AROUND CUBIC HYPERSURFACES

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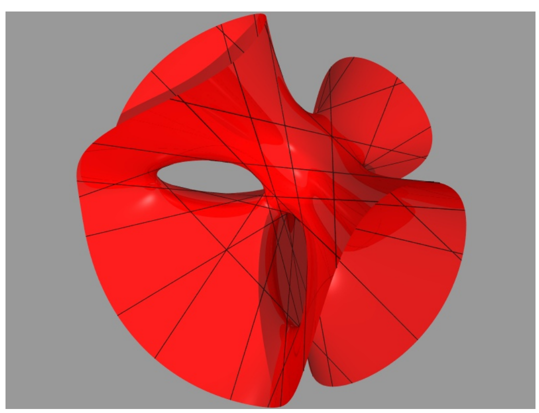
A cubic hypersurface X is defined by one polynomial equation of degree 3 in n variables with coefficients in a field $\mathbf{K}$, such as

$$1 + x_1^3 + \cdots + x_n^3 = 0.$$

One is interested in the set $X(\mathbf{K})$ of solutions $(x_1, \ldots, x_n)$ of this equation in $\mathbf{K}^n$. Depending on the field $\mathbf{K}$ (which may be for example $\mathbf{C}$, $\mathbf{R}$, $\mathbf{Q}$, or $\mathbf{F}_q$) one may ask various questions: is $X(\mathbf{K})$ nonempty? How large is it? What is the topology of $X(\mathbf{K})$? What is its geometry? Can one parametrize $X(\mathbf{K})$ by means of rational functions?

This is a very classical subject: elliptic curves (which are cubics in the plane) and cubic surfaces have fascinated mathematicians from the 19$^{\text{th}}$ century to the present day. Plaster models of the Clebsch cubic surface still decorate many mathematics libraries (or cafeterias, as in Düsseldorf) around the world. However, simple questions about cubics still remain unanswered, although actively researched.

I will explain some classical facts about cubic hypersurfaces and give some answers to the questions asked above.



The Clebsch cubic with its 27 real lines