proof of [G, Th. 2.5] R_d embeds in $a(\ker \bar\partial\partial)$. Finally, by [T1, Th. 2] any element z of $\widehat{R}_d$ can be written in a unique way as a linear combination of

$$\widehat{u}_{j_1}\cdots\widehat{u}_{j_m} \quad \text{and} \quad u_{j_1}\cdots u_{j_m}, \qquad 1\le j_1<\cdots<j_m<d \,, 1\le m<d .$$

Thus if $z \notin \operatorname{im} a$, then $h(z) \neq 0$ follows by van der Geer's result, and if $z \in \operatorname{im} a \setminus \{0\}$, then $h(z) \neq 0$ follows by embedding $R_d \otimes \mathbf{R}$. **Q**.**E**.**D**.

Using the exact sequence (2), the condition in the Lemma is that the cohomology class $c_1(E)^{d(d-1)/2}$ should not be in the image of the Beilinson regulator.

Finally by comparing Theorem 5.1 with Kühn's result [Kü, Theorem 6.1], we conjecture that the analogue of Theorem 5.4 holds in a yet to be developed Arakelov intersection theory with logarithmic singularities, extending the methods of [Kü], as described in [MR]. I.e. there should be a ring homomorphism to the Chow ring of the moduli space of abelian varieties

$$h : \mathrm{CH}^*(\overline{L}_d)_{\mathbf{Q}} \to \widehat{\mathrm{CH}}^*(\mathcal{M}_d)_{\mathbf{Q}}$$

extending the one in Theorem 5.4, and γ should provide the Green current corresponding to $\hat{c}_d(\overline{E})$. This would imply

Conjecture 5.7 *For an Arakelov intersection theory with logarithmic singularities, extending the methods of [Kü], the height of a moduli space $\mathcal{M}_d$ over* Spec **Z** *of principally polarized abelian varieties of relative dimension d is given by*

$$h(\mathcal{M}_d) = \frac{r_{d+1}}{2} \deg(\mathcal{M}_d) \ .$$

References

[B3] Bismut, J.-M.: *Equivariant immersions and Quillen metrics.* J. Diff. Geom. **41**, 53-157 (1995).

[B4] Bismut, J.-M.: Holomorphic families of immersions and higher analytic torsion forms. Astérisque **244** :SMF Paris 1997.

[BK] Bismut, J.-M., Köhler, K.: *Higher analytic torsion forms for direct images and anomaly formulas.* J. Algebraic Geom. **1**, 647-684 (1992).

[BM] Bismut, J.-M., Ma, Xiaonan: *Holomorphic immersions and equivariant torsion forms.* To appear.

[BBM] Berthelot, P., Breen, L., Messing, W., Théorie de Dieudonné Cristalline II (Lecture Notes Math. **930**): Springer 1982.

[F] Faltings, G.: Lectures on the arithmetic Riemann-Roch Theorem: Princeton University Press 1992.

[FC] Faltings, G., Chai, Ching-Li: Degeneration of abelian varieties. With an appendix by David Mumford (Erg. Math. **22**): Springer 1990.

[GS2] Gillet, H., Soulé, C.: *Arithmetic intersection theory*. Publ. Math. I.H.E.S. **72**, 94–174 (1990).

[GS3] Gillet, H., Soulé, C.: *Characteristic classes for algebraic vector bundles with hermitian metrics I, II*. Ann. of Math. **131**, 163-203, 205-238 (1990).

[GS8] Gillet, H., Soulé, C.: *An arithmetic Riemann-Roch Theorem*. Invent. Math. **110**, 473-543 (1992).

[G] van der Geer, G.: *Cycles on the moduli space of abelian varieties.* 65–89 in: Faber, C., Looijenga, E.: Moduli of curves and abelian varieties. Aspects Math. **E33**, Vieweg 1999.

[Hi1] Hirzebruch, F.: Collected Papers vol. I: Springer 1987.

[Hi2] Hirzebruch, F.: Collected Papers vol. II: Springer 1987.

[Hi3] Hirzebruch, F.: Topological Methods in Algebraic Geometry: Springer 1978.

[KK] Kaiser, Ch., Köhler, K.: *A fixed point formula of Lefschetz type in Arakelov geometry III: representations of Chevalley schemes and heights of flag varieties*. to appear in Inv. Math.

[K] Köhler, K.: *Torus fibrations and moduli spaces*, 166–195 in: Reznikov, A., Schappacher, N., eds.: Regulators in Analysis, Geometry and Number Theory. Progress in Math. **171**, Birkhäuser 2000.

[KR1] Köhler, K., Roessler, D.: *A fixed point formula of Lefschetz type in Arakelov geometry I: statement and proof*. Inv. Math. **145**, 333-396 (2001).

[KR2] Köhler, K., Roessler, D.: *A fixed point formula of Lefschetz type in Arakelov geometry II: a residue formula*. Ann. Inst. Fourier **52**, 1–23 (2002).

[KR4] Köhler, K., Roessler, D.: *A fixed point formula of Lefschetz type in Arakelov geometry IV: abelian varieties*. Prépublication de l'institut de mathématiques de Jussieu no. **264** (2000). to appear.

[Kü] Kühn, U.,: *Generalized arithmetic intersection numbers.* J. reine angew. Math. **534**, 209–236 (2001).

[Ma] Ma, Xiaonan: *Submersions and equivariant Quillen metrics*, Ann. Inst. Fourier (Grenoble) **50**, 1539–1588 (2000).

[MR] Maillot, V., Roessler, D.: *Conjectures sur les dérivées logarithmiques des fonctions L d'Artin aux entiers négatifs.* to appear.

[MiS] Milnor, J. W., Stasheff, J. D.: Characteristic Classes: Ann. Math. Stud. **76**, Princeton Univ. Press 1974.

[R] Roessler, D.: *An Adams-Riemann-Roch Theorem in Arakelov geometry.* Duke Math. J. **96** 61–126 (1999).

[S] Soulé, C.: *Hermitian vector bundles on arithmetic varieties.* In: Algebraic geometry, Santa Cruz 1995, 383–419, Proc. Sympos. Pure Math. **62**, Part 1, AMS 1997.

[SABK] Soulé, C., Abramovich, D., Burnol, J. F. and Kramer, J.: Lectures on Arakelov Geometry (Cambridge studies in math. 33): Cambridge university press 1992.

[T1] Tamvakis, H.: *Arakelov theory of the Lagrangian Grassmannian.* J. reine angew. Math. **516**, 207-223 (1999).

[T2] Tamvakis, H.: *Height formulas for homogeneous varieties.* Michigan Math. J. **48**, 593-610 (2000).

[Y] Yoshikawa, K.: *Discriminant of theta divisors and Quillen metrics.* J. Differential Geom. **52**, 73–115 (1999).